On Computability and Triviality of Well Groups*

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Abstract

The concept of *well group* in a special but important case captures homological properties of the zero set of a continuous map $f\colon K \to \mathbb{R}^n$ on a compact space K that are invariant with respect to perturbations of f. The perturbations are arbitrary continuous maps within L_∞ distance r from f for a given $r > 0$. The main drawback of the approach is that the computability of well groups was shown only when $\dim K = n$ or $n = 1$.

Our contribution to the theory of well groups is twofold: on the one hand we improve on the computability issue, but on the other hand we present a range of examples where the well groups are incomplete invariants, that is, fail to capture certain important robust properties of the zero set.

For the first part, we identify a computable subgroup of the well group that is obtained by cap product with the pullback of the orientation of $\mathbb{R}^n$ by f. In other words, well groups can be algorithmically approximated from below. When f is smooth and $\dim K < 2n - 2$, our approximation of the $(\dim\ K - n)$th well group is exact.

For the second part, we find examples of maps $f, f'\colon K \to \mathbb{R}^n$ with all well groups isomorphic but whose perturbations have different zero sets. We discuss on a possible replacement of the well groups of vector valued maps by an invariant of a better descriptive power and computability status.

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1 Introduction

In many engineering and scientific solutions, a highly desired property is the resistance against noise or perturbations. We can only name a fraction of the instances: stability in data analysis [4], robust optimization [2], image processing [14], or stability of numerical methods [16]. Some very important tools for robust design come from topology, which can capture stable properties of spaces and maps.

In this paper, we take the robustness perspective on the study of the solution set of systems of nonlinear equations, a fundamental problem in mathematics and computer science. Equations arising in mathematical modeling of real problems are usually inferred

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from observations, measurements or previous computations. We want to extract maximal information about the solution set, if an estimate of the error in the input data is given.

More formally, for a continuous map $f : K \to \mathbb{R}^n$ on a compact Hausdorff space K and $r > 0$ we want to study properties of the family of zero sets

$$Z_r(f) := \{g^{-1}(0) \colon \|f - g\| \le r\},$$

where $\|\cdot\|$ is the max-norm with respect to some fixed norm $|\cdot|$ in $\mathbb{R}^n$. The functions g with $\|f - g\| \le r$ (or $\|f - g\| < r$) will be referred to as *r-perturbations of f* (or *strict r-perturbations of f*, respectively). Quite notably, we are not restricted to *parameterized* perturbations but allow arbitrary continuous functions at most (or less than) r far from f in the max-norm.

Well groups. Recently, the concept of well groups was developed to measure "robustness of intersection" of a map $f\colon K \to Y$ with a subspace $Y' \subseteq Y$ [8].

In the special but very important case when $Y = \mathbb{R}^n$ and $Y' = \{0\}$ it is a property of $Z_r(f)$ that, informally speaking, captures "homological properties" that are common to all zero sets in $Z_r(f)$. We enhance the theory to include a *relative case*[1] that is especially convenient in the case when K is a manifold with boundary. Let $B \subseteq K$ be a pair of compact Hausdorff spaces and $f : K \to \mathbb{R}^n$ continuous. Let $X := |f|^{-1}[0, r]$ where $|f|$ denotes the function $x \mapsto |f(x)|$; this is the smallest space containing zero sets of all r-perturbations g of f. In the rest of the paper, for any space $Y \subseteq K$ we will abbreviate the pair $(Y, Y \cap B)$ by (Y, B) and, similarly for homology, $H_*(Y, Y \cap B))$ by $H_*(Y, B)$. Everywhere in the paper we use homology and cohomology groups with coefficients in $\mathbb{Z}$ unless explicitly stated otherwise. For brevity we omit the coefficients from the notation.

The jth well group $U_j(f, r)$ of f at radius r is the subgroup of $H_j(X, B)$ defined by

$$U_j(f, r) := \bigcap_{Z \in Z_r(f)} \mathrm{Im}\big(H_j(Z, B) \xrightarrow{i_*} H_j(X, B)\big),$$

where i_* is induced by the inclusion $i\colon g^{-1}(0) \hookrightarrow X$ and H refers to a convenient homology theory of compact metrizable spaces that we describe below.[2] For a simple example of a map f with $U_1(f, r)$ nontrivial see Figure 1.

Significance of well groups. We only mention a few of many interesting things mostly related to our setting. The well group in dimension zero characterizes robustness of solutions of a system of equations $f(x) = 0$. Namely, $\emptyset \in Z_r(f)$ if and only if $U_0(f, r) \cong 0$. Higher well groups capture additional robust topological properties of the zero set such as in Figure 1. Perhaps the most important is their ability to form *well diagrams* [8] – a kind of measure for robustness of the zero set (or more generally, robustness of the intersection of f with other subspace $Y' \subseteq Y$). The well diagrams are stable with respect to taking perturbations of f.[3]

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[1] Authors of [3] develop a different notion of relativity that is based on considering a pair of spaces (Y', Y'_0) instead of the single space Y'. This direction is rather orthogonal to the matters of this paper.

[2] In [8, 3], well groups were defined by means of singular homology. But then, once we allow arbitrary continuous perturbations, to the best of our knowledge, no $f\colon K \to \mathbb{R}^n$ with nontrivial $U_j(f, r)$ for $j > 0$ would be known. In particular, the main result of [3] would not hold. The correction via means of Steenrod homology was independently identified by the authors of [3].

[3] Namely, so called *bottleneck distance* between a well diagrams of f and f' is bounded by $\|f - f'\|$. The stability does not say how well the well diagrams describe the zero set. This question is also addressed in this paper.

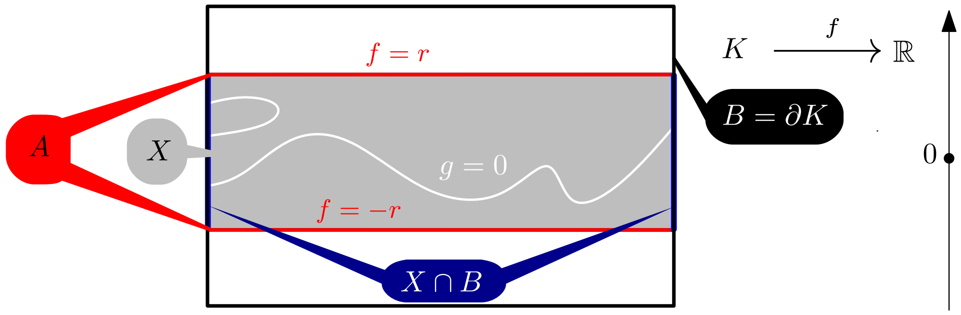

Figure 1 For the projection $f(x, y) = y$ to the vertical axis defined on a box K, the zero set of every r-perturbation is contained in $X = |f|^{-1}[0, r]$ and ∂X consists of A (upper and lower side) where $|f| = r$, and $X \cap B \subseteq \partial K$. The zero set always separates the two components of A. On the homological level, the zero set "connects" the two components of $X \cap B$ and the image of $H_1(g^{-1}(0), B)$ in $H_1(X, B)$ is always surjective and thus $U_1(f, r) \cong H_1(X, B)$. Note that the well group would be trivial with $B = \emptyset$.

