Order on Order Types*

Alexander Pilz[1] and Emo Welzl[2]

1 **Institute for Software Technology, Graz University of Technology, Austria**
apilz@ist.tugraz.at
2 **Department of Computer Science, Institute of Theoretical Computer Science ETH Zürich, Switzerland**
emo@inf.ethz.ch

Abstract

Given P and P', equally sized planar point sets in general position, we call a bijection from P to P' *crossing-preserving* if crossings of connecting segments in P are preserved in P' (extra crossings may occur in P'). If such a mapping exists, we say that P' *crossing-dominates* P, and if such a mapping exists in both directions, P and P' are called *crossing-equivalent*. The relation is transitive, and we have a partial order on the obtained equivalence classes (called *crossing types* or *x-types*). Point sets of equal order type are clearly crossing-equivalent, but not vice versa. Thus, x-types are a coarser classification than order types. (We will see, though, that a collapse of different order types to one x-type occurs for sets with triangular convex hull only.)

We argue that either the *maximal* or the *minimal x-types* are sufficient for answering many combinatorial (existential or extremal) questions on planar point sets. Motivated by this we consider basic properties of the relation. We characterize order types crossing-dominated by points in convex position. Further, we give a full characterization of minimal and maximal abstract order types. Based on that, we provide a polynomial-time algorithm to check whether a point set crossing-dominates another. Moreover, we generate all maximal and minimal x-types for small numbers of points.

1998 ACM Subject Classification G.2.1 Combinatorics, F.2.2 Nonnumerical Algorithms and Problems

Keywords and phrases point set, order type, planar graph, crossing-free geometric graph

Digital Object Identifier 10.4230/LIPIcs.SOCG.2015.285

Dedicated to Jacob E. Goodman and Richard Pollack on the occasion of their eightieth birthdays.

1 Introduction

Let us start right away with an illustrating example, which did indeed motivate our study. We came across the following nice open question, which was considered in [7] and investigated further in [3]: Given a complete geometric graph (edges as straight segments) on $2m$ points in general position in the plane, is it always possible to partition the edges into m crossing-free spanning trees? For addressing such problems, the concept of order types[1] is ubiquitously

* A.P. is supported by the ESF EUROCORES programme EuroGIGA – ComPoSe, Austrian Science Fund (FWF): I 648-N18. E.W. acknowledges support from EuroCores/EuroGiga/ComPoSe Swiss National Science Foundation (SNF) 20GG21 134318/1.
[1] The reader not familiar with the notion of order types is referred to the end of this section.

31st International Symposium on Computational Geometry (SoCG'15).
Editors: Lars Arge and János Pach; pp. 285–299
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

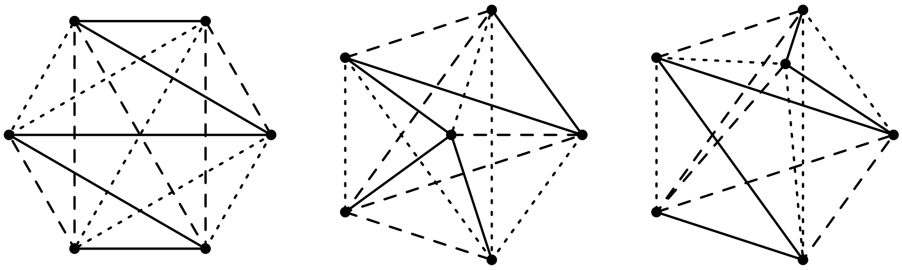

Figure 1 The three maximal order types for 6 points with a partition of the complete geometric graph into three crossing-free spanning-trees (see [3, Figure 8]).

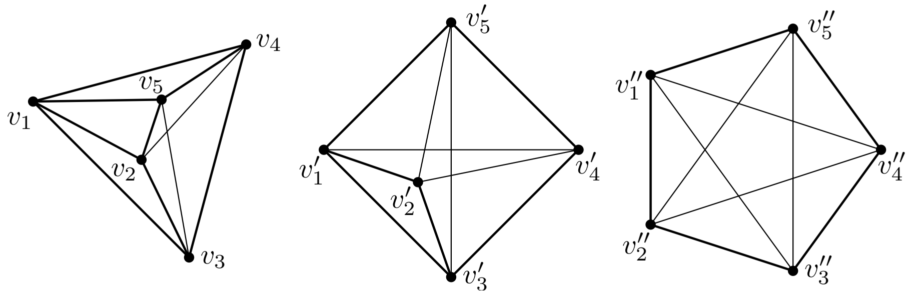

Figure 2 Points sets P, P', and P'' on five points. The mappings $p \mapsto p'$ and $p' \mapsto p''$ are x-preserving, therefore $P \leq_x P' \leq_x P''$. $P <_x P' <_x P''$ follows from the increasing number of crossings.

used in Discrete and Computational Geometry, as it allows for classifying the infinite number of point sets of a given size into a finite number of equivalence classes, capturing combinatorial properties such as which pairs of spanned line segments cross and which points define the set's convex hull. Checking the question at hand for 4 points is easy, since there are only two order types. For 6 points, there are already 16 order types to consider. By what we study below, we claim, though, that the partitions for the three order types given in Figure 1 constitute already a complete proof of the fact for 6 points. This is because the three order types are maximal w.r.t. crossing pairs of edges – a notion to be rendered more precisely and not to be confused with the maximum number of crossings, as achieved by the convex position order type only. In fact, in our example, convex position does allow a partition into crossing-free spanning paths, while this is not true for some other order type of six points. By using the techniques presented herein, we were able to experimentally confirm that such a partition exists for any point set of up to ten points, using the reduced set of order types.

We proceed to basic definitions. Given two equally sized point sets P and Q in general position in the plane, a bijection $P \to Q$, $p \mapsto p'$, is called *crossing-preserving* (or *x-preserving*) if whenever the segment pq crosses the segment rs (for points p, q, r and s in P) then $p'q'$ crosses $r's'$. If such a mapping exists, we say that Q *crossing-dominates* (*x-dominates*) P, in symbols $Q \geq_x P$ or $P \leq_x Q$. If such mappings exist in both directions, then P and Q are called *crossing-equivalent* (*x-equivalent*), in symbols $P \sim_x Q$. Finally, if $Q \geq_x P$ but Q and P are not x-equivalent, then we say that Q *strictly crossing-dominates* (*strictly x-dominates*) P, in symbols $Q >_x P$ or $P <_x Q$. The relation is transitive and it induces a partial order on the obtained equivalence classes (called *crossing types* or *x-types*). An order type or crossing type is called *crossing-maximal* (*x-maximal*), if for a set P of that type there is no point set that strictly x-dominates P; accordingly for *crossing-minimal*.

