Approximation Algorithms for Hypergraph Small Set Expansion and Small Set Vertex Expansion

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Abstract

The expansion of a hypergraph, a natural extension of the notion of expansion in graphs, is defined as the minimum over all cuts in the hypergraph of the ratio of the number of the hyperedges cut to the size of the smaller side of the cut. We study the Hypergraph Small Set Expansion problem, which, for a parameter $\delta \in (0, 1/2]$, asks to compute the cut having the least expansion while having at most δ fraction of the vertices on the smaller side of the cut. We present two algorithms. Our first algorithm gives an $\tilde{O}(\delta^{-1}\sqrt{\log n})$ approximation. The second algorithm finds a set with expansion $\tilde{O}(\delta^{-1}(\sqrt{d_{\max} r^{-1} \log r\, \phi^*} + \phi^*))$ in a r–uniform hypergraph with maximum degree $d_{\max}$ (where ϕ^* is the expansion of the optimal solution). Using these results, we also obtain algorithms for the Small Set Vertex Expansion problem: we get an $\tilde{O}(\delta^{-1}\sqrt{\log n})$ approximation algorithm and an algorithm that finds a set with vertex expansion $O\left(\delta^{-1}\sqrt{\phi^V \log d_{\max}} + \delta^{-1}\phi^V\right)$ (where ϕ^V is the vertex expansion of the optimal solution).

For $\delta = 1/2$, Hypergraph Small Set Expansion is equivalent to the hypergraph expansion problem. In this case, our approximation factor of $O(\sqrt{\log n})$ for expansion in hypergraphs matches the corresponding approximation factor for expansion in graphs due to Arora, Rao, and Vazirani.

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1 Introduction

The expansion of a hypergraph, a natural extension of the notion of expansion in graphs, is defined as follows.

▶ **Definition 1** (Hypergraph Expansion)**.** Given a hypergraph $H = (V, E)$ on n vertices (each edge $e \in E$ of H is a subset of vertices), we say that an edge $e \in E$ is cut by a set S if $e \cap S \neq \varnothing$ and $e \cap \bar{S} \neq \varnothing$ (i.e. some vertices in e lie in S and some vertices lie outside of S). We denote the set of edges cut by S by $E_{cut}(S)$. The expansion $\phi(S)$ of a set $S \subset V$ ($S \neq \varnothing$, $S \neq V$) in a hypergraph $H = (V, E)$ is defined as $\phi(S) = \frac{|E_{cut}(S)|}{\min(|S|,|\bar{S}|)}$.

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Hypergraph expansion and related hypergraph partitioning problems are of immense practical importance, having applications in parallel and distributed computing (Catalyurek and Aykanat [5]), VLSI circuit design and computer architecture (Karypis et al. [8]), scientific computing (Devine et al. [7]) and other areas. In spite of this, there has not been much theoretical work on them. In this paper, we study a generalization of the Hypergraph Expansion problem, namely the Hypergraph Small Set Expansion problem.

▶ **Problem 2** (Hypergraph Small Set Expansion Problem)**.** *Given a hypergraph $H = (V, E)$ and a parameter $\delta \in (0, 1/2]$, the Hypergraph Small Set Expansion problem (H-SSE) is to find a set $S \subset V$ of size at most δn that minimizes $\phi(S)$. The value of the optimal solution to H-SSE is called the small set expansion of H. That is, for $\delta \in (0, 1/2]$, the small set expansion $\phi^*_{H,\delta}$ of a hypergraph $H = (V, E)$ is defined as $\phi^*_{H,\delta} = \min_{\substack{S \subset V \\ 0<|S|\leq \delta n}} \phi(S)$.*

Note that for $\delta = 1/2$, the Hypergraph Small Set Expansion Problem is the Hypergraph Expansion Problem.

Small Set Expansion in graphs has attracted a lot of attention recently. The problem was introduced by Raghavendra and Steurer [15], who showed that it is closely related to the Unique Games problem. Raghavendra, Steurer and Tetali [16] designed an algorithm for SSE that finds a set of size $O(\delta n)$ with expansion $O(\sqrt{\phi^* d \log(1/\delta)})$ in d regular graphs (where ϕ^* is the expansion of the optimal solution). Later Bansal, Feige, Krauthgamer, Makarychev, Nagarajan, Naor, and Schwartz gave a $O(\sqrt{\log n \log(1/\delta)})$ approximation algorithm for the problem.