Homology theory. For the foundation of well groups we need a homology theory on compact Hausdorff spaces that satisfies some additional properties that we specify later in Section 2. Roughly speaking, we want that the homology theory behaves well with respect to infinite intersections. Without these properties we would have to consider only "well behaved" perturbations of a given f in order to be able to obtain some nontrivial well groups in dimension greater than zero. We explain this in more detail also in Section 2. For the moment it is enough to say that the *Čech* homology can be used and that for any computational purposes it behaves like simplicial homology. In Section 2 we explain why using singular homology would make the notion of well groups trivial.

A basic ingredient of our methods is the notion of *cap product*

$$\frown: H^n(X, A) \otimes H_k(X, A \cup B) \to H_{k-n}(X, B)$$

between cohomology and homology. We refer the reader to [21, Section 2.2] and [15, p. 239] for its properties and to [11, Appendix E] for its construction in Čech (co)homology. Again, it behaves like the simplicial cap product when applied to simplicial complexes. For an algorithmic implementation, one can use its simplicial definition from [21].

1.1 Computability results

Computer representation. To speak about computability, we need to fix some computer representation of the input. Here we assume the simple but general setting of [10], namely, K is a finite simplicial complex, $B \subseteq K$ a subcomplex, f is simplexwise linear with rational values on vertices[4] and the norm $|\cdot|$ in $\mathbb{R}^n$ can be (but is not restricted to) ℓ_1, ℓ_2 or ℓ_∞ norm.

Previous results. The algorithm for the computation of well groups was developed only in the particular cases of $n = 1$ [3] or $\dim K = n$ [5]. In [10] we settled the computational complexity of the well group $U_0(f, r)$. The complexity is essentially identical to deciding

[4] We emphasize that the considered r-perturbations of f need not be neither simplexwise linear nor have rational values on the vertices.

whether the restriction $f|_A \colon A \to S^{n-1}$ can be extended to $X \to S^{n-1}$ for $A = |f|^{-1}(r)$, or equivalently, $A = f^{-1}(S^{n-1})$. The extendability problem can be decided as long as $\dim K \leq 2n-3$ or $n = 1, 2$ or n is even. On the contrary, the extendability of maps into a sphere – as well as triviality of $U_0(f,r)$ – cannot be decided for $\dim K \geq 2n-2$ and n odd, see [10].[5] In this paper we shift our attention to higher well groups.

Higher well groups – extendability revisited. The main idea of our study of well groups is based on the following. We try to find r-perturbations of f with as small zero set as possible, that is, avoiding zero on X' for $X' \subseteq X$ as large as possible. It is shown in [11, Lemma D.1] that for each strict r-perturbation g of f we can find an extension $e \colon X \to \mathbb{R}^n$ of $f|_A$ with $g^{-1}(0) = e^{-1}(0)$ and vice versa. Thus equivalently, we try to extend $f|_A$ to a map $X' \to S^{n-1}$ for X' as large as possible. The higher skeleton[6] of X we cover, the more well groups we kill.

▶ **Observation 1.1.** *Let $f \colon K \to \mathbb{R}^n$ be a map on a compact space. Assume that the pair of spaces $A \subseteq X$ defined as $|f|^{-1}(r) \subseteq |f|^{-1}[0,r]$, respectively, can be triangulated and* $\dim X = m$. *If the map $f|_A$ can be extended to a map $A \cup X^{(i-1)} \to S^{n-1}$ then $U_j(f,r)$ is trivial for $j > m - i$.*

Assume, in addition, that there is no extension $A \cup X^{(i)} \to S^{n-1}$. By the connectivity of the sphere S^{n-1}, we have $i \geq n$. Does the lack of extendability to $X^{(i)}$ relate to higher well groups, especially $U_{m-i}(f,r)$? The answer is *yes* when $i = n$ as we show in our computability results below. On the other hand, when $i > n$, the lack of extendability *is not* necessarily reflected by $U_{m-i}(f,r)$. This leads to the incompleteness results we show in the second part of the paper.

The first obstruction. The lack of extendability of $f|_A$ to the n-skeleton is measured by the so called *first obstruction* that is defined in terms of cohomology theory as follows. We can view f as a map of pairs $(X,A) \to (B^n, S^{n-1})$ where B^n is the ball bounded by the sphere $S^{n-1} := \{x \colon |x| = r\}$. Then the first obstruction ϕ_f is equal to the pullback $f^*(\xi) \in H^n(X,A)$ of the fundamental cohomology class $\xi^n \in H^n(B^n, S^{n-1})$.[7]

▶ **Theorem 1.2.** *Let $B \subseteq K$ be compact spaces and let $f : K \to \mathbb{R}^n$ be continuous. Let $|f|^{-1}[0,r]$ and $|f|^{-1}(r)$ be denoted by X and A, respectively, and ϕ_f be the first obstruction. Then $\phi_f \frown H_k(X, A \cup B)$ is a subgroup of $U_{k-n}(f,r)$ for each $k \geq n$.*

Our assumptions on computer representation allow for simplicial approximation of X, A and f. The pullback of $\xi^n \in H^n(B^n, S^{n-1})$ and the cap product can be computed by the standard formulas. This together with more details worked out in the proof in Section 2 gives the following.

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[5] We cannot even approximate the "robustness of roots": it is undecidable, given a simplicial complex K and a simplexwise linear map $f \colon K \to \mathbb{R}^n$, whether there exists $\epsilon > 0$ such that $U_0(f,\epsilon)$ is nontrivial or whether $U_0(f,1)$ is trivial. The extendability can always be decided for n even, however, the problem is less likely tractable for $\dim K > 2n-2$.

[6] The i-skeleton $X^{(i)}$ of a simplicial (cell) complex X is the subspace of X containing all simplices (cells) of dimension at most i.

[7] This is the global description of the first obstruction as presented in [25]. It can be shown that ϕ_f depends on the homotopy class of $f|_A$ only. Another way of defining the first obstruction is the following. It is represented by the so-called *obstruction cocycle* $z_f \in Z^n(X,A)$ that assigns to each n-simplex $\sigma \in X$ the element $[f|_{\partial\sigma}] \in \pi_{n-1}(S^{n-1}) \cong \mathbb{Z}$ [21, Chap. 3]. Through this definition it is not difficult to derive that the map $f|_A$ can be extended to $X^{(n)} \to S^{n-1}$ if and only if $\phi_f = 0$, see also [21, Chap. 3].

▶ **Theorem 1.3.** *Under the assumption on computer representation of K, B and f as above, the subgroup $\phi_f \frown H_k(X, A \cup B)$ of $U_{k-n}(f,r)$ (as in Theorem 1.2) can be computed.*

The gap between U_{k-n} and $\phi_f \frown H_k(X, A \cup B)$. There are maps f with ϕ_f trivial but nontrivial $U_0(f,r)$.[8] But this can be detected by the above mentioned extendability criterion. We do not present an example where $U_{k-n}(f,r) \neq \phi_f \frown H_k(X, A \cup B)$ for $k-n > 0$, although the inequality is possible in general. In the rest of the paper we work in the other direction to show that there is no gap in various cases and various dimensions.