Figure 2 shows three 5-point sets P, P', and P'' with x-preserving mappings ($p \mapsto p'$ and $p' \mapsto p''$, respectively) witnessing $P \leq_x P' \leq_x P''$. Obviously, $P \leq_x Q$ entails that the

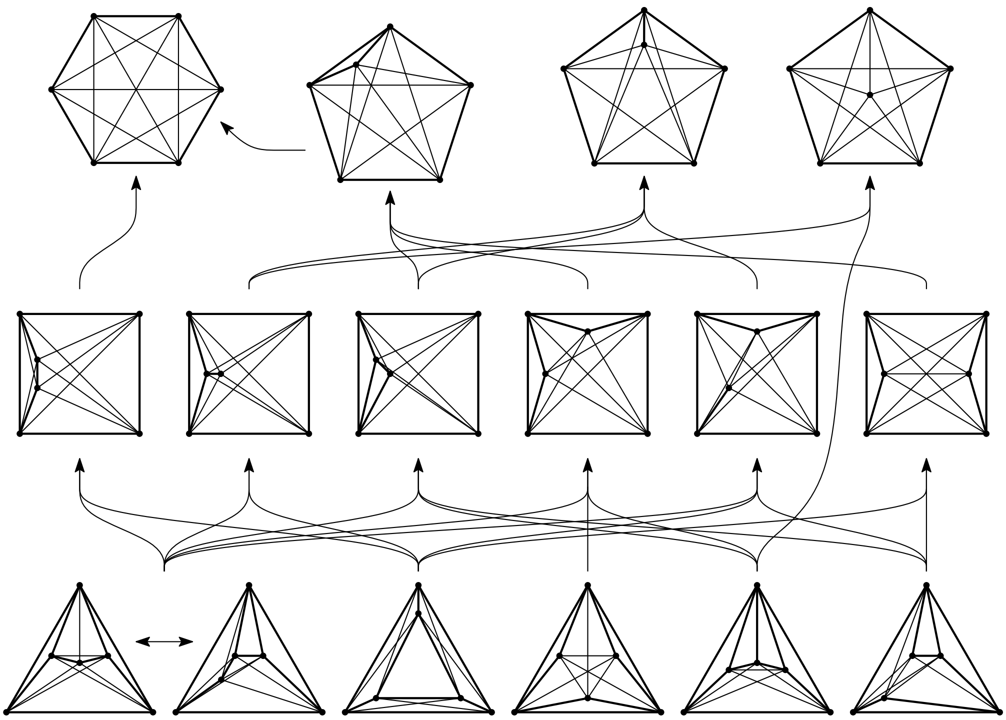

Figure 3 The 16 order types for 6 points with the Hasse-diagram for the x-dominance relation.

complete geometric graph on Q has at least as many crossings as the complete graph for P. We can therefore conclude, in fact, $P <_x P' <_x P''$.

Figure 3 displays 16 point sets representing the order types on six points, with the Hasse-diagram for the x-dominance relation. There, we can make the following observations.

- There are two order types (at the lower left of Figure 3) that merge to one x-type. We will show that such a collapse happens only if the order types have three extreme points.
- The number of maximal order types is 3, all these maximal order types are also x-types. There are six minimal order-types and five minimal x-types. Exploiting the basic properties we develop, we were able to determine (by a computer program) the values in Table 1. Two of these necessary basic properties are listed next.
- If Q strictly x-dominates P in Figure 3, then Q has strictly more extreme points than P. We will show that this is true in general.
- There are only four order types that are *not* x-dominated by sets in convex position. We will develop a necessary and sufficient criterion for x-dominance by sets in convex position. This makes this property easy to check without explicitly providing an x-preserving mapping (the property is that there has to be a Hamiltonian cycle of so-called unavoidable edges). For sets not dominated by sets in convex position, we give a detailed characterization, and show how to efficiently obtain an x-preserving mapping between two point sets, if one exists.

It often suffices to check x-minimal or x-maximal x-types. In Combinatorial Geometry, one is often concerned with estimating the minimum or maximum number of certain (mainly plane) geometric graphs a point set admits. Such combinatorial questions usually depend only on the set's order type. A complete enumeration of the order types of small point sets

Table 1 The number of small crossing-minimal and crossing-maximal order types.

#	order types [1, 5]	crossing-maximal		crossing-minimal	
4	2	1	50%	1	50%
5	3	1	33%	1	33%
6	16	3	19%	6	38%
7	135	17	13%	49	36%
8	3′315	489	15%	1′179	36%
9	158′817	28′103	18%	55′278	35%
10	14′309′547	2′866′895	20%	4′888′160	34%
11	2′334′512′907	[503′727′394, 504′463′503]	22%	[787′697′700, 787′720′845]	34%

by Aichholzer, Aurenhammer, and Krasser [1] allows for investigating such problems for small instances. Our concept of crossing types enables us to only consider a subset of all order types for several combinatorial problems. We give an incomplete list of examples.

- The number of crossing-free graphs of a certain type (e.g., spanning trees, polygonizations, or perfect matchings) is minimized for a maximal x-type and it is maximized for a minimal x-type.
- Similarly, the smallest number $e_k(n)$ such that any graph with at least $e_k(n)$ edges contains $k+1$ pairwise disjoint edges (see, e.g., [16]) is determined by a maximal x-type.
- The maximal number of points in convex position is minimized for a minimal x-type.
- The size of the largest crossing family (see e.g., [6]) is minimized for a minimal x-type.
- The minimal number of crossings in a straight line drawing of the complete graph is realized on some minimal crossing type.

Indeed, point sets in convex position minimize the number of various classes of plane graphs [4]. However, there are classes where the crossing-preserving mapping does not maintain membership in the class, the most obvious example being triangulations. Hence, the number of triangulations may not be minimized by a maximal x-type.

Order types, orientation-equivalence. Given a sequence pqr of three distinct non-collinear points, we define its *orientation* ∇pqr as $+1$ if the sequence (p, q, r) traverses the triangle bounding the convex hull of $\{p, q, r\}$ (denoted by Δpqr) in counterclockwise direction and as -1 if this orientation is clockwise. Given two equally sized point sets P and Q in general position, a bijection $P \to Q$, $p \mapsto p'$, is called *order-preserving* if there is an $\varepsilon \in \{-1, +1\}$ such that $\nabla p'q'r' = \varepsilon \nabla pqr$ for all sequences pqr of three distinct points in P. If such a mapping exists, we say that P and Q are *order-equivalent*. The resulting equivalence classes are called *order types* [9] (and "*being of the same order type*" is mostly used for what we called here "order-equivalent"). For every natural number n there is only a finite number of order types; complete databases are available up to $n = 11$, see [1, 5].

Segment pq crosses segment rs iff both $\nabla pqr \cdot \nabla pqs = -1$ (i.e., r and s lie on different sides of the line through p and q) and $\nabla rsp \cdot \nabla rsq = -1$. Hence, order-equivalent sets are also x-equivalent (as we have indicated already, there are examples that show the reverse implication not to be true).