We present analogs of the results of Bansal et al. [4] and Raghavendra, Steurer and Tetali [16] for hypergraphs. Our first result is an $\tilde{O}(\delta^{-1}\sqrt{\log n})$ approximation algorithm[1] for H-SSE (see Theorem 3). Our second result is an algorithm that finds a set with expansion at most $\tilde{O}\left(\delta^{-1}\left(\sqrt{d_{\max}\frac{\log r}{r}\phi^*_{H,\delta}} + \phi^*_{H,\delta}\right)\right)$ if H is an r–uniform hypergraph with maximum degree $d_{\max}$ (see Theorem 4; the result also applies to non-uniform hypergraphs, see Theorem 21).

We note that H-SSE can be reduced to SSE (small set expansion in graphs) if all hyperedges have bounded size. Let r be the size of the largest hyperedge in H. Construct an auxiliary (weighted) graph F on V as follows: pick a vertex in each hyperedge e and connect it in F to all other vertices of e (i.e. replace e with a star); let the weight of an edge f in F be the total weight of the hyperedges $e \in E$ for which f is part of the representative star of e. Then solve SSE in the graph F. It is easy to see that if we solve SSE using an α approximation algorithm, then we get $(r-1)\alpha$ approximation for H-SSE. This approach gives $O(\sqrt{\log n \log(1/\delta)})$ approximation if r is bounded. However, if H is an arbitrary hypergraph, we only get an $O(n\sqrt{\log n \log(1/\delta)})$ approximation. The goal of this paper is to give an approximation guarantee valid for hypergraphs with hyperedges of arbitrary size. We now formally state our main results.

▶ **Theorem 3.** *There is a randomized polynomial-time approximation algorithm for the Hypergraph Small Set Expansion problem that given a hypergraph $H = (V, E)$, and parameters $\varepsilon \in (0, 1)$ and $\delta \in (0, 1/2)$, finds a set $S \subset V$ of size at most $(1 + \varepsilon)\delta n$ such that*

$$\phi(S) \leq O_\varepsilon\left(\delta^{-1} \log \delta^{-1} \log\log \delta^{-1} \cdot \sqrt{\log n} \cdot \phi^*_{H,\delta}\right) = \tilde{O}_\varepsilon\left(\delta^{-1}\sqrt{\log n}\, \phi^*_{H,\delta}\right),$$

(where the constant in the O notation depends polynomially on $1/\varepsilon$). That is, the algorithm gives $O(\sqrt{\log n})$ approximation when δ and ε are fixed.

[1] The $\tilde{O}$–notation hides a $\log \delta^{-1} \log\log \delta^{-1}$ term.

We state our second result, Theorem 4, for r-uniform hypergraphs. We present and prove a more general Theorem 21 that applies to any hypergraphs in Section 6.

▶ **Theorem 4.** *There is a randomized polynomial-time algorithm for the Hypergraph Small Set Expansion problem that given an r–uniform hypergraph $H = (V, E)$ with maximum degree d_{max}, and parameters $\varepsilon \in (0, 1)$ and $\delta \in (0, 1/2)$ finds a set $S \subset V$ of size at most $(1 + \varepsilon)\delta n$ such that*

$$\phi(S) \leq \tilde{O}_\varepsilon \left(\delta^{-1} \left(\sqrt{d_{max} \frac{\log r}{r} \phi^*_{H,\delta}} + \phi^*_{H,\delta} \right) \right).$$

Our algorithms for H-SSE are bi-criteria approximation algorithms in that they output a set S of size at most $(1 + \varepsilon)\delta n$. We note that this is similar to the algorithm of Bansal et al. [4] for SSE, which also finds a set of size at most $(1 + \varepsilon)\delta n$ rather than a set of size at most δn. The algorithm of Raghavendra, Steurer and Tetali [16] finds a set of size $O(\delta n)$. The approximation factor of our first algorithm does not depend on the size of hyperedges in the input hypergraph. It has the same dependence on n as the algorithm of Bansal et al. [4] for SSE. However, the dependence on $1/\delta$ is quasi-linear; whereas it is logarithmic in the algorithm of Bansal et al. [4]. In fact, we show that the integrality gap of the standard SDP relaxation for H-SSE is at least linear in $1/\delta$ (Theorem 22). The approximation guarantee of our second algorithm is analogous to that of the algorithm of Raghavendra, Steurer and Tetali [16].