An important instance of Theorem 1.2 is the case when X can be equipped with the structure of a smooth orientable manifold.

▶ **Theorem 1.4.** *Let $f\colon K \to \mathbb{R}^n$ and X, A be as above. Assume that X can be equipped with a smooth orientable manifold structure, $A = \partial X$, $B = \emptyset$ and $n+1 \leq m \leq 2n-3$ for $m = \dim X$. Then*

$$U_{m-n}(f,r) = \phi_f \frown H_m(X, \partial X).$$

When $m = n$, the well group $U_0(f,r)$ can be strictly larger than $\phi_f \frown H_n(X, \partial X)$ but it can be computed.

We believe that the same claim holds when X is an orientable PL manifold. It remains open whether the last equation holds also for $m > 2n-3$. Throughout the proof of Theorem 1.4, we will show that if $g : K \to \mathbb{R}^n$ is a smooth r-perturbation of f transverse to 0, then the fundamental class of $g^{-1}(0)$ is mapped to the Poincaré dual of the first obstruction. This also holds if $B \neq \emptyset$ and in all dimensions.

1.2 Well groups $U_*(f,r)$ are incomplete as an invariant of $Z_r(f)$

A simple example illustrating Theorem 1.4 is the map $f\colon S^2 \times B^3 \to \mathbb{R}^3$ defined by $f(x,y) := y$ with B^3 considered as the unit ball in $\mathbb{R}^3$. It is easy to show that

for every 1-perturbation g of f and every $x \in S^2$ there is a root of g in $\{x\} \times B^3$. (1)

This robust property is nicely captured by (and can be also derived from) the fact $U_2(f,1) \cong \mathbb{Z}$.

The main question of Section 3 is what happens, when the first obstruction ϕ_f is trivial – and thus $f|_A$ can be extended to $X^{(n)}$ – but the map $f|_A$ cannot be extended to whole of X. The zero set of f can still have various robust properties such as (1). It is the case of $f\colon S^2 \times B^4 \to \mathbb{R}^3$ defined by $f(x,y) := |y|\eta(y/|y|)$ where $\eta\colon S^3 \to S^2$ is a homotopically nontrivial map such as the Hopf map. The zero set of each r-perturbation g of f intersects each section $\{x\} \times B_4$, but unlike in the example before, well groups do not capture this property. All well groups $U_j(f,r)$ are trivial for $j > 0$ and,[9] consequently, they cannot distinguish f from another f' having only a single robust root in X. We will describe the construction of such f' for a wider range examples.

In the following, B^i_q will denote the i-dimensional ball of radius q, that is, $B^i_q = \{y \in \mathbb{R}^i\colon |y| \leq q\}$. We also emphasize that this and the following theorem hold for arbitrary coefficient group of the homology theory H_*.

[8] This is the case for $f\colon \mathbb{R}^4 \to \mathbb{R}^3$ given by $f(x) := |x|\eta(x/|x|)$ where $\eta\colon S^3 \to S^2$ is the Hopf map.

[9] Namely $U_2(f,r) \cong 0$ as is shown by the r-perturbation $g(x,y) = f(x,y) - rx$ with the zero set homeomorphic to the 3-sphere.

▶ **Theorem 1.5.** *Let $i, m, n \in \mathbb{N}$ be such that $m - i < n < i < (m + n + 1)/2$ and both $\pi_{i-1}(S^{n-1})$ and $\pi_{m-1}(S^{n-1})$ are nontrivial. Then on $K = S^{m-i} \times B^i_1$ we can define two maps $f, f' \colon K \to \mathbb{R}^n$ such that for all $r \in (0, 1]$*

- *f, f' induce the same $X = S^{m-i} \times B^i_r$ and $A = \partial X$ and have the same well groups for any coefficient group of the homology theory H_* defining the well groups,*
- *but $Z_r(f) \neq Z_r(f')$.*

In particular, the property

$$\text{for each } Z \in Z_r(.) \text{ and } x \in S^{m-i} \text{ there exists } y \in B^i_r \text{ such that } (x, y) \in Z$$

is satisfied for f but not for f'. Namely, $Z_\epsilon(f')$ contains a singleton for each $\epsilon > 0$.

The lack of extendability not reflected by $U_{m-i}(f, r)$. The key property of the example of Theorem 1.5 is that the maps $f|_A$ and $f'|_A$ can be extended to the $(i-1)$-skeleton $X^{(i-1)}$ of X, for $i > n$. The difference between the maps lies in the extendability to $X^{(i)}$. Unlike in the case when $i = n$, the lack of extendability is not reflected by the well groups. The crucial part is the triviality of the well groups in dimension $m - i$ and[10] this triviality holds in greater generality:

▶ **Theorem 1.6.** *Let $f \colon K \to \mathbb{R}^n$, $B \subseteq K$, $X := |f|^{-1}[0, r]$ and $A := |f|^{-1}\{r\}$. Assume that the pair (X, A) can be finitely triangulated.[11] Further assume that $f|_A$ can be extended to a map $h : A \cup X^{(i-1)} \to S^{n-1}$ for some i such that $m - i < n < i < (m+n)/2$ for $m := \dim X$. Then $U_{m-i}(f, r) = 0$ for any coefficient group of the homology theory H_*.*

The whole proof is in [11, Appendix C] but its core idea is already contained in the proof of Theorem 1.5. There we also comment on the possibility of finding pairs of maps f and f' with the same well groups but different robust properties of their zero sets in this more general situation.

Our subjective judgment on well groups of $\mathbb{R}^n$-valued maps. We find the problem of the computability of well groups interesting and challenging with connections to homotopy theory (see also Proposition 1.7 below). Moreover, we acknowledge that well groups may be accessible for non-topologists: they are based on the language of homology theory that is relatively intuitive and easy to understand. On the other hand, well groups may not have sufficient descriptive power for various situations (Theorems 1.5 and 1.6). Furthermore, despite all the effort, the computability of well groups seems far from being solved. In the following paragraphs, we propose an alternative based on homotopy and obstruction theory that addresses these drawbacks.

1.3 Related work

A replacement of well groups of $\mathbb{R}^n$-valued maps. In a companion paper [20], we find a complete invariant for an enriched version of $Z_r(f)$. The starting point is the surprising claim that $Z_r(f)$ – an object of a geometric nature – is determined by terms of homotopy theory.

[10] This dimension is somewhat important as all higher well groups are trivial by [11, Lemma C.2] and all lower homology groups of X may be trivial as is the case in Theorem 1.5. On the other hand, $H_{m-i}\big(X, \pi_{i-1}(S^{n-1})\big)$ has to be nontrivial in the case when X is a manifold for the reasons following from obstruction theory and Poincaré duality.

[11] That is, there exist finite simplicial complexes $A^\Delta \subseteq X^\Delta$ and a homeomorphism $(X^\Delta, A^\Delta) \to (X, A)$.