Given a point set, the orientation of each point triple is clearly defined by the containment of points in the half-planes given by the supporting lines of all point pairs. In a *generalized configuration of points*, these supporting lines of point pairs are replaced by supporting pseudo-lines (i.e., bi-infinite simple Jordan curves such that each pair of pseudo-lines intersects once – in a crossing, not tangentially). The orientation of a point triple is defined by the half-planes given by these supporting pseudo-lines. See, e.g., [10] for a formal definition (in the projective plane). This concept generalizes order types to *abstract order types*. An abstract order type of a point set (with straight supporting lines) is *realizable*.

We will see that there are point sets that are x-dominated by an abstract order type, but not by a realizable one. For abstract order types, we will obtain a complete characterization of the generalized configurations of points x-dominating and x-dominated by a given one. Since it is ∃ℝ-complete to decide whether an abstract order type is realizable [12] (see also [15]), there is not much hope for obtaining the same result for (realizable) order types.

Rotation systems. Let P be a point set of n points in general position. For any point $p \in P$, consider a ray r_p starting at p. When rotating r_p counterclockwise around p, the points $P \setminus \{p\}$ are traversed by r_p in a fixed circular order, called the *rotation* of p. The *rotation system* of P is the set of the rotations of all points of P. Similar to order types, we consider two rotation systems to be equivalent if one can be obtained from the other by relabeling and mirroring. The following result (whose origin will be discussed later in this paragraph) gives a tight relation between the rotation system and crossing-equivalence.

▶ **Corollary 1** (Kynčl [11, Proposition 6]). *Two point sets have the same rotation system iff they are crossing-equivalent.*

However, our main concern in this work is not crossing-equivalence, but rather the partial order on the set of all order types defined by crossing dominance.

Clearly, the order type of P determines its rotation system. The reverse problem, i.e., reconstructing the order type of P when given only its rotation system, has been considered in connection with applications in robotics (see, e.g., [17]). Wismath [18] gives a simple example of a rotation system on four elements that can be obtained by two different point sets with labels. (He then describes a method to reconstruct the point set if additional information is available.) However, when disregarding the labels in Wismath's example, the two different point sets have the same order type. An example of two different order types producing the same rotation system is given by the two point sets in the lower left corner of Figure 3. Aichholzer et al. [2] show that, essentially, the order type can be reconstructed from the rotation system if there are more than three extreme points, or if the extreme points are given. Their reconstruction method is applicable even if an unknown number of rotations have been reversed. Further, they give tight bounds on the number of (labeled) order types with a common rotation system based on properties of the point set. (We will revisit these properties in Section 5.)

Let K_P be the complete geometric graph on the point set P. The rotation system of P determines the order in which the edges emanate from each vertex of K_P. Straight-edge drawings of the complete graph are generalized by so-called *good drawings*. In a good drawing of a graph, vertices are represented by distinct points, and edges are drawn as simple Jordan arcs, where two edges intersect in at most one single point that may be their common endpoint or a proper crossing. Kynčl [11] shows that, for good drawings of the complete graph, a valid set of crossing edge pairs fully determines the rotation system of the good drawing, and that the rotation system determines whether two edges cross. Corollary 1 is therefore a special case of that result.[2]

[2] In general, the rotation system does not determine the crossings for non-complete graphs. The problem of determining the crossing number of a graph with a given rotation system is NP-complete [13].

Further related work. A complementary topic to characterizing crossing-maximal sets is the one for finding universal point sets. An *n-universal point set* admits a straight-line embedding of any planar graph with n vertices. Cardinal, Hoffmann, and Kusters [8] showed that for $n \leq 10$ there exists an n-universal point set of size n, and that for $n \geq 15$ no such set can exist. There is a certain relation to crossing-minimal sets, but observe that our setting is more constrained, as there is a bijection between the vertices/points.

Notation. The function $\mathsf{conv}(P)$ denotes the convex hull and $\mathsf{extr}(P)$ denotes the set of extreme points in a set P of points. Let Conv_n denote the order type of all sets of n points in convex position (i.e., sets P with $|\mathsf{extr}(P)| = |P| = n$). Throughout the paper, let $p \mapsto p'$ be an x-preserving mapping from a finite set P of points in the plane (in general position) to another point set $P' = \{p' \,|\, p \in P\}$ in general position. For $A \subseteq P$, we write A' for $\{p' \,|\, p \in A\}$.

2 Crossing-Dominance, Convex Position, and Inner Points

In a set of n points in convex position, every 4-tuple of points determines exactly one crossing pair of segments. Hence there are $\binom{n}{4}$ such crossing pairs, which is obviously the largest possible number for n points. Therefore, no set can strictly x-dominate a set in Conv_n, and Conv_n is an x-maximal order type. We characterize the sets that are x-dominated by sets in convex position. For that purpose, given a set P in general position, we call a pair $\{p, q\}$ of two distinct points in P *unavoidable* if no segment determined by two points in P crosses the segment pq. The term "unavoidable" stems from the fact that every triangulation of P must use all unavoidable pairs as edges. Clearly, the edges of the convex hull give rise to such unavoidable pairs, but other possibilities occur. In fact, the number of unavoidable pairs in a set of n points can be as large as $2n - 2$, see [14].

▶ **Theorem 2.** *$P \leq_x Q \in \mathsf{Conv}_n$ iff the unavoidable pairs in P contain a Hamiltonian cycle.*

Proof. Suppose $P \leq_x Q \in \mathsf{Conv}_n$ with $Q = \{q_0, q_1, \ldots, q_{n-1}\}$ such that points of Q appear in order $(q_0, q_1, \ldots, q_{n-1})$ along the boundary of $\mathsf{conv}(Q)$ in counterclockwise order, and let $P = \{p_0, p_1, \ldots, p_{n-1}\}$ such that $p_i \mapsto q_i$ is x-preserving. Then all pairs $\{p_i, p_{(i+1) \bmod n}\}$, $i = 0, 1, \ldots, n-1$, have to be unavoidable, since only unavoidable pairs can map to unavoidable pairs under an x-preserving mapping. Hence, a Hamiltonian cycle of unavoidable pairs exists for P. For the other direction, suppose the unavoidable pairs in P allow a Hamiltonian cycle. Let $P = \{p_0, p_1, \ldots, p_{n-1}\}$ with pairs $\{p_i, p_{(i+1) \bmod n}\}$, $i = 0, 1, \ldots, n-1$, forming a spanning cycle of unavoidable pairs. Note that the geometric realization of this cycle is a simple polygon that is crossed by no segment connecting points in P. That is, every segment connecting points in P is either completely inside or completely outside the polygon (or part of the polygon). Suppose segment $p_i p_j$ crosses $p_k p_\ell$. Then they have to be (i) either both inside the unavoidable polygon or both outside the unavoidable polygon and (ii) the appearance of points $\{p_i, p_j\}$ and $\{p_k, p_\ell\}$ alternate along the unavoidable cycle. Since two segments in a convex polygon cross iff their endpoints alternate along the convex polygon, an x-preserving mapping from P to Q readily follows. ◀

For point sets not dominated by Conv_n, we can identify the following property, which gives a rather strong condition for x-dominance.

