An Approximate Version of the Tree Packing Conjecture via Random Embeddings*

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Abstract

We prove that for any pair of constants $\varepsilon > 0$ and Δ and for n sufficiently large, every family of trees of orders at most n, maximum degrees at most Δ, and with at most $\binom{n}{2}$ edges in total, packs into $K_{(1+\varepsilon)n}$. This implies asymptotic versions of the well-known tree packing conjecture of Gyárfás from 1976 and another tree packing conjecture of Ringel from 1963 for trees with bounded maximum degree. A novel random tree embedding process combined with the nibble method forms the core of the proof.

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1 Introduction

Tree embeddings and packings, albeit their simple formulation, have proven to be among the most difficult tasks in graph theory. In 1963, Erdős and Sós conjectured that every graph with average degree larger than $k-1$ must contain a copy of every tree on $k+1$ vertices. In close vicinity to this problem, Loebl, Komlós and Sós conjectured in 1995 that the same holds when substituting the median degree for the average degree. A solution to the first conjecture has been announced by Ajtai, Komlós, Simonovits and Szemerédi in the early 1990s. In 2008, the dense case of the second conjecture has been proven to be true by

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Hladký and Piguet [14] and Cooley [9], and an approximate version of the general case was confirmed recently by Hladký, Komlós, Piguet, Simonovits, Stein, and Szemerédi [13].

The focus of this paper is on packing trees, which generalises the notion of embeddings to finding several subgraphs simultaneously. A family of graphs $\mathcal{H} = (H_1, \dots, H_k)$ is said to *pack* into a graph G if there exist pairwise edge-disjoint copies of $H_1, \dots, H_k$ in G. In 1976, Gyárfás (who, according to his own words, was fascinated and motivated by the fact that $\sum_{i=1}^{n} i = n(n-1)/2$) proposed the following conjecture that is often referred to as the Tree Packing Conjecture.

▶ **Conjecture 1.** *Any family $(T_1, T_2, \dots, T_n)$ of trees, with $v(T_j) = j$ with $j \in [n]$, packs into K_n.*

A related conjecture of Ringel [18], dating back to 1963, deals with packing many copies of the same tree.

▶ **Conjecture 2.** *Any $2n+1$ identical copies of any tree of order $n+1$ pack into K_{2n+1}.*

Note that this conjecture, too, proposes the existence of a *perfect packing*, which means that the number of edges in the host graph equals to the total number of edges to be packed.

In this extended abstract, we outline a proof of a common generalisation that confirms the approximate correctness of Conjectures 1 and 2 for bounded-degree trees, without needing any further requirement than just the obvious upper bound on the total number of edges.

▶ **Theorem 3.** *For any $\varepsilon > 0$ and any $\Delta \in \mathbb{N}$, there is an $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ the following holds. Any family of trees $\mathcal{T} = (T_i)_{i\in[k]}$ such that T_i has maximum degree at most Δ and order at most n for each $i \in [k]$, and $\sum_{i\in[k]} e(T_i) \leq \binom{n}{2}$ packs into $K_{(1+\varepsilon)n}$.*

So far other major steps towards the resolution of these two conjectures have been comparatively limited. We briefly review them in the following (see also the slightly outdated survey by Hobbs [15]). Focussing on the initial set of smaller trees appearing in Conjecture 1, Bollobás [4] proved that any family of trees $T_1, \dots, T_s$ with $v(T_i) = i$ and $s < n/\sqrt{2}$ can be packed into K_n. Moreover, he observed that the validity of Erdős–Sós conjecture would imply that one can improve the bound to $s < \frac{1}{2}\sqrt{3}n$. Yuster [22] considered packings of trees into complete bipartite graphs and proved that any sequence of trees $T_1, \dots, T_s$, $s < \sqrt{5/8}n$ can be packed into $K_{n-1,n/2}$. This improves upon a result of Caro and Roditty [6] and is related to a conjecture of Hobbs, Bourgeois and Kasiraj [16]. Furthermore, a result of Caro and Yuster [7] implies the existence of a perfect packing of a family of trees into K_n, provided that the trees are very small compared to n.