▶ **Proposition 1.7** ([20]). *Let $f\colon K \to \mathbb{R}^n$ be a continuous map on a compact Hausdorff domain, $r > 0$, and let us denote the space $|f|^{-1}[r,\infty]$ by A_r. Then the set $Z_r(f) := \{g^{-1}(0)\colon \|g-f\| \le r\}$ is determined by the pair (K, A_r) and the homotopy class of $f|_{A_r}$ in $[A_r, \{x \in \mathbb{R}^n\colon \|x\| \ge r\}] \cong [A_r, S^{n-1}]$.*[12]

The complete proof can be found in [11, Appendix D] and will also appear in [20].

Note that since the well groups is a property of $Z_r(f)$, they are determined by the pair (K, A_r) and the homotopy class $[f|_{A_r}]$. Thus the homotopy class has a greater descriptive power and the examples from the previous section show that this inequality is strict. If K is a simplicial complex, f is simplexwise linear and dim $A_r \le 2n-4$ then $[A_r, S^{n-1}]$ has a natural structure of an Abelian group denoted by $\pi^{n-1}(A_r)$. The restriction $\dim A_r \le 2n-4$ does not apply when $n = 1, 2$ and[13] otherwise we could replace $[A_r, S^{n-1}]$ with $[A_r^{(2n-4)}, S^{n-1}]$ which contains less information but is computable. The isomorphism type of $\pi^{n-1}(A_r)$ together with the distinguished element $[f|_{A_r}]$ can be computed essentially by [23, Thm 1.1]. Moreover, the inclusions $A_s \subseteq A_r$ for $s \ge r$ induce computable homomorphisms between the corresponding pointed Abelian groups. Thus for a given f we obtain a sequence of pointed Abelian groups $\pi^{n-1}(A_r), r > 0$ and it can be easily shown that the interleaving distance of the sequences $\pi^{n-1}\big(A_*(f)\big)$ and $\pi^{n-1}\big(A_*(g)\big)$ is bounded by $\|g-f\|$. Thus after tensoring the groups by an arbitrary field, we get persistence diagrams (with a distinguished bar) that will be stable with respect to the bottleneck distance and the L_∞ norm. The construction will be detailed in [20].

The computation of the cohomotopy group $\pi^{n-1}(A)$ is naturally segmented into a hierarchy of approximations of growing computational complexity. Therefore our proposal allows for compromise between the running time and the descriptive power of the outcome. The first level of this hierarchy is the primary obstruction ϕ_f. One could form similar modules of cohomology groups with a distinguished element as we did with the cohomotopy groups above. However, in this paper we passed to homology via cap product in order to relate it to the established well groups. In the "generic" case when X is a manifold no information is lost as from the Poincaré dual $\phi_f \frown [X]$ we can reconstruct the primary obstruction ϕ_f back.

The cap-image groups. The groups $\phi_f \frown H_k(X, A)$ (with $B = \emptyset$) has been studied by Amit K. Patel under the name *cap-image groups.* In fact, his setting is slightly more complex with $\mathbb{R}^n$ replaced by arbitrary manifold Y. Instead of the zero sets, he considers preimages of all points of Y simultaneously in some sense. Although his ideas have not been published yet, they influenced our research; the application of the cap product in the context of well groups should be attributed to Patel.[14]

Verification of zeros. An important topic in the interval computation community is the verification of the (non)existence of zeros of a given function [19]. While the nonexistence can be often verified by interval arithmetic alone, a proof of existence requires additional

[12] Here $[A_r, S^{n-1}]$ denotes the set of all homotopy classes of maps from A_r to S^{n-1}, that is, the cohomotopy group $\pi^{n-1}(A_r)$ when $\dim A_r \le 2n-4$.

[13] Note that for $n = 1$ the structure of the set $[A, S^{n-1}]$ is very simple and for $n = 2$ we have $[A, S^{n-1}] \cong H^1(A; \mathbb{Z})$ no matter what the dimension of A_r is.

[14] We originally proved that when K is a triangulated orientable manifold, the Poincaré dual of ϕ_f is contained in $U_{m-n}(f, r)$. Expanding the proof was not difficult, but the preceding inspiration of replacing the Poincaré duality by cap product came from Patel. The cap product provides a nice generalization to an arbitrary simplicial complex K.

methods which often include topological considerations. In the case of continuous maps $f : B^n \to \mathbb{R}^n$, Miranda's or Borsuk's theorem can be used for zero verification [13, 1], or the computation of the topological degree [17, 6, 12]. Fulfilled assumptions of these tests not only yield a zero in B^n but also a "robust" zero and a nontrivial 0th well group $U_0(f, r)$ for some $r > 0$. Recently, topological degree has been used for simplification of vector fields [22].

The first obstruction ϕ_f is the analog of the degree for underdetermined systems, that is, when $\dim K > n$ in our setting. To the best of our knowledge, this tool has not been algorithmically utilized.

2 Computing lower bounds on well groups

Homology theory behind the well groups. For computing the approximation $\phi_f \frown H_k(X, A \cup B)$ of well group $U_{k-n}(f)$ we only have to work with simplicial complexes and simplicial maps for which all homology theories satisfying the Eilenberg–Steenrod axioms are naturally equivalent. Hence, regardless of the homology theory H_* used, we can do the computations in simplicial homology. Therefore the standard algorithms of computational topology [7] and the formula for the cap product of a simplicial cycle and cocycle [21, Section 2.2] will do the job.

The need for a carefully chosen homology theory stems from the courageous claim that the zero set Z of *arbitrary* continuous perturbation *supports* $\phi_f \frown \beta$ for any $\beta \in H_*(X, A \cup B)$, i.e. some element of $H_*(Z, B)$ is mapped by the inclusion-induced map to $\phi_f \frown \beta$. Without more restrictions on the perturbations, the zero sets can be "wild" non-triangulable topological spaces that can fool singular homology and render this claim false and – to the best of our knowledge – make well groups trivial. See an example after the proof of Theorem 1.2.

For the purpose of the work with the general zero sets, we will require that our homology theory satisfies the Eilenberg-sequenc-Steenrod axioms with a possible exception of the exactness axiom, and these additional properties:

1. *Weak continuity property:* for an inverse sequence of compact pairs $(X_0, B_0) \supset (X_1, B_1) \supset \ldots$ the homomorphism $H_* \varprojlim (X_i, B_i) \to \varprojlim H_*(X_i, B_i)$ induced by the family of inclusion $\varprojlim (X_i, B_i) = \bigcap (X_i, B_i) \hookrightarrow (X_j, B_j)$ *is surjective.*
2. *Strong excision:* Let $f\colon (X, X') \to (Y, Y')$ be a map of compact pairs that maps $X \setminus X'$ homeomorphically onto $Y \setminus Y'$. Then $f_*\colon H_*(X, X') \to H_*(Y, Y')$ *is an isomorphism.*

Čech homology theory satisfies these properties as well as the Eilenberg–Steenrod axioms with the exception of the exactness axiom, and coincides with simplicial homology for triangulable spaces [24, Chapter 6].