▶ **Theorem 3.** *Suppose $P \leq_x P'$ and there exists point $p' \notin \mathsf{extr}(P')$. Then the rotation of p in P is equivalent to the rotation of p' in P'.*

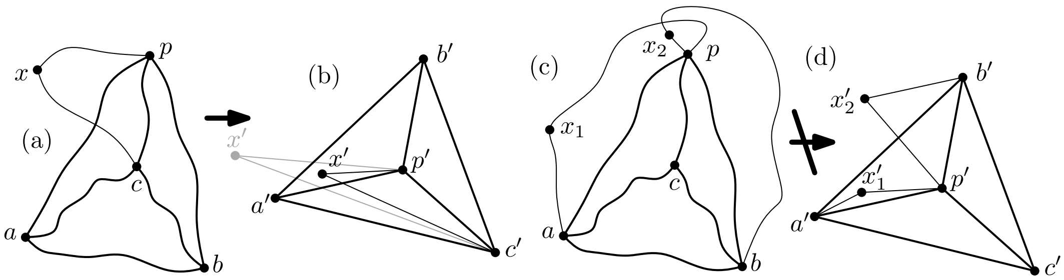

Figure 4 For a point p' in the interior of a triangle, the rotation is equivalent to the one of its preimage. (a) The path from p via x to c has to cross an edge of the triangle Δpab. This is only possible in P' if x' is at the same relative position in the rotation around p'. (b) If the images of x_1 and x_2 would change their relative position in the rotation, we would loose a crossing between the paths (p, x_1, a) and (p, x_2, b).

Proof. The statement is obviously true for four points; suppose therefore that $|P| \geq 5$. Since p' is an inner point of P', there exists at least one triangle $\Delta a'b'c'$ that contains p'. The complete graph on $\{a, b, c, p\}$ is also crossing-free. (We do not know whether p is inside Δabc, but this is not needed for our argument. To prevent confusion by geometric artifacts, one may even consider K_P to be projected onto a sphere.) Suppose there is a point x that is, w.l.o.g, separated from c by a and b in the rotation around p. Then the path (p, x, c) has to leave the triangle Δabp by crossing one of its edges (see Figure 4 (a)). This crossing also has to be present in P' and this is only possible if x' is also separated from c' by a' and b' in the rotation around p', as shown in Figure 4 (b). Hence, the rotation around p is the same w.r.t. (a, b, c) for every fifth point x. We are therefore left with the case where there are two points x'_1 and x'_2, separated, w.l.o.g., from c' by a' and b'; the situation is the same for their preimages in P' by the previous arguments. Let the subsequence in the rotation of p' be (a', x'_1, x'_2, b'), and suppose it is (a, x_2, x_1, b) in P. Then the path (p, x_1, a) intersects the path (p, x_2, b), as sketched in Figure 4 (c). But the images of such paths, shown in Figure 4 (d), are always non-intersecting in P', a contradiction. ◀

3 Crossing-Dominance Needs More Extreme Points

The x-dominance relation exhibits the following monotonicity properties.

▶ **Proposition 4.** (1) *If $P \leq_x Q$, then the complete straight line drawing K_Q of the complete graph on Q has at least as many crossing pairs of edges as K_P does.* (2) *If $P \leq_x Q$, then* $|\mathsf{extr}(P)| \leq |\mathsf{extr}(Q)|$.

Proof. (1) follows directly from the definition of an x-preserving mapping. For (2) remember that a triangulation of a point set (in general position) with n points and h extreme points has exactly $3n - 3 - h$ edges. Now consider an x-preserving bijection $p \mapsto p'$ from P to Q and some triangulation of Q, which has $3n - 3 - |\mathsf{extr}(Q)|$ edges. The preimage of this triangulation is a crossing-free graph on P which is contained in some triangulation of P with $3n - 3 - |\mathsf{extr}(P)|$ edges. Therefore, $3n - 3 - |\mathsf{extr}(Q)| \leq 3n - 3 - |\mathsf{extr}(P)|$. ◀

The main purpose of this section is to shed some extra light on property (2). In particular we will show that (i) $P <_x Q$ implies $|\mathsf{extr}(P)| < |\mathsf{extr}(Q)|$ (Theorem 13). Moreover, given an x-preserving mapping from P to Q, (ii) the inverse is also x-preserving iff $|\mathsf{extr}(P)| = |\mathsf{extr}(Q)|$

(Theorem 14) and (iii) the mapping is order-preserving iff $\mathsf{extr}(P)' = \mathsf{extr}(Q)$ (Theorem 10). We will see that a triangular convex hull is a situation that needs special attention.

Switching to crossing. We discriminate the relative position of two distinct non-crossing segments pq and rs on points in a set S in general position as follows: They are called *incident* if they share an endpoint, they are called *parallel* if the lines supporting them intersect in a point outside of both segments, and they are called *stabbing* if the line of one of the two segments crosses the the other segment. “Crossing”, “parallel”, “stabbing”, and “incident” exhaust all possibilities for two segments connecting points in general position.

Note that in an x-preserving mapping the image of non-crossing segments can of course be crossing – not so for parallel segments, though.

▶ **Lemma 5.** *(1) If pq and rs are parallel, then $p'q'$ and $r's'$ are parallel. (2) If pq and rs are non-crossing and $p'q'$ and $r's'$ are crossing, then pq and rs are stabbing (i.e., $\{p, q, r, s\}$ is not in convex position).*

Diagonals stay. A segment pq connecting two points p and q in a point set S is called a *diagonal of S* if p and q are extreme points in S and pq is not an edge of the convex hull of S.

▶ **Lemma 6.** *If pq is a diagonal of P, then $p'q'$ is a diagonal of P'.*

Proof. Note that pq is crossed by some segment rs (take any two points in $P \setminus \{p, q\}$ on opposite sides of the line through pq). Hence, $p'q'$ cannot be an edge of the convex hull. Therefore, if $p'q'$ is not a diagonal of P', then the line containing $p'q'$ must intersect some edge $a'b'$ (in its interior) of the convex hull of P'. Note that $p'q'$ and $a'b'$ do not cross, so pq and ab do not cross, and therefore a and b must lie on the same side of the line h containing pq (because p and q are extreme). Now, since pq is a diagonal, there must be a point c on the other side of this line h; we know that both ac and bc cross pq. But it is not possible that both $a'c'$ and $b'c'$ cross $p'q'$, since a' and b' lie on opposite sides of the line through p' and q'; a contradiction to $p \mapsto p'$ being x-preserving. ◀

If there are at least four extreme points then every extreme point participates in a diagonal. Therefore, an x-preserving mapping maps extreme points to extreme points.