In contrast, packing the larger trees that appear in Conjecture 1 has turned out to be a far more challenging task. Recently, Balogh and Palmer [2] proved that any family of trees T_n, T_{n-1}, $\dots$, $T_{n-\frac{1}{10}n^{1/4}}$, $v(T_i) = i$ packs into K_{n+1}.

In addition, special classes of tree families have been investigated. Progress so far mainly concerns trees that are similar to stars or paths. Already in the starting paper [12], Conjecture 1 is proven to hold in the special case when all the trees are stars and paths. Dobson [10] and Hobbs, Bourgeois and Kasiraj [16] consider packings of trees with small diameter.

Finally, we remark that it is known that the degree sequences of the trees appearing in Conjecture 1 can be matched up to fit into the complete graph: Fishburn [11] proved that if we fill up each tree T_i by adding $n-i$ isolated vertices and let $d_{i,1}, \dots, d_{i,n}$ denote the degree sequence of the resulting forest, then there are permutations $\pi_1, \dots, \pi_n$ such that $\sum_i d_{i,\pi_i(j)} = n-1$ for all $j \in [n]$.

2 Strategy and Preliminaries

In rough terms the basic idea for our proof of Theorem 3 is as follows. We use a random process to pack the trees. During this process, we keep checking that the remaining host graph (composed of the edges where no tree edge has been embedded yet) continues to satisfy certain quasirandom properties with high probability. The quasirandomness guarantees that we can carry on with our embedding as before.

In the following we will flesh out this agenda a bit further. In Section 3 we then recapitulate the main steps of the proof.

2.1 Quasirandomness

We start by recalling the concept of quasirandom graphs, which goes back to Thomason [21], and Chung, Graham, and Wilson [8].

▶ **Definition 4** (Quasirandom). We say that a graph G of order n is *α-quasirandom of density d* if for every $B \subseteq V(G)$ we have $e(B) = d\binom{|B|}{2} \pm \alpha n^2$ edges.

A well-known feature of quasirandom graphs that is particularly important for our purposes is that we can control the size of the joint neighbourhood of almost all sets of vertices of size ℓ for constant ℓ.

▶ **Lemma 5.** *For every $\beta > 0$ and every integer $\Delta \geq 1$ there is a constant $\alpha > 0$ so that in every α-quasirandom graph $G = (V, E)$ of density $d \geq \beta$ for every given set $B \subseteq V$ and any $1 \leq \ell \leq \Delta$ all but at most $\beta\binom{n}{\ell}$ sets $\{v_1, \dots, v_\ell\} \subseteq V$ have a joint neighbourhood of size $(d^\ell \pm \beta)|B|$ in B.*

For the proof of Theorem 3 it is convenient to disregard vertices contained in too many sets that are exceptional in the sense of Lemma 5. This leads to the following definition.

▶ **Definition 6** (Superquasirandom). We say that a graph $G = (V, E)$ is *(α, Δ)-superquasirandom* if for all $v \in V$, and for all $p \in [\Delta]$, we have

$$\left|\left\{S \in \tbinom{V}{p-1} : \mathrm{N}(S \cup \{v\}) \neq (1 \pm \alpha) d^p |V|\right\}\right| \leq \alpha \tbinom{|V|}{p-1} ,$$

where $\mathrm{N}(X)$ denotes the joint neighbourhood of vertices in the set X.

A consequence of Lemma 5 is that each quasirandom graph contains an almost spanning superquasirandom graph (where Δ is fixed and the parameter α is slightly worse than the original quasirandomness parameter).