In addition, we need a cohomology theory H^* that satisfies the Eilenberg–Steenrod axioms and is paired with H_* via a cap product $H^n(X, A) \otimes H_k(X, A \cup B) \xrightarrow{\frown} H_{k-n}(X, B)$ that is natural[15] and coincides with the simplicial cap product when applied to simplicial complexes. We have not found any reference for the definition of cap product in Čech (co)homology, so we present our own construction in [11, Appendix E].

Proof of Theorem 1.2. We need to show that for any map g with $\|g - f\| \le r$, the image of the inclusion-induced map

$$H_*(g^{-1}(0), B) \to H_*(X, B)$$

[15] Naturality of the cap product means that if $f : (X, A \cup B, A) \to (X; A' \cup B', A')$ is continuous, then $f_*(f^*(\tilde{\alpha}) \frown \beta) = \tilde{\alpha} \frown f_*(\beta)$ for any $\beta \in H_*(X, A \cup B)$ and $\tilde{\alpha} \in H^*(X', A')$.

contains the cap product of the first obstruction $\phi_f := f^*(\xi)$ with all relative homology classes of $(X, A \cup B)$. Let us first restrict to the less technical case of g being a strict r-perturbation, that is, $\|g - f\| < r$.

Let us denote $X_0 := X = |f|^{-1}[0, r]$ and $A_0 := A = |f|^{-1}(r)$. Next we choose a decreasing positive sequence $\epsilon_1 > \epsilon_2 > \dots$ with $\lim_{i\to\infty} \epsilon_i = 0$ and with $\epsilon_1 < r - \|f - g\|$. Thus $X_1 := |g|^{-1}[0, \epsilon_1] \subseteq X_0$ and $A'_0 := |g|^{-1}[\epsilon_2, \infty] \cap X_0 \supseteq |g|^{-1}[\epsilon_2, \epsilon_1]$. Then we for each $i > 0$ we define

- $X_i := |g|^{-1}[0, \epsilon_i]$,
- and its subspaces $A_i := |g|^{-1}[\epsilon_{i+1}, \epsilon_i]$, $A'_i := |g|^{-1}[\epsilon_{i+2}, \epsilon_i]$ and $B_i := B \cap X_i$.

Note that $\bigcap_i X_i = g^{-1}(0)$, and consequently, $\bigcap_i B_i = g^{-1}(0) \cap B$. For any given $\beta \in H_k(X, A \cup B)$, our strategy is to find homology classes $\alpha_i \in H_{k-n}(X_i, B_i)$, with $\alpha_0 = \phi_f \frown \beta$, that fit into the sequence of maps $H_{k-n}(X_0, B_0) \leftarrow H_{k-n}(X_1, B_1) \leftarrow \dots$ induced by inclusions. This gives an element in $\varprojlim H_{k-n}(X_i, B_i)$, and consequently by the weak continuity property (requirement 1 above), we get the desired element $\alpha \in H_{k-n}(g^{-1}(0), B)$.

The elements α_i will be constructed as cap products. To that end, we need to obtain "analogs" of β and for that we will need a more complicated sequence of maps. It is the zig-zag sequence

$$X_0 \overset{\text{id}}{\to} X_0 \overset{\text{incl}}{\hookleftarrow} X_1 \overset{\text{id}}{\to} X_1 \overset{\text{incl}}{\hookleftarrow} X_2 \overset{\text{id}}{\to} \cdots \tag{2}$$

that restricts to the zig-zags

$$A_0 \overset{\text{incl}}{\hookrightarrow} A'_0 \overset{\text{incl}}{\hookleftarrow} A_1 \overset{\text{incl}}{\hookrightarrow} A'_1 \overset{\text{incl}}{\hookleftarrow} A_2 \overset{\text{incl}}{\hookrightarrow} \cdots \tag{3}$$

and

$$A_0 \cup B_0 \overset{\text{incl}}{\hookrightarrow} A'_0 \cup B_0 \overset{\text{incl}}{\hookleftarrow} A_1 \cup B_1 \overset{\text{incl}}{\hookrightarrow} A'_1 \cup B_1 \overset{\text{incl}}{\hookleftarrow} A_2 \cup B_2 \overset{\text{incl}}{\hookrightarrow} \cdots \tag{4}$$

The pair $(X_{i+1}, A_{i+1} \cup B_{i+1})$ is obtained from $(X_i, A'_i \cup B_i)$ by excision of $|g|^{-1}(\epsilon_{i+1}, \epsilon_i]$, that is, $X_{i+1} = X_i \setminus |g|^{-1}(\epsilon_{i+1}, \epsilon_i]$ and $A_{i+1} \cup B_{i+1} = (A'_i \cup B_i) \setminus |g|^{-1}(\epsilon_{i+1}, \epsilon_i]$. Hence by excision,[16] each inclusion of the pairs $(X_i, A'_i \cup B_i) \hookrightarrow (X_{i+1}, A_{i+1} \cup B_{i+1})$ induces isomorphism on relative homology groups. Therefore the zig-zag sequences (2) and (4) induce a sequence

$$\begin{array}{ccccccccc}
H_k(X_0, A_0 \cup B_0) & \to & H_k(X_0, A'_0 \cup B_0) & \cong & H_k(X_1, A_1 \cup B_1) & \to & H_k(X_1, A'_1 \cup B_1) & \cong & \cdots \\
\Cup & & \Cup & & \Cup & & \Cup & & \\
\beta_0 := \beta & & \beta'_0 & & \beta_1 & & \beta'_1 & & \cdots
\end{array}$$

that can be made pointed by choosing the distinguished homology classes $\beta_i \in H_k(X_i, A_i \cup B_i)$ and $\beta'_i \in H_k(X_i, A'_i \cup B_i)$ that are the images of $\beta_0 := \beta \in H_k(X, A \cup B)$ in this sequence.

Similarly, we want to construct a pointed zig-zag sequence in cohomology induced by (2) and (3). The distinguished elements $\phi_i \in H^n(X_i, A_i)$ and $\phi'_i \in H^n(X_i, A'_i)$ are defined as the pullbacks of the fundamental cohomology class $\xi \in H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ by the restrictions of g. Because of the functoriality of cohomology, ϕ_i and ϕ'_i fit into the sequence induced by (2) and (3):

$$\begin{array}{ccccccccc}
H^n(X_0, A_0) & \leftarrow & H^n(X_0, A'_0) & \to & H^n(X_1, A_1) & \leftarrow & H^n(X_1, A'_1) & \to & \cdots \\
\Cup & & \Cup & & \Cup & & \Cup & & \\
\phi_0 & & \phi'_0 & & \phi_1 & & \phi'_1 & & \cdots
\end{array}$$

[16] Because of our careful choice of the spaces A_i and A'_i we do not need the strong excision here. However, we do not know how to avoid it in the case when $\|g - f\| = r$.

Since g is an r-perturbation of f and thus $g|_{(X,A)}$ is homotopic to $f|_{(X,A)}$ via the straight line homotopy, we have that $\phi_0 = \phi_f \in H^n(X,A)$.