▶ **Corollary 7.** *If $|\mathsf{extr}(P)| \geq 4$ then $(\mathsf{extr}(P))' \subseteq \mathsf{extr}(P')$.*

The aforementioned argument cannot be used if there are only three extreme points in P. In fact, the implication in the corollary simply is not true in this case (see the x-equivalent sets in Figure 3).

Jumping out of a triangle. As we have learned, if pq and rs do not cross, but $p'q'$ and $r's'$ do, then pq and rs must be stabbing and therefore p, q, r, s is not in convex position. W.l.o.g., let p be in $\mathsf{conv}(\{q, r, s\})$. If, indeed, $p'q'$ and $r's'$ cross, then clearly $p' \notin \mathsf{conv}(\{q', r', s'\})$. This “jumping out of a triangle” immediately has further implications for the location of p', as the following lemma states.

▶ **Lemma 8.** *If $p \in \mathsf{conv}(\{q, r, s\})$ and $p' \notin \mathsf{conv}(\{q', r', s'\})$ then $p' \notin \mathsf{conv}(A')$ with A being the set of points in P not in the interior of $\mathsf{conv}(\{q, r, s\})$. In particular, $p' \notin \mathsf{conv}(\mathsf{extr}(P)')$.*

Proof. Consider some point $t \in P$ not in the interior of $\mathsf{conv}(\{q, r, s\})$, i.e., this point lies either outside $\mathsf{conv}(\{q, r, s\})$ or is one of the points q, r, or s. Now consider p', which is outside $\mathsf{conv}(\{q', r', s'\})$ and therefore a line h separating p' from $\mathsf{conv}(\{q', r', s'\})$ exists.

Clearly, if t is among q, r, s, then h separates p' from t'. If t is outside $\mathsf{conv}(\{q, r, s\})$, then pt crosses one of the segments qr, rs, or sq, say it is qr. Now $p't'$ must cross $q'r'$; therefore p' and t' must be on opposite sides of h and this line h separates all of A from p. ◀

Observe now that if $P' >_x P$, then the mapping must turn a non-crossing pair into a crossing pair (otherwise P and P' are x-equivalent). Hence some point p has to leave some triangle under the x-preserving mapping and therefore P' must have some new extreme point u' (not necessarily p' itself), i.e., $u' \in \mathsf{extr}(P')$ but $u \notin \mathsf{extr}(P)$. In combination with Corollary 7 this yields the following.

▶ **Corollary 9.** *If* $|\mathsf{extr}(P)| \geq 4$ *and* $P' >_x P$*, then* $(\mathsf{extr}(P))' \subsetneq \mathsf{extr}(P')$.

Hence, the number of extreme points has to increase. The following theorem is an important implication of the "jumping out of a triangle" observation.

▶ **Theorem 10.** *The x-preserving mapping* $p \mapsto p'$ *is order-preserving iff* $\mathsf{extr}(P)' = \mathsf{extr}(P')$.

Corollary 7 and Theorem 10 imply that order types and crossing types coincide for point sets with at least 4 extreme points. This can also be seen by combining Corollary 1 and the fact that, given the rotation system and the extreme points of a point set, its order type is determined (as shown in [2]).

▶ **Corollary 11.** *If* $P \sim_x P'$ *and* $|\mathsf{extr}(P)| \geq 4$*, then* P *and* P' *are of the same order type.*

In Figure 3 we have seen an example witnessing that the condition $|\mathsf{extr}(P)| \geq 4$ is essential in Corollary 11. It is evident that $|\mathsf{extr}(P)| = 3$ needs special attention.

Three extreme points. While we have examples where $(\mathsf{extr}(P))' \subseteq \mathsf{extr}(P')$ is not true (if $|\mathsf{extr}(P)| = 3$), we can still show that $P' >_x P$ always implies $|\mathsf{extr}(P')| > |\mathsf{extr}(P)|$.

▶ **Lemma 12.** *If* $|\mathsf{extr}(P)| = |\mathsf{extr}(P')| = 3$ *for sets* P *and* P' *with* $P \leq_x P'$*, then* $P \sim_x P'$.

We can finally conclude (from Corollary 9 and Lemma 12) that strict x-dominance goes with a strictly larger set of extreme points.

▶ **Theorem 13.** *If* $P <_x P'$*, then* $|\mathsf{extr}(P)| < |\mathsf{extr}(P')|$. ◀

It remains to show x-equivalence for related sets with the same number of extreme points.

▶ **Theorem 14.** *The inverse of the x-preserving mapping* $p \mapsto p'$ *is x-preserving iff* $|\mathsf{extr}(P)| = |\mathsf{extr}(P')|$.

Proof. If the inverse of $p \mapsto p'$ is x-preserving, then $P \sim_x P'$ and we know that $|\mathsf{extr}(P)| = |\mathsf{extr}(P')|$ holds by Proposition 4. If the inverse of $p \mapsto p'$ is not x-preserving, then there has to be a crossing in P' that is not present in the preimage P, i.e., there are strictly more crossings in $K_{P'}$ than in K_P. Therefore, P' strictly x-dominates P and, by Theorem 13, we have $|\mathsf{extr}(P)| < |\mathsf{extr}(P')|$, contradicting the assumption that $|\mathsf{extr}(P)| = |\mathsf{extr}(P')|$. ◀

4 (Sufficient) Conditions for Crossing-Dominance

In Section 2 we gave a characterization of the point sets dominated by Conv_n. Together with Theorem 13, this immediately gives us the following result.

▶ **Corollary 15.** *If* $|\mathsf{extr}(P)| = |P| - 1$*, then* P *is x-maximal iff it has no Hamiltonian cycle of unavoidable edges.*

While for (realizable) point sets not dominated by Conv_n we cannot give such a complete characterization, we can give properties that witness x-maximality of a point set, based on unavoidable edges. Consider two point sets $P \leq_x P'$ with $\mathsf{extr}(P)' \subsetneq \mathsf{extr}(P')$. Then there is an edge ab of $\mathsf{conv}(P)$ such that a' and b' are not consecutive on the boundary of $\mathsf{conv}(P')$. Let $C' = (a', \ldots, b')$ be the chain from a' to b' on the boundary of $\mathsf{conv}(P')$ whose preimage C does not contain any points of $\mathsf{extr}(P)$ except from a and b. Clearly, the edges of C have to be unavoidable in P. We call such a chain of unavoidable edges C between a and b an *unavoidable detour* and call the points in $C \setminus \{a, b\}$ its *elements*. Using Corollary 9 we immediately get the following result.

▶ **Theorem 16.** *If P with $|\mathsf{extr}(P)| \geq 4$ has no unavoidable detour, then it is x-maximal.* ◀

This further implies

▶ **Theorem 17.** *For any given number m, there exists a number n such that among all order types of size n there are at least m crossing-maximal ones.*

General properties of unavoidable detours. Since unavoidable detours are fundamental for x-dominance, we identify some of their properties. For the following lemmas, let P be a point set containing an unavoidable detour C between two distinct extreme points a and b (recall that a and b are neighbored on the convex hull boundary of P and observe that there cannot exist a chain of unavoidable edges between two non-neighbored extreme points that does not use other extreme points of the set).