2.2 Probabilistic Tools

For the analysis we shall use only two relatively standard bounds, McDiarmid's Inequality and Suen's Inequality, which we now introduce. Suppose that $\Omega = \prod_{i=1}^{k} \Omega_i$ is a product probability space. A measurable function $f : \Omega \to \mathbb{R}$ is said to be *C-Lipschitz* if for each $\omega_1 \in \Omega_1, \omega_2 \in \Omega_2, \dots, \omega_i, \omega_i' \in \Omega_i, \dots, \omega_k \in \Omega_k$ we have

$$|f(\omega_1, \omega_2, \dots, \omega_i, \dots, \omega_k) - f(\omega_1, \omega_2, \dots, \omega_i', \dots, \omega_k)| \leq C .$$

McDiarmid's Inequality (see [17]) states that Lipschitz functions are concentrated around their expectation.

▶ **Lemma 7** (McDiarmid's Inequality). *Let $f : \Omega \to \mathbb{R}$ be a C-Lipschitz function defined on a product probability space $\Omega = \prod_{i=1}^{k} \Omega_i$. Then for each $t > 0$ we have*

$$\mathbb{P}\big[|f - \mathbb{E}[f]| > t\big] \leq 2\exp\left(-\frac{2t^2}{C^2 k}\right) .$$

Next, we state Suen's inequality ([20], see also [1, p. 128]). Let $\{B_i \subseteq \Omega\}_{i\in I}$ be a finite collection of events in an arbitrary probability space Ω. A *superdependency graph* for $\{B_i\}_{i\in I}$ is an arbitrary graph on the vertex set I whose edges satisfy the following. Let $I_1, I_2 \subseteq I$ be any two disjoint sets with no edge crossing from I_1 to I_2. Then any Boolean combination of the events $\{B_i\}_{i\in I_1}$ is independent of any Boolean combination of the events $\{B_i\}_{i\in I_2}$. In this setting (and only in this setting) we write $i \sim j$ to denote that ij forms an edge. Suen's Inequality allows us to approximate $\mathbb{P}[\bigwedge \overline{B_i}]$ by $\prod \mathbb{P}[\overline{B_i}]$.

▶ **Lemma 8** (Suen's Inequality). *Using the above notation, we have*

$$\Big|\mathbb{P}\Big[\bigwedge \overline{B_i}\Big] - \prod \mathbb{P}[\overline{B_i}]\Big| \leq \prod \mathbb{P}[\overline{B_i}] \cdot \Big(\exp\big(\sum_{i\sim j} \nu_{i,j}\big) - 1\Big),$$

where $\nu_{i,j} = \frac{\mathbb{P}[B_i \wedge B_j] + \mathbb{P}[B_i]\mathbb{P}[B_j]}{\prod_{\ell \,\sim\, i \text{ or } \ell \,\sim\, j}(1-\mathbb{P}[B_\ell])}$.

2.3 Nibble Rounds

In this subsection we specify two natural ways to design a random embedding process for trees. The random embedding process we use in our proof requires some further variations, which we shall describe in the next subsection.

First, consider the following approach: successively build a packing h of the trees, edge by edge, starting with an arbitrary edge in an arbitrary tree and then following the structure of the trees. Here, when embedding an edge xy of some tree T_i, with x already embedded to $h(x)$, we choose a random neighbour $v \in V(G)$ of $h(x)$ that is not contained in the set $U_i \subseteq V(G)$ of T_i-images so far, and embed y to $h(y) := v$. After embedding xy, we remove the edge uv from G and add v to U_i. Clearly, this process produces a proper packing unless we get stuck, that is, unless the set $\mathrm{N}_G\big(h(x)\big) \setminus U_i$ gets empty. But if, during the evolution, the host graph G always remains sufficiently quasirandom, then with high probability $\mathrm{N}_G\big(h(x)\big) \setminus U_i$ should not get empty (because $e(K_{(1+\varepsilon)n}) - \sum_{i\in[k]} e(T_i) \geq \varepsilon n^2$ implies that G has positive density throughout).