Small Set Vertex Expansion

Our techniques can also be used to obtain an approximation algorithm for Small Set Vertex Expansion (SSVE) in graphs.

▶ **Problem 5** (Small Set Vertex Expansion Problem)**.** *Given graph $G = (V, E)$, the vertex expansion of a set $S \subset V$ is defined as*

$$\phi^V(S) = \frac{|\{u \in \bar{S} : \exists\, v \in S \text{ such that } \{u, v\} \in E\}|}{|S|}$$

Given a parameter $\delta \in (0, 1/2]$, the Small Set Vertex Expansion problem (SSVE) is to find a set $S \subset V$ of size at most δn that minimizes $\phi^V(S)$. The value of the optimal solution to SSVE is called the small set vertex expansion of G. That is, for $\delta \in (0, 1/2]$, the small set expansion $\phi^V_{G,\delta}$ of a graph $G = (V, E)$ is defined as

$$\phi^V_{G,\delta} = \min_{\substack{S \subset V \\ 0 < |S| \leq \delta n}} \phi^V(S).$$

Small Set Vertex Expansion recently gained interest due to its connection to obtaining sub-exponential-time, constant factor approximation algorithms for many combinatorial problems like Sparsest Cut and Graph Coloring [1, 12]. Using a reduction from vertex expansion in graphs to hypergraph expansion, we can get an approximation algorithm for SSVE having the same approximation guarantee as that for H-SSE.

▶ **Theorem 6.** *There exist absolute constants $c_1, c_2 \in \mathbb{R}^+$ such that for every graph $G = (V, E)$, there exists a polynomial time computable hypergraph $H = (V', E')$ such that $c_1\phi^*_{H,\delta} \leq \phi^V_{G,\delta} \leq c_2\phi^*_{H,\delta}$. Also, $\eta^H_{max} \leq \log_2(d_{max} + 1)$, where d_{max} is the maximum degree of G (where η^H_{max} is defined in Definition 20).*

From this theorem, Theorem 3 and Theorem 21 we immediately get algorithms for SSVE.

▶ **Theorem 7** (Corollary to Theorem 3 and Theorem 6). *There is a randomized polynomial-time approximation algorithm for the Small Set Vertex Expansion problem that given a graph $G = (V, E)$, and parameters $\varepsilon \in (0, 1)$ and $\delta \in (0, 1/2))$ finds a set $S \subset V$ of size at most $(1 + \varepsilon)\delta n$ such that*

$$\phi^V(S) \leq O_\varepsilon\left(\sqrt{\log n}\,\delta^{-1}\log\delta^{-1}\log\log\delta^{-1}\cdot\phi^V_{G,\delta}\right),$$

That is, the algorithm gives $O(\sqrt{\log n})$ approximation when δ and ε are fixed.

▶ **Theorem 8** (Corollary to Theorem 21 and Theorem 6). *There is a randomized polynomial-time algorithm for the Small Set Vertex Expansion problem that given a graph $G = (V, E)$ of maximum degree d_{max}, parameters $\varepsilon \in (0, 1)$ and $\delta \in (0, 1/2)$ finds a set $S \subset V$ of size at most $(1 + \varepsilon)\delta n$ such that*

$$\begin{aligned}\phi^V(S) &\leq O_\varepsilon\left(\sqrt{\phi^V_{G,\delta}\log d_{max}}\cdot\delta^{-1}\log\delta^{-1}\log\log\delta^{-1} + \delta^{-1}\phi^V_{G,\delta}\right)\\ &= \tilde{O}_\varepsilon\left(\delta^{-1}\sqrt{\phi^V_{G,\delta}\log d_{max}} + \delta^{-1}\phi^V_{G,\delta}\right).\end{aligned}$$

We note that the Small Set Vertex Expansion problem for $\delta = 1/2$ is just the Vertex Expansion problem. In that case, Theorem 8 gives the same approximation guarantee as the algorithm of Louis, Raghavendra and Vempala [13].