▶ **Lemma 18.** *The region bounded by the cycle $C \cup ab$ does not contain any point of $P \setminus C$.*

▶ **Lemma 19.** *In the rotation of any point $p \in P \setminus C$, the elements of C occur in the order defined by C and are consecutive among $C \cup \mathsf{extr}(P)$.*

▶ **Lemma 20.** *All points of $P \setminus C$ are on the same side of any two points $p, q \in C$.*

▶ **Lemma 21.** *No two points in $P \setminus C$ have a supporting line intersecting C more than once.*

Unavoidable detours and x-dominating sets. We can construct examples where not only the elements of C jump out of a triangle. However, the points that jump out of a triangle are not arbitrary.

▶ **Lemma 22.** *Suppose we have two point sets $P \leq_x P'$ with an unavoidable detour $C = (a, \ldots, b)$ s.t. $\mathsf{extr}(P') = (\mathsf{extr}(P) \cup C)'$. Let J be the set of points that jump out of a triangle in that mapping. Then the line defined by a point $j \in J$ and any other point $p \in P \setminus C$ intersects ab.*

▶ **Theorem 23.** *If a point set P has an unavoidable detour then it is x-dominated by an abstract order type (that may not be realizable by a point set).*

Proof. Let C be any unavoidable detour in P between two extreme points a and b. We construct a generalized configuration P' of points that x-dominates P such that $\mathsf{extr}(P')$ consists of $\mathsf{extr}(P)'$ plus the images of the elements of C. We transform $P \setminus C$ and its set of supporting lines into a generalized configuration of points. Since C is an unavoidable detour, all points of $P \setminus C$ are on the same side of any two points of C by Lemma 20. We replace C by a Jordan arc between a and b. This pseudo-segment intersects exactly those supporting lines of $P \setminus C$ as the initial edge ab, since any supporting line of $P \setminus C$ intersected C at most once by Lemma 21. Therefore, it can be extended to a pseudo-line intersecting each supporting line exactly once along the initial supporting line of ab. The supporting lines that intersected C now intersect the pseudo-segment $a'b'$ in the same order as along C. We can therefore place the convex chain C' and the relevant parts of its supporting pseudo-lines arbitrarily close to the pseudo-segment. Again, an extension of the supporting lines is done appropriately along the initial supporting line of ab, making the resulting point set and its pseudo-line arrangement a valid generalized configuration of points that x-dominates P. ◀

In contrast to that, we have the following result, which implies that realizability of abstract order types is crucial in connection with x-maximal point sets.

▶ **Theorem 24.** *There are point sets that have at least four extreme points and an unavoidable detour, but are still x-maximal.*

The construction in the proof of Theorem 23 gives, in general, one out of many abstract order types that dominate P, depending on where in $\mathsf{conv}(C')$ we place the points of $J' \setminus C'$. However, we have the following restriction.

▶ **Proposition 25.** *Suppose we have two point sets $P \leq_x P'$ with $|\mathsf{extr}(P)| \geq 4$ and an unavoidable detour $C = (a, \ldots, b)$ s.t. $\mathsf{extr}(P') = (\mathsf{extr}(P) \cup C)'$. Then $P \setminus C$ and $P' \setminus C'$ have the same order type.*

In the previous statements, we considered only single unavoidable detours. However, the results can again be applied to the dominating set if it contains an unavoidable detour. While this is fine when working with abstract order types, keep in mind that there may be non-realizable dominating abstract order types that are again dominated by a realizable one.

5 Different Extreme Points in the Dominating Set

While in Section 4 we gave a characterization of crossing-domination for the case where all images of the extreme points are again extreme points of the dominating set, we consider now, for $P \leq_x P'$, the case $\mathsf{extr}(P)' \not\subseteq \mathsf{extr}(P')$. Corollary 9 tells us that we have $|\mathsf{extr}(P)| = 3$. We classify the different cases by the number of common extreme points (see Figure 5 for examples). For two of the cases, we will exploit the properties of crossing-equivalent subsets.

Properties of crossing-equivalent sets. It became obvious that the case $|\mathsf{extr}(P')| = 3$ needs special attention. For that we try to understand the scenario when P and P' have a triangular convex hull, but some extreme points of P do not map to extreme points in P'.

As already discussed, crossing-equivalence of two point sets implies that they have the same rotation system. Point sets with the same rotation system have been discussed by Aichholzer et al. [2]. They show that if two order types with the same rotation system have no common extreme point (under the given bijection) these two order types are the only ones in that equivalence class, as there are exactly two triangles of unavoidable edges that can be transformed to the convex hull. (They give an upper bound of $|P| - 1$ on the number of different (labeled) order types in that equivalence class when there are common extreme points.) The following lemma states, in a different formulation, a result that is also given in [2].

▶ **Lemma 26.** *If $\mathsf{extr}(P') = \{a', b', c'\} \neq (\mathsf{extr}(P))'$, then $|\mathsf{extr}(S)| = 3$ for all sets S with $\{a, b, c\} \subseteq S \subseteq P$. More precisely, P can be partitioned into three sets $P_a = \{a_0, \ldots, a_{n_a}\}$, $P_b = \{b_0, \ldots, b_{n_b}\}$, and $P_c = \{c_0, \ldots, c_{n_c}\}$, such that $a_0 = a$, $b_0 = b$, and $c_0 = c$ and for all nonnegative integers i, j, k with $i \leq n_a$, $j \leq n_b$, and $k \leq n_c$, $\mathsf{conv}(\{a_i, b_j, c_k\}) \cap P = \{a_0, \ldots, a_i\} \cup \{b_0, \ldots, b_j\} \cup \{c_0, \ldots, c_k\}$.*

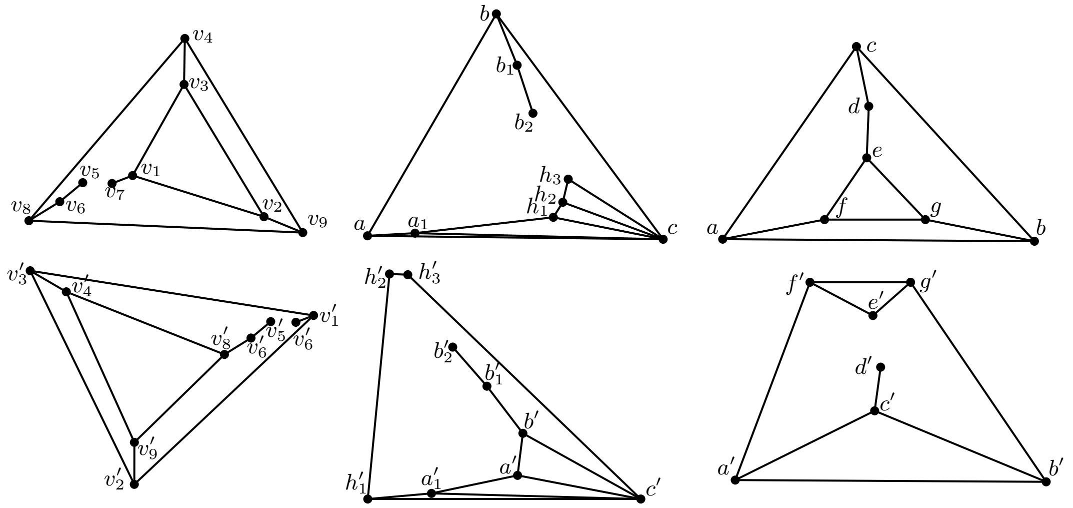

Figure 5 x-Preserving mappings where only zero (left), one (middle), and two (right) extreme points stay the same. Unavoidable edges are drawn.