It seems likely hat the host graph does indeed remain quasirandom in this process, but unfortunately graph processes like this one are rather difficult to analyse because of their dynamically evolving environment in each step. To circumvent these difficulties we have adopted a *nibble* approach by proceeding in constantly many rounds and updating the environment only after each round. This method was pioneered by Rödl [19] to prove the existence of asymptotically optimal Steiner systems (see [1] for an exposition). Since then it has served as an important ingredient for several breakthroughs in combinatorics. In our setting the nibble method could amount to the following approach for embedding $T_1, \dots, T_k$ into $G = K_{(1+\varepsilon)n}$:

- Pack the trees in r rounds (with r big but constant). For this purpose, cut each tree T_i into small equally sized forests F_i^j with $j \in [r]$. In round j embed exactly one forest of each tree T_i, i.e., the forests $F_1^j, F_2^j, \dots, F_k^j$.

- In round j, for each i construct a *random homomorphism* from the forest F_i^j to G as follows. First, randomly embed some forest vertex x, then choose a neighbour v uniformly at random in $\mathrm{N}_G(h(x)) \setminus U_i$, where the *forbidden set* $U_i \subseteq V(G)$ are vertices used by T_i in previous rounds. Then continue with the next vertex in F_i^j, following again the structure of T_i.
- After round j, delete all the edges from G to which some forest edges were mapped in this round and add to U_i all images of vertices of F_i^j.

In other words, the difference between this approach and the random process described in the beginning of this subsection, is that the host graph G and the sets U_i are not updated after the embedding of each single vertex, but only at the end of each round.

Obviously, this procedure may not produce a proper packing of the trees: Firstly, it could create *vertex collisions*, where two vertices of some tree T_i are mapped to the same vertex of the host graph G. Secondly, there could be *edge collisions*, where two edges of different trees are mapped to the same edge. But since all forests F_i^j are small, this will create only a small proportion of vertex and edge collisions in each round, and the vertex and edge deletions at the end of each round guarantee that there are no collisions between rounds. Because our host graph has $(1+\varepsilon)n$ vertices it turns out that these few collisions are easy to "repair" by reembedding vertices with the help of a simple greedy strategy (see also Section 3).

2.4 Dependencies

The difficulty with analysing the random homomorphisms described above (sometimes called *tree-indexed random walks*) is that dependencies between embedded vertices are difficult to control. Recently, Barber and Long developed techniques that allow to handle these dependencies. In particular, in [3] they show that the image of a bounded-degree tree of order $\Theta(n^2)$ in a dense quasirandom graph of order n using a random homomorphism is again a quasirandom graph with high probability. It seems likely that the techniques from [3] could be used to prove the similar but more complicated properties that form the core of our proof.

Our approach (which was developed before the techniques of Barber and Long) is different. We instead use the following construction of random homomorphisms, which we call *limping homomorphisms*, in round j of the nibble approach described above.

- For each i, call one of the colour classes of F_i^j the set P of *primary vertices*, and the other the set S of *secondary vertices*. First map all primary vertices randomly to vertices of $V(G) \setminus U_i$. Then map each secondary vertex randomly into the common $(G - U_i)$-neighbourhood of the images of its forest neighbours — unless the size of the common neighbourhood is not as expected, in which case we simply *skip* this secondary vertex.

One crucial observation is that Lemma 5 asserts that if G is quasirandom in each round, then few vertices are skipped. We shall argue that this is the case in Section 2.6.

For the arguments presented below, notice that a realization of the limping homomorphism can be represented by an element of the probability space

$$\Omega_F = V^P \times [0,1]^S \,. \tag{1}$$

Here, the $[0,1]^S$-component indicates the relative positions of the images of the secondary vertices in the list of the common neighbours of the images of the respective primary vertices.