From the naturality of the cap product we get that the elements $\phi_i \frown \beta_i$ and $\phi'_i \frown \beta'_i$ fit into the sequence

$$\begin{array}{ccccccccc}
H_{k-n}(X_0,B_0) & \overset{\mathrm{id}}{\cong} & H_{k-n}(X_0,B_0) & \leftarrow & H_{k-n}(X_1,B_1) & \overset{\mathrm{id}}{\cong} & H_{k-n}(X_1,B_1) & \leftarrow & \cdots \\
\Cup & & \Cup & & \Cup & & \Cup & & \\
\phi_0 \frown \beta_0 & & \phi'_o \frown \beta'_0 & & \phi_1 \frown \beta_1 & & \phi'_1 \frown \beta'_1 & & \cdots \\
\| & & & & & & & & \\
\phi_f \frown \beta & & & & & & & &
\end{array}$$

that is induced by (2), that is, each $H_{k-n}(X_i,B_i) \overset{\mathrm{id}}{\cong} H_{k-n}(X_i,B_i)$ is induced by the identity $X_i \overset{\cong}{\to} X_i$ and each map $H_{k-n}(X_i,B_i) \leftarrow H_{k-n}(X_{i+1},B_{i+1})$ is induced by the inclusion $X_i \hookleftarrow X_{i+1}$. Hence $\alpha_i := \phi_i \frown \beta_i$ are the desired elements and thus there is an element $\tilde{\alpha} := (\alpha_0, \alpha_1, \ldots)$ in $\varprojlim H_{k-n}(X_i,B_i)$.

We recall that the weak continuity property of the homology theory H_* assures the surjectivity of the the map

$$(\iota_i)_{i\ge 0}\colon H_{k-n}\Big(\bigcap X_i, B\Big) \to \varprojlim H_{k-n}(X_i,B) \tag{5}$$

where each component ι_i is induced by the inclusion $\bigcap_i X_i \hookrightarrow X_i$. Let $\alpha \in H_{k-n}(g^{-1}(0),B)$ be arbitrary preimage of $\tilde{\alpha}$ under the surjection (5). By construction, α is mapped to $\alpha_0 = \phi_f \frown \beta$ by the map ι_0.

It remains to prove the theorem in the case when $\|g-f\| = r$. The proof goes along the same lines with only the following differences:

- For arbitrary decreasing sequence $1 = \epsilon_0 > \epsilon_1 > \epsilon_2 > \ldots$ with $\lim \epsilon_i = 0$ we define $h_i := \epsilon_i f + (1-\epsilon_i) g$ for $i \ge 0$. We will furthermore need that $2\epsilon_{i+1} > \epsilon_i$ for every $i \ge 0$. Let
$$\begin{aligned}
&X_i := |h_i|^{-1}[0, \epsilon_i r],\\
&\cup\!\!\mid\\
&A'_i := \{x \in X\colon |h_i(x)| \le \epsilon_i r \text{ and } |h_{i+1}(x)| \ge \epsilon_{i+1} r\} \text{ and}\\
&\cup\!\!\mid\\
&A_i := |h_i|^{-1}(\epsilon_i r).
\end{aligned}$$
We have $A_i \subseteq A'_i$ because by definition $\|h_i - h_{i+1}\| \le (\epsilon_i - \epsilon_{i+1})r$ and thus $|h_i(x)| = \epsilon_i r$ implies $|h_{i+1}(x)| \ge \epsilon_{i+1} r$. Similarly $A_{i+1} \subseteq A'_i$ and $X_{i+1} \subseteq X_i$. Therefore as before, the zig-zag sequence (2) restricts to (3) and (4).
- The homology classes β_i and β'_i are defined as above. We only need to use the strong excision for the inclusion $(X_i, A'_i \cup B_i) \hookleftarrow (X_{i+1}, A_{i+1} \cup B_{i+1})$.
- We define the cohomology classes $\phi_i := h_i^*(\xi)$ and $\phi'_i := h_{i+1}^*(\xi)$. We only need to check that h_i is homotopic to h_{i+1} as a map of pairs $(X_i, A'_i) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. Indeed, they are homotopic via the straight-line homotopy since $|h_{i+1}(x)| \ge \epsilon_{i+1} r$ implies $|h_i(x)| \ge \epsilon_{i+1} r - (\epsilon_i - \epsilon_{i+1}) r = (2\epsilon_{i+1} - \epsilon_i) r > 0$. We used the inequality $2\epsilon_{i+1} > \epsilon_i$ which was our requirement on the sequence $(\epsilon_i)_{i>0}$. We also have $\phi_0 = \phi_f$ as $h_0 = f$ and $(X_0, A_0) = (X, A)$.
- We continue by defining cap products α_i, their limit $\tilde{\alpha}$ and its preimage α under the surjection $H_{k-n}(\bigcap_i X_i, B) \to \varprojlim_i H_{k-n}(X_i, B)$. To finish the proof we claim that $\bigcap_i X_i = g^{-1}(0)$. Indeed, $g(x) = 0$ implies $h_i(x) \le \|h_i - g\| = \epsilon_i r$ for each i and $g(x) > 0$ implies $h_i(x) > 0$ for i such that $2\epsilon_i r < |g(x)|$.

◀

The surjectivity of (5) and the strong excision is not only a crucial step for Theorem 1.2 but implicitly also for the results stated in [3, p. 16]. If we defined well groups by means of singular homology, then even in a basic example $f(x, y) = x^2 + y^2 - 2$ and $r = 1$, the first well group $U_1(f, r)$ would be trivial. The zero set of any 1-perturbation g is contained in the annulus $X := \{(x, y) : 1 \le x^2 + y^2 \le 3\}$ and the two components of ∂X are not in the same connected components of $\{x \in X : g(x) \neq 0\}$. However, we could construct a "wild" 1-perturbation g of f such that $g^{-1}(0)$ is a Warsaw circle [18] which is, roughly speaking, a circle with infinite length, trivial first singular homology, but nontrivial Čech homology. Thus Čech homology serves as a better theoretical basis for the well groups. Another solution to avoid problems with wild zero sets would be to restrict ourselves to "nice" perturbations, for example piecewise linear or smooth and transverse to 0. Such approach would lead, to the best of our knowledge, to identical results.