Observe that this constrains the position of the points to a high extent. Still, we can actually take an instance of any order type, scale it appropriately, and use it as one of the three subsets when constructing such a point set. The following theorem shows that this characterization also provides a construction for all sets in the equivalence class, meaning that if one of them is realizable then all of them are realizable.

▶ **Theorem 27.** *If there exists a triangle Δabc in a point set P such that P can be partitioned into three sets as in Lemma 26, then we can construct a point set P' with $P \sim_x P'$ and* $\mathsf{extr}(P') = \{a', b', c'\}$.

No common extreme point. For the case $\mathsf{extr}(P)' \cap \mathsf{extr}(P') = \emptyset$, we have

▶ **Theorem 28.** *If, for $P \leq_x P'$, $\mathsf{extr}(P)' \cap \mathsf{extr}(P') = \emptyset$, then $|\mathsf{extr}(P')| = 3$ and $P \sim_x P'$.*

Therefore, the two point sets have a different order type but the same rotation system. As already mentioned, there are only two order types in such an equivalence class [2].

One common extreme point. Let $\mathsf{extr}(P) = \{a, b, c\}$ with $a', b' \notin \mathsf{extr}(P')$, i.e., c is the only extreme point of P whose image is also an extreme point of P'. We denote the cycle on the boundary of $\mathsf{conv}(P')$ by $H' = (h'_0, \ldots, h'_k)$ and define $h_0 = c$. Further, we consider the triangle Δabc and the cycle H to be oriented counterclockwise. Since the segment between c and any other point of P must not cross the chain $(h_1, \ldots, h_k)$, the region where the other points of P can be placed is partitioned into two disjoint parts. Let $A \subset P$ be the points to the right of h_0h_1 (in particular, $a \in A$), and let $B \subset P$ be the points to the left of h_0h_k (implying $b \in B$). Observe that the cycle H may not be in convex position, but the radial order of its interior points around both a and b is $h_1, \ldots, h_k$.

▶ **Lemma 29.** *The interior of the convex hull of $h'_1, \ldots, h'_k$ is empty.*

Since, by Lemma 29, we obtain a triangular convex hull when removing the vertices $h_2, \ldots, h_{k-1}$, Lemma 12 directly gives us the following result.

▶ **Corollary 30.** *If the convex hulls of P and P' share exactly one point, then the rotations around all vertices remain the same when removing the elements $h_2, \dots, h_{k-1}$ and their images from P and P', respectively.*

Corollary 30 is a powerful tool for inspecting the structure of P and P'. It allows us to apply Lemma 26 to see that there is a hierarchy among the points of P in A and B. For example, in the rotation around a, all points of $B \setminus \{b\}$ are between b and h_k, and vice versa.

▶ **Corollary 31.** *No point of $P' \setminus \{c'\}$ is on the same side of $a'b'$ as c'.*

Also, the triangle Δabc behaves as in the case of crossing-equivalent sets and therefore its image is unavoidable when removing the points $\{h_2, \dots, h_{k-1}\}$.

▶ **Corollary 32.** *There is no point $p' \in P' \setminus H'$ s.t. the edge $c'p'$ intersects the edge $a'b'$.*

Let us partition A into subsets A_i s.t. A_i consists of the points in A that are in the interior of the triangle $\Delta a h_i h_{i+1}$, and do the same for B. Note that there is a line ℓ that separates A and B from H.

▶ **Lemma 33.** *Let i be the highest index such that $A_i \neq \emptyset$. Analogously, let j be the lowest index such that $B_j \neq \emptyset$. Then $i \leq j$.*

Corollary 30, in combination with Lemma 26, tells us that, after removing, say, a, the resulting subset has again a triangular convex hull; therefore the previous lemma also holds for the new extreme point a_1. Hence, we get

▶ **Corollary 34.** *Let i be the highest index such that the supporting line of two points $a_k, a_l \in A$ intersects the edge $h_i h_{i+1}$, and let j be the lowest index such that the supporting line of two points $b_q b_r \in B$ intersects the edge $h_j h_{j+1}$. Then $i \leq j$.*

It is easy to construct examples where there are multiple sets that strictly dominate P and that have the image of H as extreme points by having different rotations around the images of c. Still, Corollary 34 completes the characterization of the sets P and P' up to stretchability by similar reasoning as in the proof of Theorem 23.

Two common extreme points. If the convex hull boundary of P' contains two images of extreme points of P, then there has to be an unavoidable detour connecting such points a and b. This unavoidable detour gives the description of an abstract order type that strictly dominates P by Theorem 23, just like for sets with more extreme points. However, as the three order types with five points show (e.g., by swapping the labels v_1' and v_2' in Figure 2), we might not have $\mathsf{extr}(P)' \subset \mathsf{extr}(P')$. There may therefore be several abstract order types that strictly dominate P, and these may not be constructed the same way as in the proof of Theorem 23. However, we know the following.

▶ **Theorem 35.** *Let $\mathsf{extr}(P) = \{a, b, c\}$ and $P \leq_x P'$. If $a', b' \in \mathsf{extr}(P')$ and $c' \notin \mathsf{extr}(P')$, then $a'b'$ is an edge of $\mathsf{conv}(P')$.*

6 Algorithms and the Order Type Data Base

So far, we did not address the algorithmic problem of deciding whether a point set crossing-dominates another. We do so in this section. Further, we explain how we used the properties of for x-dominance to extract all crossing-maximal and crossing-minimal order types for up to 11 points. Theorem 3 is the key result for checking whether a point set not in convex position x-dominates another. Since Theorem 2 also gives a description of sets dominated by Conv_n, we can devise fast algorithms for checking x-dominance.

▶ **Lemma 36.** *The existence of a Hamiltonian cycle in the set of unavoidable edges of a point set P can be decided in polynomial time.*

▶ **Theorem 37.** *Given two point sets P and P' of size n, it can be decided in polynomial time whether $P \leq_x P'$.*

As a practical result of our work, we generated files containing exactly the realizations of the crossing-maximal and crossing-minimal order types for up to 10 points. For 11 points, such files were also extracted, but are likely to contain a small fraction of false-positives, which could not be filtered out by our methods due to the wast number of order types of size 11. We sketch our approach that allowed us to eventually generate these data bases within few CPU hours by extracting them from the Point Set Order Type Data Base (see [1, 5]).