In Section 2.5, we illustrate some basic properties of limping homomorphisms. Then, in Section 2.6 we give, as an illustration of our proof method, a short proof of the main result of [3] when tree-indexed random walks are replaced by limping homomorphisms. This result forms one main component of our proof in a simplified setting.

2.5 Limping Homomorphism on Quasirandom Graphs

In this section we show that limping homomorphisms on quasirandom graphs behave very much like tree-indexed random walks. In particular, vertices of the images are spread uniformly over the graph, and so are the edges. Moreover, there are very few skipped vertices.

▶ **Lemma 9.** *Suppose that we are given $\alpha \in (0, \frac{1}{4})$, a tree T of maximum degree at most Δ with a bipartition into primary and secondary vertices, and an (α, Δ)-superquasirandom graph $G = (V, E)$ of density d. Let h be the limping homomorphism from T to G. Let $u, v \in V$, let $x \in V(T)$ be an arbitrary primary vertex, let $y \in V(T)$ be an arbitrary secondary vertex. Then the following statements hold.*

(a) $\mathbb{P}[h(x) = v] = \frac{1}{|V|}$.

(b) $\mathbb{P}[y \text{ is skipped}] \le \alpha$.

(c) $\mathbb{P}[h(y) = v] = \frac{\left(1 \pm \alpha\left(\frac{2}{d}\right)^{\Delta}\right)^{\Delta+3}}{|V|}$.

(d) *Suppose that $xy \in E(T)$ and $uv \in E$. Then* $\mathbb{P}[h(x) = u \text{ and } h(y) = v] = \frac{\left(1 \pm \alpha\left(\frac{2}{d}\right)^{\Delta}\right)^{\Delta+2}}{d|V|^2}$.

(e) $\mathbb{P}\left[\exists z \in V(T) \setminus \{x\} \,:\, h(x) = h(z)\right] \le \frac{v(T)}{|V|}$.

(f) $\mathbb{P}\left[\exists z \in V(T) \setminus \{y\} \,:\, h(y) = h(z)\right] \le \frac{2v(T)}{d^{\Delta}|V|}$.

(g) *For the number of colliding vertices* $\mathrm{VC} = \{z \in V(T) \colon \exists z' \,:\, h(z) = h(z')\}$ *and every $t > 0$ we have* $\mathbb{P}\left[|\mathrm{VC}| \ge \frac{2v(T)^2}{d^{\Delta}|V|} + t\right] \le 2\exp\left(-\frac{t^2}{2(\Delta+1)^2 v(T)}\right)$.

As an example of the methods used for obtaining this lemma, we include the proofs of 4 and 7.

Proof of Lemma 94 and 7.

4 Let A be the event that x gets mapped to u, let B be the event that y gets mapped to v, let C be the event that y is not skipped, and let D be the event that v is in the common neighbourhood of $h(\mathrm{N}_T(y) \setminus \{x\})$. Note that $B \subseteq C \cap D$. Let $\mathcal{E}_q$ be the event that $|h(\mathrm{N}_T(y))| = q + 1$. As D and A are independent even if we condition on $\mathcal{E}_q$, we have

$$\begin{aligned}
\mathbb{P}\left[A \cap B | \mathcal{E}_q\right] &= \mathbb{P}\left[A \cap B \cap C \cap D | \mathcal{E}_q\right] \\
&= \mathbb{P}\left[A | \mathcal{E}_q\right] \cdot \mathbb{P}\left[D | A \cap \mathcal{E}_q\right] \cdot \mathbb{P}\left[C | \mathcal{E}_q \cap D \cap A\right] \cdot \mathbb{P}\left[B | \mathcal{E}_q \cap C \cap D \cap A\right] \\
&= \mathbb{P}\left[A | \mathcal{E}_q\right] \cdot \mathbb{P}\left[D | \mathcal{E}_q\right] \cdot \mathbb{P}\left[C | \mathcal{E}_q \cap D \cap A\right] \cdot \mathbb{P}\left[B | \mathcal{E}_q \cap C \cap D \cap A\right] \,. \qquad (2)
\end{aligned}$$