Techniques. Our general approach to solving H-SSE is similar to the approach of Bansal et al. [4]. We recall how the algorithm of Bansal et al. [4] for (graph) SSE works. The algorithm solves a semidefinite programming relaxation for SSE and gets an SDP solution. The SDP solution assigns a vector $\bar{u}$ to each vertex u. Then the algorithm generates *an orthogonal separator*. Informally, an orthogonal separator S with distortion D is a random subset of vertices such that

- **(a)** If $\bar{u}$ and $\bar{v}$ are close to each other then the probability that u and v are separated by S is small; namely, it is at most $\alpha D\|\bar{u} - \bar{v}\|^2$, where α is a normalization factor such that $\Pr(u \in S) = \alpha\|\bar{u}\|^2$.
- **(b)** If the angle between $\bar{u}$ and $\bar{v}$ is larger than a certain threshold, then the probability that both u and v are in S is much smaller than the probability that one of them is in S.

Bansal et al. [4] showed that condition (b) together with SDP constraints implies that S is of size at most $(1 + \varepsilon)\delta n$ with sufficiently high probability. Then condition (a) implies that the expected number of cut edges is at most D times the SDP value. That means that S is a D–approximate solution to SSE.

If we run this algorithm on an instance of H-SSE, we will still find a set of size at most $(1 + \varepsilon)\delta n$, but the cost of the solution might be very high. Indeed, consider a hyperedge e. Even though every two vertices u and v in e are *unlikely* to be separated by S, at least one pair out of $\binom{|e|}{2}$ pairs of vertices is quite likely to be separated by S; hence, e is quite likely to be cut by S. To deal with this problem, we develop *hypergraph orthogonal separators*. In the definition of a hypergraph orthogonal separator, we strengthen condition (a) by requiring that a hyperedge e is cut by S with small probability if all vertices in e are close to each other. Specifically, we require that

$$\Pr(e \text{ is cut by } S) \leq \alpha D \max_{u,v\in e}\|\bar{u} - \bar{v}\|^2. \tag{1}$$

We show that there is a hypergraph orthogonal separator with distortion proportional to $\sqrt{\log n}$ (the distortion also depends on parameters of the orthogonal separator). Plugging this hypergraph orthogonal separator in the algortihm of Bansal et al. [4], we get Theorem 3. We also develop another variant of hypergraph orthogonal separators, ℓ_2–ℓ_2^2 orthogonal separators. An ℓ_2–ℓ_2^2 orthogonal separator with ℓ_2–distortion $D_{\ell_2}(r)$ and ℓ_2^2–distortion $D_{\ell_2^2}$ satisfies the following condition[2]

$$\Pr(e \text{ is cut by } S) \le \alpha D_{\ell_2}(|e|) \cdot \min_{w\in E} \|\bar{w}\| \cdot \max_{u,v\in e} \|\bar{u}-\bar{v}\| + \alpha D_{\ell_2^2} \cdot \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2. \tag{2}$$

We show that there is an ℓ_2-ℓ_2^2 hypergraph orthogonal separator whose ℓ_2 and ℓ_2^2 distortions do not depend on n (in contrast, there is no hypergraph orthogonal separator whose distortion does not depend on n). This result yields Theorem 4.

We now give a brief conceptual overview of our construction of hypergraph orthogonal separators. We use the framework developed by Chlamtac, Makarychev, and Makarychev in [6, Section 4.3] for (graph) orthogonal separators. For simplicity, we ignore vector normalization steps in this overview; we do not explain how we take into account vector lengths. Note, however, that these normalization steps are crucial. We first design a procedure that partitions the hypergraph into two pieces (the procedure labels every vertex with either 0 or 1). In a sense, each set S in the partition is a "very weak" hypergraph orthogonal separator. It satisfies property (1) with $D_0 \sim \sqrt{\log n}\log\log(1/\delta)$ and $\alpha_0 = 1/2$ and a weak variant of property (b): if the angle between vectors $\bar{u}$ and $\bar{v}$ is larger than the threshold then events $u \in S$ and $v \in S$ are "almost" independent. We repeat the procedure $l = \log_2(1/\delta) + O(1)$ times and obtain a partition of graph into $2^l = O(1/\delta)$ pieces. Then we randomly choose one set S among them; this set S is our hypergraph orthogonal separator. Note that by running the procedure many times we decrease exponentially in l the probability that two vertices, as in condition (b), belong to S. So condition (b) holds for S. Also, we affect the distortion in (1) in two ways. First, the probability that the edge is cut increases by a factor of l. That is, we get $\Pr(e \text{ is cut by } S) \le l \times \alpha_0 D_0 \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2$. Second, the probability that we choose a vertex u goes down from $\|\bar{u}\|^2/2$ to $\Omega(\delta)\|\bar{u}\|^2$ since roughly speaking we choose one set S among $O(1/\delta)$ possible sets. That is, the parameter α of S is $\Omega(\delta)$. Therefore, $\Pr(e \text{ is cut by } S) \le \alpha(\alpha_0 l D_0/\alpha)\max_{u,v\in e} \|\bar{u}-\bar{v}\|^2$. That is, we get a hypergraph orthogonal separator with distortion $(\alpha_0 l D_0/\alpha) \sim \tilde{O}(\delta^{-1}\sqrt{\log n})$. The construction of ℓ_2–ℓ_2^2 orthogonal separators is similar but a bit more technical.