Theorem 23 states that every point set P containing an unavoidable detour is dominated by an abstract order type. We use this to quickly find a point set in the Order Type Data Base that dominates a given one. First, we enumerate all cycles that consist of unavoidable edges that contain all extreme points of P. For each such cycle, we consider the abstract order type that has this cycle as convex hull boundary; all other point triples are oriented in the same way as in P. By the proof of Theorem 23, this abstract order type x-dominates P. (Note that this does not produce all possible abstract order types x-dominating P; we do not get those sets where points with non-extreme image jump out of a triangle.) We get the lexicographically smallest λ-matrix as a fingerprint (see [9]) from each such abstract order type and search for this matrix in the data base. Using this method, most of the non-maximal sets could be identified quickly (we can perform a binary search for the matrix in the data base). For some of the sets, no realizable order type could be found this way. Therefore, for up to 10 points, we used a second iteration in which these sets where checked against all other ones that were not identified to be non-maximal in the first iteration. Further, all sets with triangular convex hull that contain a cycle of unavoidable edges of length at least four not violating the conditions of Corollary 34 were checked in this phase. Similarly, the following characterization allows us to identify crossing-minimal sets.

▶ **Theorem 38.** *Let $P, |\mathsf{extr}(P)| \geq 4$, be a set of points such that, for every sequence $H = (h_i, \ldots, h_j), |H| \geq 3$, of consecutive points on the convex hull boundary of P, $\mathsf{conv}(H)$ contains a pair $p, q \in P \setminus H$ s.t. pq does not stab $h_i h_j$. Then P is crossing-minimal.*

For sets that have at least one chain C, $|C| \leq |\mathsf{extr}(P)| - 1$, on the boundary of the convex hull that may be an unavoidable detour in a dominated set, we obtain a corresponding abstract order type. In such an abstract order type, the chain C is made reflex (which is, in general, not the only possibility), and for all points inside $\mathsf{conv}(C)$, the triple orientations change accordingly. This way, many dominated order types can be found quickly.

Calculating the λ-matrix of an implicitly given abstract order type is a rather involved and therefore error-prone task, and even obtaining a bijection between two point sets using Theorem 3 is not completely fail-safe. However, when given the bijection between two point sets, we can, in a brute-force way, compare all 4-tuples to check crossing-dominance, a comparatively simple task. Once given the point set that witnesses non-maximality or non-minimality of another one, these sets can be compared quickly. This separate check was used to verify the resulting data.

Acknowledgements. We want to thank Oswin Aichholzer and Thomas Hackl for valuable discussions, in particular on handling the Order Type Data Base and on partitioning edges into spanning trees for 10 points, as well as for setting up computations on the Graz University of Technology HPC system in early stages. The authors would like to acknowledge the use of HPC resources provided by the ZID of Graz University of Technology. This work was initiated during the 10th European Research Week on Geometric Graphs, held in St. Karl, Switzerland, from September 9 to September 13, 2013.

References

1. Oswin Aichholzer, Franz Aurenhammer, and Hannes Krasser. Enumerating order types for small point sets with applications. *Order*, 19(3):265–281, 2002.
2. Oswin Aichholzer, Jean Cardinal, Vincent Kusters, Stefan Langerman, and Pavel Valtr. Reconstructing point set order types from radial orderings. In *ISAAC 2014*, volume 8889 of *LNCS*, pages 15–26. Springer, 2014.
3. Oswin Aichholzer, Thomas Hackl, Matias Korman, Marc van Kreveld, Maarten Löffler, Alexander Pilz, Bettina Speckmann, and Emo Welzl. Packing plane spanning trees and paths in complete geometric graphs. In *Proc. 26th Canadian Conference on Computational Geometry (CCCG 2014)*, 2014.
4. Oswin Aichholzer, Thomas Hackl, Birgit Vogtenhuber, Clemens Huemer, Ferran Hurtado, and Hannes Krasser. On the number of plane graphs. In *Proc. 17th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2006)*, pages 504–513. ACM Press, 2006.
5. Oswin Aichholzer and Hannes Krasser. Abstract order type extension and new results on the rectilinear crossing number. *Comput. Geom.*, 36(1):2–15, 2007.
6. Boris Aronov, Paul Erdős, Wayne Goddard, Daniel J. Kleitman, Michael Klugerman, János Pach, and Leonard J. Schulman. Crossing families. *Combinatorica*, 14(2):127–134, 1994.
7. Prosenjit Bose, Ferran Hurtado, Eduardo Rivera-Campo, and David R. Wood. Partitions of complete geometric graphs into plane trees. *Comput. Geom.*, 34(2):116–125, 2006.
8. Jean Cardinal, Michael Hoffmann, and Vincent Kusters. On universal point sets for planar graphs. In *Computational Geometry and Graphs - Thailand-Japan Joint Conference (TJJCCGG 2012)*, volume 8296 of *LNCS*, pages 30–41. Springer, 2012.
9. Jacob E. Goodman and Richard Pollack. Multidimensional sorting. *SIAM J. Comput.*, 12(3):484–507, 1983.
10. Jacob E. Goodman and Richard Pollack. Semispaces of configurations, cell complexes of arrangements. *J. Combin. Theory Ser. A*, 37(3):257–293, 1984.
11. Jan Kynčl. Enumeration of simple complete topological graphs. *Eur. J. Comb.*, 30(7):1676–1685, 2009.
12. Nicolai E. Mnëv. The universality theorems on the classification problem of configuration varieties and convex polytope varieties. In *Topology and Geometry—Rohlin Seminar*, volume 1346 of *Lecture Notes in Math.*, pages 527–544. Springer, 1988.
13. Michael J. Pelsmajer, Marcus Schaefer, and Daniel Štefankovič. Crossing numbers of graphs with rotation systems. *Algorithmica*, 60(3):679–702, 2011.
14. Gerhard Ringel. Extremal problems in the theory of graphs. In *Theory of Graphs and its Applications (Smolenice, 1963)*, pages 85–90. Publ. House Czechoslovak Acad. Sci., Prague, 1964.
15. Marcus Schaefer. Complexity of some geometric and topological problems. In *Graph Drawing*, volume 5849 of *LNCS*, pages 334–344. Springer, 2009.
16. Géza Tóth and Pavel Valtr. Geometric graphs with few disjoint edges. In *Symposium on Computational Geometry*, pages 184–191, 1998.
17. Benjamín Tovar, Luigi Freda, and Steven M. LaValle. Learning combinatorial map information from permutations of landmarks. *I. J. Robotic Res.*, 30(9):1143–1156, 2011.
18. Stephen K. Wismath. Point and line segment reconstruction from visibility information. *Int. J. Comput. Geometry Appl.*, 10(2):189–200, 2000.