Proof of Theorem 1.3. Under the assumption on computer representation of K and f, the pair (X, A) is homeomorphic to a computable simplicial pair (X', A') such that X' is a subcomplex of a subdivision K' of K [10, Lemma 3.4]. Therefore, the induced triangulation B' of $B \cap X'$ is a subcomplex of X'. Furthermore, a simplicial approximation $f' : A' \to S'$ of $f|_A : A \to S^{n-1}$ can be computed. The computation is implicit in the proof of Theorem 1.2 in [10] where the sphere S^{n-1} is approximated by the boundary S' of the n-dimensional cross polytope B'. The simplicial approximation $(X', A') \to (B', S')$ of $f|_X$ can be constructed consequently by sending each vertex of $X \setminus A$ to an arbitrary point in the interior of the cross polytope, say $0 \in \mathbb{R}^n$. The pullback of a cohomology class can be computed by standard algorithms. Therefore ϕ_f and $H_*(X, B)$ can be computed and the explicit formula for the cap product in [21, Section 2.1] yields the computation of $\phi_f \frown H_*(X, B)$. All this can be done without any restriction on the dimensions of the considered simplicial complexes. ◀

Well diagram associated with $\boldsymbol{\phi \frown H_*(X, A \cup B)}$. Let $r_1 > r_2 > 0$ and let X_1, X_2, A_1, A_2 be $|f|^{-1}[0, r_1]$, $|f|^{-1}[0, r_2]$, $|f|^{-1}\{r_1\}$, $|f|^{-1}\{r_2\}$ respectively, ϕ_1, ϕ_2 be the respective obstructions. Further, let $A_1' := |f|^{-1}[r_2, r_1]$ and $\phi_1' = f^*(\xi) \in H^n(X_1, A_1')$ be the pullback of the fundamental class $\xi \in H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. The inclusions $(X_1, A_1) \subseteq (X_1, A_1') \supseteq (X_2, A_2)$ induce cohomology maps that take ϕ_1' to ϕ_1 resp. ϕ_2. Let us denote, for simplicity, by V_1 the group $\phi_1 \frown H_*(X_1, A_1 \cup B)$, $V_2 := \phi_2 \frown H_*(X_2, A_2 \cup B)$ and $V_1' := \phi_1' \frown H_*(X_1, A_1' \cup B)$. Further, let U_1 resp. U_2 be the well groups $U(f, r_1)$ resp. $U(f, r_2)$.

In this section, we analyze the relation between V_1 and V_2. First let i_1 be a map from V_1 to V_1' that maps $\phi_1 \frown \beta_1$ to $\phi_1' \frown i_*(\beta_1)$. By the naturality of cap product, $\phi_1 \frown \beta_1 = \phi_1' \frown i_*(\beta_1)$, so i_1 is an inclusion.

$$\begin{tikzcd}
V_2 \arrow[d, hook] & & V_1 \arrow[ll, "\iota_{12}"'] \arrow[d, hook] \\
U_2 \arrow[r, "a"] \arrow[d, hook] & U_2/(U_2 \cap \ker i_{21}) & U_1 . \arrow[l, "b"'] \arrow[d, hook] \\
H_*(X_2, B) \arrow[r, "i_{21}"] & H_*(X_1, B) & i_{21}(U_2) \arrow[l, hook'] \arrow[ul, "\cong"']
\end{tikzcd}$$

By excision, there is an inclusion-induced isomorphisms $i_1' : H_*(X_2, A_2 \cup B) \xrightarrow{\sim} H_*(X_1, A_1' \cup B)$ and its inverse induces an isomorphism $i_2 : V_1' \xrightarrow{\sim} V_2$ by mapping $\phi_1' \frown \beta_1'$ to $\phi_2 \frown (i_1')^{-1}(\beta_1')$. The composition $i_2 \circ i_1 =: \iota_{12}$ is a homomorphism from V_1 to V_2. Being the composition of an inclusion and an isomorphism, ι_{12} is an injection and one easily verifies that the inclusion-induced map $i_{21} : H_*(X_2, B) \to H_*(X_1, B)$ satisfies $i_{21} \circ \iota_{12} = \mathrm{id}|_{V_1}$. It follows that $\{V(r_i), \iota_{i,i+1}\}_{r_i > r_{i+1}}$ is a persistence module consisting of shrinking abelian groups and injections $V_i \to V_{i+1}$ for $r_i > r_{i+1}$. The relation between ι and well diagrams described in [9] is reflected by the commutative diagram above.

The idea behind the proof of Theorem 1.4. In the special case when X is a smooth m-manifold with $A = \partial X$, the zero set of any smooth r-perturbation g transverse to 0 is an $(m-n)$-dimensional smooth submanifold of X. It is not so difficult to show that its fundamental class $[g^{-1}(0)]$ is mapped by the inclusion-induced map to $\phi_f \frown [X]$, where $[X] \in H_m(X, \partial X)$ is the fundamental class of X. If $g^{-1}(0)$ is connected, then $H_{m-n}(g^{-1}(0))$ is generated by its fundamental class and we immediately obtain the reverse inclusion $\phi_f \frown H_m(X, A) \supseteq U_{m-n}(f, r)$. The nontrivial part in the proof of Theorem 1.4 is to show that in the indicated dimension range, we can find a perturbation g so that $g^{-1}(0)$ is connected. The full proof is in [11, Appendix B].

3 Incompleteness of well groups

In this section, we study the case when the first obstruction ϕ_f is trivial and thus the map $f|_A$ can be extended to a map $f^{(n)} \colon X^{(n)} \to S^{n-1}$ on the n-skeleton $X^{(n)}$ of X. Observation 1.1 (proved in [11, Appendix C]) implies that the only possibly nontrivial well groups are $U_j(f, r)$ for $j \le m - n - 1$.

The following lemma summarizes the necessary tools for the constructions of this section. They directly follow from Lemma D.1 in [11, Appendix D] and from [10, Lemma 3.3].

▶ **Lemma 3.1.** *Let $f \colon K \to \mathbb{R}^n$ be a map on a compact Hausdorff space, $r > 0$, and let us denote the pair of spaces $|f|^{-1}[0, r]$ and $|f|^{-1}\{r\}$ by X and A, respectively. Then*

1. *for each extension $e \colon X \to \mathbb{R}^n$ of $f|_A$ we can find a strict r-perturbation g of f with $g^{-1}(0) = e^{-1}(0)$;*
2. *for each r-perturbation g of f without a root there is an extension $e \colon X \to \mathbb{R}^n \setminus \{0\}$ of $f|_A$ (without a root).*

In the following we want to show that well groups can fail to distinguish between maps with intrinsically different families of zero sets. Namely, in the following examples we present maps f and f' with $U_0(f, r) = U_0(f', r) = \mathbb{Z}$ for each $r \le 1$ and $U_i(f, r) = U_i(f, r) = 0$ for each $r \le 1$ and $i > 0$. However, $Z_r(f)$ will be significantly different from $Z_r(f')$.

Proof of Theorem 1.5. We have that $B = \emptyset$ and $K = S^j \times B^i$, where B^i is represented by the unit ball in $\mathbb{R}^i$ and $j = m - i$. Let the maps $f, f' : K \to \mathbb{R}^n$ be defined by

$$f(x, y) := |y|\, \varphi(x, y/|y|) \quad \text{and} \quad f'(x, y) := |y| \varphi'(x, y/|y|)$$

where $\varphi, \varphi' \colon S^j \times S^{i-1} \to S^{n-1} \subseteq \mathbb{R}^n$ are defined by

- $\varphi(x, y) := \mu(y)$ where $\mu : S^{i-1} \to S^{n-1}$ is an arbitrary nontrivial map.
- φ' is defined as the composition $S^j \times S^{i-1} \to S^{m-1} \xrightarrow{\nu} S^{n-1}$ where the first map is the quotient map $S^j \times S^{i-1} \to S^j \wedge S^{i-1} \cong S^{m-1}$ and ν is an arbitrary nontrivial map. In other words, we require that the composition $\varphi'\Phi$ – where Φ denotes the characteristic map of the $(m-1)$-cell of $S^j \times S^{i-1}$ – is equal to the composition νq, where q is the quotient map $B^{m-1} \to B^{m-1}/(\partial B^{m-1}) \cong S^{m-1}$.