Organization. We present our SDP relaxation and introduce our main technique, hypergraph orthogonal separators, in Section 2. We describe our first algorithm for H-SSE in Section 3, and then describe an algorithm that generates hypergraph orthogonal separators in Section 4. We define ℓ_2–ℓ_2^2 hypergraph orthogonal separators, give an algorithm that generates them, and then present our second algorithm for H-SSE in Section 5 and Section 6. Finally, we show a simple SDP integrality gap for H-SSE in Section 7. This integrality gap also gives a lower bound on the quality of m-orthogonal separators. We give a proof of Theorem 6 in Section 8.

[2] It may look strange that we have two terms in the bound. One may expect that we can either have only term $D_{\ell_2^2}\max_{u,v\in e}\|\bar{u}-\bar{v}\|^2$ (as in the previous definition) or only term $D_{\ell_2}(|e|)\cdot\min_{w\in E}\|\bar{w}\|\cdot\max_{u,v\in e}\|\bar{u}-\bar{v}\|$. However, the latter is not possible — there is no ℓ_2–ℓ_2^2 separator with $D_{\ell_2^2}=0$.

$$\text{minimize} \sum_{e\in E} \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2 \tag{3}$$

subject to:

$$\sum_{v\in V} \langle \bar{u},\bar{v}\rangle \le \delta n \cdot \|\bar{u}\|^2 \qquad \text{for every } u\in V \tag{4}$$

$$\sum_{u\in V} \|\bar{u}\|^2 = 1 \tag{5}$$

$$\|\bar{u}-\bar{v}\|^2 + \|\bar{v}-\bar{w}\|^2 \ge \|\bar{u}-\bar{w}\|^2 \qquad \text{for every } u,v,w\in V \tag{6}$$

$$0 \le \langle \bar{u},\bar{v}\rangle \le \|u\|^2 \qquad \text{for every } u,v\in V. \tag{7}$$

Figure 1 SDP relaxation for H-SSE.

2 Preliminaries

2.1 SDP Relaxation for Hypergraph Small Set Expansion

We use the SDP relaxation for H-SSE shown in Figure 1. There is an SDP variable $\bar{u}$ for every vertex $u \in V$.

Every combinatorial solution S (with $|S| \le \delta n$) defines the corresponding (intended) SDP solution: $\bar{u} = \frac{e}{\sqrt{|S|}}$, if $u \in S$; $\bar{u} = 0$, otherwise, where e is a fixed unit vector. It is easy to see that this solution satisfies all SDP constraints. Note that $\max_{u,v\in e} \|\bar{u}-\bar{v}\|^2$ is equal to $1/|S|$, if e is cut, and to 0, otherwise. Therefore, the objective function equals

$$\sum_{e\in E} \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2 = \sum_{e\in E_{cut}(S)} \frac{1}{|S|} = \frac{E_{cut}(S)}{S} = \phi(S).$$

Thus our SDP for H-SSE is indeed a relaxation.

2.2 Hypergraph Orthogonal Separators

The main technical tool for proving Theorem 3 is *hypergraph orthogonal separators*. *Orthogonal separators* were introduced by Chlamtac, Makarychev, and Makarychev [6] (see also Bansal et al. [4], Louis and Makarychev [9], and Makarychev and Makarychev [14]) and were previously used for solving Unique Games and various graph partitioning problems. In this paper, we extend the technique of orthogonal separators to hypergraphs and introduce hypergraph orthogonal separators. We then use hypergraph orthogonal separators to solve H-SSE. In Section 5, we introduce another version of hypergraph orthogonal separators, ℓ_2–ℓ_2^2 hypergraph orthogonal separators, and then use them to prove Theorem 4 and Theorem 21.