We have $\mathbb{P}\left[A | \mathcal{E}_q\right] = \mathbb{P}\left[A\right] = \frac{1}{|V|}$. As G is (α, Δ)-superquasirandom, $\deg(v) = (1 \pm \alpha) d|V|$. Consequently, $\mathbb{P}\left[D | \mathcal{E}_q\right] = \left((1 \pm \alpha) d\right)^q$. The number of q-sets $\{v_1, \ldots, v_q\}$ in $\mathrm{N}(v)$ with $|\mathrm{N}(v_1, \ldots, v_q, u)| \ne (1 \pm \alpha) d^{q+1} |V|$ is at most $\alpha\binom{|V|}{q}$. As $|\mathrm{N}(v)| \ge (1 - \alpha) d|V|$, the total number of $(q + 1)$-sets that contain u and have the remaining vertices in $\mathrm{N}(v)$ is at least $\binom{(1-\alpha)d|V|}{q}$. We thus get

$$1 \ge \mathbb{P}\left[C | \mathcal{E}_q \cap D \cap A\right] \ge 1 - \frac{\alpha\binom{|V|}{q}}{\binom{(1-\alpha)d|V|}{q}} \ge 1 - \alpha\left(\tfrac{2}{d}\right)^{\Delta} \,,$$

where we use $(1 - \alpha) d|V| - q \ge \frac{1}{2} d|V|$, which follows from $|V| \ge 4\Delta/d$. Finally, if y is not skipped, then $|N(h(\mathrm{N}_T(y)))| = (1 \pm \alpha) d^{q+1} |V|$, implying that

$$\mathbb{P}\left[B | \mathcal{E}_q \cap C \cap D \cap A\right] = \left((1 \pm \alpha) d^{q+1} |V|\right)^{-1} .$$

The claimed bound now follows by substituting the above estimates into (2).

7 Using the bounds from 5 and 6, we get $\mathbb{E}\left[|\mathrm{VC}|\right] \leq \frac{2v(T)^2}{d^{\Delta}|V|}$. To prove concentration, consider the product space Ω_T from (1) and view $|\mathrm{VC}|$ as a function τ from Ω_T to $\mathbb{R}$. We claim that $|\mathrm{VC}|$ is $2(\Delta+1)$-Lipschitz. Indeed, if the random real $\tau(y)$ changes for a single secondary vertex y, this only affects the embedding of y and hence changes $|\mathrm{VC}|$ by at most 2. If, on the other hand, for a single primary vertex x the random choice of $h(x)$ changes, then only the embedding of x and possibly its neighbours is affected. In this case $|\mathrm{VC}|$ changes by at most $2(\Delta+1)$. McDiarmid's Inequality (Lemma 7) implies then that

$$\mathbb{P}\left[|\mathrm{VC}| \geq \tfrac{2v(T)^2}{d^{\Delta}|V|} + t\right] \leq \mathbb{P}\Big[|\mathrm{VC}| \geq \mathbb{E}\big[|\mathrm{VC}|\big] + t\Big] \leq 2\exp\left(-\frac{2t^2}{(2(\Delta+1))^2 v(T)}\right) . \blacktriangleleft$$

2.6 Images of Limping Homomorphisms are Quasirandom

To illustrate our techniques, we prove that the image of the limping homomorphism of a bounded-degree forest of order $\Theta(n^2)$ in a quasirandom graph of order n is typically again a quasirandom graph. Even though this statement directly does not appear in our proof of Theorem 3, some more involved variants of it do. We emphasise that the extremely short range of correlations in limping homomorphisms allow for a relatively easy proof, which is in particular much shorter than the proof of the corresponding statement for tree-indexed random walks verified in [3].