Well groups computation. Next we prove that the well groups of $U_*(f,r)$ and $U_*(f',r)$ are the same for $r \in (0,1]$, namely, nonzero only in dimension 0, where they are isomorphic to $\mathbb{Z}$. We obviously have $X = S^j \times \{y \in \mathbb{R}^i : |y| \le r\} \simeq S^j \times B^i$ and $A = \partial X$ for both maps. The restriction $f|_A$ and $f'|_A$ are equal to φ and φ' (after normalization). We first prove that $U_0(f,1) \cong U_0(f',1) \cong \mathbb{Z}$. This fact follows from $H_0(X) \cong \mathbb{Z}$, from non-extendability of φ and φ' and from Lemma 3.1 part 2 (or [10, Lemma 3.3]).

▶ **Lemma 3.2.** *The map φ' cannot be extended to a map $X \to S^{n-1}$.*

The proof can be found in [11, Appendix A]. Since the map $\mu\colon S^{i-1} \to S^{n-1}$ cannot be extended to $B^i \supset S^{i-1}$, also φ cannot be extended to X.

Since then only the jth homology group of X is nontrivial, the remaining task is to show that $U_j(f,1) \cong U_j(f',1) \cong 0$. We do so by presenting two r-perturbations g and g' of f and f', respectively:

- $g(x,y) := f(x,y) - rx = |y|\mu(y/|y|) - rx$ where we consider $S^j \subseteq \mathbb{R}^{j+1}$ as a subset of $\mathbb{R}^n$ naturally embedded in the first $j+1$ coordinates (here we need that $j = m-i < n$).
- We first construct an extension $e'\colon X \to \mathbb{R}^n$ of $\varphi' = f'|_A$ and then the r-perturbation g' is obtained by Lemma 3.1 part 1. The extension e' is defined as constant on the single i-cell of X, that is, $e'(x_0, y)$ is put equal to the basepoint of $S^{n-1} \subseteq \mathbb{R}^n$. On the remaining m-cell $B^m \cong \{z \in \mathbb{R}^m : |z| \le 1\}$ of X we define $e'(z) := |z|e'(z/|z|)$, where each point z is identified with a point of X via the characteristic map $\Psi_1\colon B^m \to X$ of the m-cell B^m.[17]

By definition the only root of g' is the single point $\Psi_1(0)$ of the interior of X. Therefore $U_j(f,1) \cong 0$. Note that the role of $\Psi_1(0)$ could be played by an arbitrary point in the interior of X.[18]

The zero set $g^{-1}(0) = \{(x,y) : |y| = r \text{ and } \mu(y/|y|) = x\}$ is by definition homeomorphic to the pullback (i.e., a limit) of the diagram

$$\begin{array}{ccc} & & S^{i-1} \\ & & \big\downarrow{\scriptstyle\mu} \\ S^j & \xrightarrow{\iota} & S^{n-1} \end{array} \tag{6}$$

where ι is the equatorial embedding, i.e., sends each element x to $(x,0,0,\ldots)$. In plain words, the zero set is the μ-preimage of the equatorial j-subsphere of S^{n-1}. We will prove that under our assumptions on dimensions, this is the $(m-n)$-sphere S^{m-n}. Then from $m-n > m-i = j$ it will follow that $H_j(g^{-1}(0)) \cong 0$ which proves Theorem 1.5.

The topology of the pullback is particularly easy to see in the case when $j = n-1$ and ι is the identity. There it is simply the domain of μ, that is, S^{i-1} where $i-1 = m-j-1 = m-n$.

In the general case, the only additional tool we use to identify the pullback is the Freudenthal suspension theorem. The pullback is homeomorphic to the μ-preimage of the equatorial subsphere $S^{m-i} \subseteq S^{n-1}$. By Freudenthal suspension theorem μ is homotopic to an iterated suspension $\Sigma^a\eta$ for some $\eta\colon S^{i-1-a} \to S^{n-1-a}$ assuming $i-1-a \le 2(n-1-a)-1$. We want to choose a so that $n-1-a = m-i$ and thus images $\mathrm{Im}(\eta) = S^{n-1-a}$ and $\mathrm{Im}(\iota) = S^j \subseteq S^{n-1}$ coincide (since $j = m-i$ by definition). The last inequality with the choice $a = n-1-m+i$ is equivalent to the bound $i \le (m+n-1)/2$ from the hypotheses of

[17] Thus the formal definition is $e'(\Psi_1(z)) := |z|e'\big(\Psi_1(z/|z|)\big)$.

[18] With more effort we could show that for *any* point z of X there is an r-perturbation of f' with z being its only zero point.

the theorem. In our example, we may have chosen f in such a way that $\mu = \Sigma^a \eta$. But even for the choices of μ only homotopic to $\Sigma^a \eta$ we could have changed f on a neighborhood of ∂K by a suitable homotopy. To finish the proof we use the fact that, by the definition of suspension, the μ-preimage of $S^{m-i} \subseteq S^{n-1}$ is identical to the η-preimage of S^{m-i}, that is $S^{i-1-j} = S^{m-n}$.

Difference between $Z_r(f)$ and $Z_r(f')$. Because the map μ is homotopically nontrivial, the zero set of each extension $e\colon X \to \mathbb{R}^n$ of $f|_A$ intersects each "section" $\{x\} \times B^i$ of X. By Lemma 3.1 part 2 (or [10, Lemma 3.3]) applied to each restriction $f|_{\{x\}\times B^i}$, the same holds for r-perturbations g of f as well. In other words, the formula "for each $x \in S^j$ there is $y \in B^i$ such that $f(x, y) = 0$" is *satisfied robustly*, that is

$$\forall Z \in Z_r(f) : \forall x \in S^j : \exists y \in B^i : (x, y) \in Z$$

is satisfied. The above formula is obviously not true for f' as can be seen on the r-perturbations g'. In particular, for every $r \in (0, 1]$ the family $Z_r(f')$ contains a singleton. ◀

As an example of another relevant property of $Z_r(f)$ not captured by the well groups, we mention the following. For any given $u\colon K \to \mathbb{R}$, we may want to know what is the *r-robust maximum of u over the zero set of f*, i.e., $\inf_{Z\in Z_r(f)} \max_{z\in Z} u(z)$. Let, for instance, $u(x, y) = u(x)$ depend on the first coordinate only. Then the r-robust maximum for f is equal to $\max_{x\in S^j} u(x)$ as follows from the discussion in the previous paragraph. On the other hand, the r-robust maximum for f' is equal to $\min_x u(x)$ and is attained in g' when we set the value $\Psi_1(0) := (\arg\min_{x\in S^j} u(x), 0)$ from the proof above. This holds for r arbitrarily small. The robust optima constitutes another and, in our opinion, practically relevant quantity whose approximation cannot be derived from well groups.

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