▶ **Definition 9** (Hypergraph Orthogonal Separators). Let $\{\bar{u} : u \in V\}$ be a set of vectors in the unit ball that satisfy ℓ_2^2–triangle inequalities (6) and (7). We say that a random set $S \subset V$ is *a hypergraph m-orthogonal separator* with distortion $D \ge 1$, probability scale $\alpha > 0$, and separation threshold $\beta \in (0,1)$ if it satisfies the following properties.

1. For every $u \in V$, $\Pr(u \in S) = \alpha\|\bar{u}\|^2$.
2. For every u and v such that $\|\bar{u}-\bar{v}\|^2 \ge \beta \min(\|\bar{u}\|^2, \|\bar{v}\|^2)$

$$\Pr\left(u \in S \text{ and } v \in S\right) \le \alpha \frac{\min(\|\bar{u}\|^2, \|\bar{v}\|^2)}{m}.$$

3. For every $e \subset V$, $\Pr\left(e \text{ is cut by } S\right) \le \alpha D \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2$.

The definition of a hypergraph m-orthogonal separator is similar to that of a (graph) m-orthogonal separator: a random set S is an m-orthogonal separator if it satisfies properties 1, 2, and property 3′, which is property 3 restricted to edges e of size 2.

3′. For every (u,v), $\Pr(e \text{ is cut by } S) \le \alpha D\|\bar{u}-\bar{v}\|^2$.

In this paper, we design an algorithm that generates a hypergraph m-orthogonal separator with distortion $O_\beta(\sqrt{\log n}\cdot m\log m\log\log m)$. We note that the distortion of *any* hypergraph orthogonal separator must depend on m at least linearly (see Section 7). We remark that there are two constructions of (graph) orthogonal separators, "orthogonal separators via ℓ_1" and "orthogonal separators via ℓ_2", with distortions, $O_\beta(\sqrt{\log n}\log m)$ and $O_\beta(\sqrt{\log n\log m})$, respectively (presented in [6]). Our construction of hypergraph orthogonal separators uses the framework of orthogonal separators via ℓ_1. We prove the following theorem in Section 4.

▶ **Theorem 10.** *There is a polynomial-time randomized algorithm that given a set of vertices V, a set of vectors $\{\bar{u}\}$ satisfying ℓ_2^2–triangle inequalities (6) and (7), parameters $m \ge 2$ and $\beta \in (0,1)$, generates a hypergraph m-orthogonal separator with probability scale $\alpha \ge 1/n$ and distortion $D = O(\beta^{-1} m\log m\log\log m \times \sqrt{\log n})$.*

3 Algorithm for Hypergraph Small Set Expansion

In this section, we present our algorithm for Hypergraph Small Set Expansion. Our algorithm uses hypergraph orthogonal separators that we describe in Section 4. We use the approach of Bansal et al. [4]. Suppose that we are given a polynomial-time algorithm that generates hypergraph m-orthogonal separators with distortion $D(m,\beta)$ (with probability scale $\alpha > 1/\operatorname{poly}(n)$). We show how to get a $D^* = 4D(4/(\varepsilon\delta),\varepsilon/4)$ approximation for H-SSE.

▶ **Theorem 11.** *There is a randomized polynomial-time approximation algorithm for the Hypergraph Small Set Expansion problem that given a hypergraph $H=(V,E)$, and parameters $\varepsilon \in (0,1)$ and $\delta \in (0,1/2)$ finds a set $S \subset V$ of size at most $(1+\varepsilon)\delta n$ such that $\phi(S) \le 4D(4/(\varepsilon\delta),\varepsilon/4)\cdot\phi^*_{H,\delta}$.*

Proof. We solve the SDP relaxation for H-SSE and obtain an SDP solution $\{\bar{u}\}$. Denote the SDP value by sdp-cost. Consider a hypergraph orthogonal separator S with $m = 4/(\varepsilon\delta)$ and $\beta = \varepsilon/4$. Define a set S':

$$S' = \begin{cases} S, & \text{if } |S| \le (1+\varepsilon)\delta n,\\ \varnothing, & \text{otherwise.}\end{cases}$$