Suppose we are given an (α,Δ)-quasirandom graph G of density d on n vertices, for $0 < \alpha \ll \beta \ll d$. Let F be a forest of total size γn^2 whose degrees are bounded by a constant Δ, where $\gamma \in (0,1]$ is arbitrary. Let H be its image in G under the limping homomorphism. We prove that with high probability the graph H is a β-quasirandom graph.

Let α_1 and α_2 be such that $\alpha \ll \alpha_1 \ll \alpha_2 \ll \beta$. Suppose that $f \in E(F)$ and $e \in E(G)$. Let $A_{f,e}$ be the event that f is not mapped to e. An application of Lemma 94 gives that for each $f \in E(F)$ and most edges $e \in E(G)$ we have $\mathbb{P}[A_{f,e}] = 1-(1\pm\alpha')\frac{2}{dn^2}$. Now, fix such an edge e, and build an auxiliary superdependency graph on the vertex set $E(F)$ by connecting $f_1 \in E(F)$ to $f_2 \in E(F)$ if $\mathrm{dist}(f_1,f_2) \leq 2$. This is indeed a superdependency graph for the events $\left\{A_{f,e}\right\}_{f\in E(F)}$ with respect to the limping homomorphism. (This is the first point where we benefit from working with limping homomorphisms instead of random F-indexed walks.) Suen's Inequality gives us $\mathbb{P}[\wedge_f A_{f,e}] = (1\pm\alpha_1)(1-(1\pm\alpha')\frac{2}{dn^2})^{e(F)} = (1\pm\alpha_2)\exp(-\frac{2\gamma}{d})$. In particular, this quantity does not depend on the choice of the edge e.

Therefore, for each $B \subseteq V(G)$ we have $\mathbb{E}[|E(H)\cap\binom{B}{2}|] = d\binom{|B|}{2}\cdot(1-\exp(-\frac{2\gamma}{d})) \pm \frac{\beta n^2}{2}$. Further, observe that the quantity $|E(H)\cap\binom{B}{2}|$ is Δ^2-Lipschitz. (Here again we make use of the short range of dependencies in limping walks.) Indeed, changing a position of a secondary vertex only changes the images of at most Δ edges incident to it. Changing a position of a primary vertex changes only the images of the edges at distance zero or one from that vertex. Thus, McDiarmid's Inequality gives

$$\mathbb{P}\left[\left|E(H)\cap\tbinom{B}{2}\right| \neq d\tbinom{|B|}{2}\cdot(1-\exp(-\tfrac{2\gamma}{d})) \pm \beta n^2\right] \leq 2\exp\left(-\frac{2(\frac{\beta n}{2})^2}{\Delta^4\cdot\gamma n^2}\right) = \exp(-\Theta(n^2)) .$$

In particular, we can apply the union bound over all 2^n choices of the set B, and see that with high probability for each such set we have $\left|E(H)\cap\binom{B}{2}\right| = d\binom{|B|}{2}\cdot(1-\exp(-\frac{2\gamma}{d}))\pm\beta n^2$. This proves the quasirandomness of H.

The above computation can be considered as a simplified version of a key argument in our proof, where we take G as the graph at the beginning of a nibble round j, and

$F = F_1^j \cup \ldots \cup F_k^j$. The simplification comes from the fact that we ignore the role of the forbidden sets U_i.

3 Outline of the Proof of Theorem 3

In this section we sketch how the ideas developed in Section 2 lead to a proof of Theorem 3. We follow the plan outlined in Section 2.3. That is, we cut the given bounded-degree trees $\{T_i\}_{i=1}^k$ into forests $\{F_i^1, \ldots, F_i^r\}_{i=1}^k$, where r is a fixed constant depending on ε and Δ only. We require for each $i \in [k]$ and $j \in [r]$ that the graph $F_i^1 \cup \ldots \cup F_i^j$ is a tree, and that $v(F_i^j) \approx \frac{v(F_i)}{r}$. This is possible because the trees T_i have bounded degrees.

We embed these forests into $K_{(1+\varepsilon)n}$ in r rounds. We start in round $j = 1$ with $G = K_{(1+\varepsilon)n}$. (Note that G is quasirandom.) In that round, we embed the forests $F_1^1, F_2^1, \ldots, F_k^1$ using limping homomorphisms. After the round, we update G by deleting the edges used by $F_1^1, F_2^1, \ldots, F_k^1$. Further, we create the forbidden sets $U_1, \ldots, U_k \subseteq V(K_{(1+\varepsilon)n})$ corresponding to the vertex images of $F_1^1, \ldots, F_k^1$. Using the techniques (simplified versions of which) we presented in Sections 2.5 and 2.6, we prove that, with high probability, G remains quasirandom (albeit with a worse parameter), the sets U_i are distributed in a random-like fashion over $V(K_{(1+\varepsilon)n})$, and the numbers of (vertex- and edge-) collisions and of skipped vertices are small. We then iterate this step in the next round. Throughout the whole embedding process, the graph G keeps getting sparser, but remains quasirandom and the forbidden sets U_i keep growing as further parts of the tree T_i are being added, but stay spread in a random-like way.

To take care of the vertex and edge collisions and of the skipped vertices, we set aside $\varepsilon n/2$ reserve vertices R of our original host graph $K_{(1+\varepsilon)n}$ before we actually start the embedding rounds described above. Throughout the nibble rounds, the limping homomorphisms avoid the set R. Then, at the end, a simple greedy strategy can be used to relocate vertices in collisions (and skipped vertices) to R, thus obtaining a proper packing. To make the greedy strategy work, we also need to guarantee that the collisions are well distributed over the host graph, implying that further invariants need to be controlled in the nibble rounds above.

4 Concluding Remarks

4.1 Strengthenings of Theorem 3

Theorem 3 does not hold for $\varepsilon = 0$. In [5] we construct an infinite sequence $\{\mathcal{T}_n\}_{n\in\mathcal{I}}$ of families of trees with maximum degree Δ, where $\mathcal{T}_n$ contains trees of orders n and has $\binom{n}{2}$ edges in total. Yet we show that $\mathcal{T}_n$ does not pack into K_n.

On the other hand, the following strengthening of Theorem 3 may be true: Any family of trees of orders at most n and maximum degrees at most Δ whose total number of edges is at most $\binom{n}{2}$ packs into K_{n+C_Δ}, for a suitable constant C_Δ depending on Δ only.

We are convinced that, at an expense of a more involved analysis, our techniques would allow to prove a version of Theorem 3 (for each fixed $\varepsilon > 0$) for Δ growing with n, possibly as big as $\Delta = O(\log^\alpha n)$ for some $\alpha > 0$.

Moreover, it could well be that Theorem 3 holds even for $\Delta = \frac{n}{2}$, but new techniques would be necessary for a proof. It can be shown (see [5]) that the family of $\ell := \lfloor \binom{n}{2} / ((\frac{1}{2} + 2\sqrt{\varepsilon})n) \rfloor$ copies of the star of order $(\frac{1}{2} + 2\sqrt{\varepsilon})n + 1$ does not pack in $K_{(1+\varepsilon)n}$. This shows that the $\frac{n}{2}$ barrier can essentially not be exceeded.

4.2 The Tree-packing Process

We expect that the random embedding process described in Section 2 performs well even as a dynamic process on an evolving graph. That is, we believe that the quasirandomness of the host graph is also maintained by a sequential random embedding of the trees, where we forbid the edges (globally) and vertices (just for that particular tree) immediately after they are used. This would yield another proof of Theorem 3, but we believe the analysis of this process would also be interesting in its own right.

4.3 Eliminating Dependencies

The key technical ingredient in our proof was to replace tree-indexed random walks by another process which behaves very similarly, but in which independence is regained extremely quickly. Such an approach may be useful elsewhere, in particular in the analysis of randomised algorithms.

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