Clearly, $|S'| \le (1+\varepsilon)\delta n$. Bansal et al. [4] showed that $\Pr(u\in S') \in \left[\frac{\alpha}{2}\|\bar{u}\|^2, \alpha\|\bar{u}\|^2\right]$ for every $u \in V$ (see also Theorem A.1 in [14]). Note that

$$\Pr(S' \text{ cuts edge } e) \le \Pr(S \text{ cuts edge } e) \le \alpha D^* \max_{u,v\in e}\|\bar{u}-\bar{v}\|^2.$$

where D^* denotes $D(4/(\varepsilon\delta),\varepsilon/4)$ for the sake of brevity. Let $Z = |S'| - \frac{|E_{cut}(S')|}{4D^*\cdot\text{sdp-cost}}$. We have,

$$\mathbb{E}[Z] = \mathbb{E}[|S'|] - \frac{\mathbb{E}[|E_{cut}(S')|]}{4D^*\cdot\text{sdp-cost}} \ge \sum_{u\in V}\frac{\alpha}{2}\cdot\|\bar{u}\|^2 - \frac{\sum_{e\in E}\alpha D^*\max_{u,v\in e}\|\bar{u}-\bar{v}\|^2}{4D^*\cdot\text{sdp-cost}}$$
$$= \frac{\alpha}{2} - \frac{1}{4D^*\cdot\text{sdp-cost}}\times\alpha D^*\,\text{sdp-cost} = \frac{\alpha}{4}.$$

Since $Z \le |S'| \le (1+\varepsilon)\delta n < n$ (always), by Markov's inequality, we have $\Pr(Z > 0) \ge \alpha/(4n)$ and hence $\Pr(|E_{cut}(S')|/|S'| < 4D^* \cdot \text{sdp-cost}) \ge \alpha/(4n)$.

We sample S independently $4n/\alpha$ times and return the first set S' such that $\frac{|E_{cut}(S')|}{|S'|} < 4D^* \cdot \text{sdp-cost}$. This gives a set S' such that $|S'| \le (1+\varepsilon)\delta n$, and $\phi(S') \le 4D^*\phi^*_{H,\delta}$. The algorithm succeeds (finds such a set S') with a constant probability. By repeating the algorithm n times, we can make the success probability exponentially close to 1. ◀

In Section 4, we describe how to generate an m-hypergraph orthogonal separator with distortion $D = O\big(\sqrt{\log n} \times \beta^{-1} m \log m \log\log m\big)$. That gives us an algorithm for H-SSE with approximation factor $O_\varepsilon\left(\delta^{-1}\log\delta^{-1}\log\log\delta^{-1} \times \sqrt{\log n}\right)$.

4 Generating Hypergraph Orthogonal Separators

In this section, we present an algorithm that generates a hypergraph m-orthogonal separator. At the high level, the algorithm is similar to the algorithm for generating orthogonal separators from Section 4.3 in [6]. We use a different procedure for generating words $W(u)$ (see below) and set parameters differently; also the analysis of our algorithm is different.

In our algorithm, we use a "normalization" map φ from [6]. Map φ maps a set $\{\bar{u}\}$ of vectors satisfying ℓ_2^2–triangle inequalities (6) and (7) to $\mathbb{R}^n$. It has the following properties.

1. For all vertices u, v, w, $\|\varphi(\bar{u}) - \varphi(\bar{v})\|_2^2 + \|\varphi(\bar{v}) - \varphi(\bar{w})\|_2^2 \ge \|\varphi(\bar{u}) - \varphi(\bar{w})\|_2^2$.
2. For all nonzero vertices u and v, $\langle\varphi(\bar{u}), \varphi(\bar{v})\rangle = \frac{\langle\bar{u},\bar{v}\rangle}{\max(\|\bar{u}\|^2, \|\bar{v}\|^2)}$.
3. In particular, for every $\bar{u} \ne 0$, $\|\varphi(\bar{u})\|_2^2 = \langle\varphi(\bar{u}), \varphi(\bar{u})\rangle = 1$. Also, $\varphi(0) = 0$.
4. For all non-zero vectors $\bar{u}$ and $\bar{v}$, $\|\varphi(\bar{u}) - \varphi(\bar{v})\|_2^2 \le \frac{2\,\|\bar{u}-\bar{v}\|^2}{\max(\|\bar{u}\|^2, \|\bar{v}\|^2)}$.

We also use the following theorem of Arora, Lee, and Naor [2] (see also [3]).

▶ **Theorem 12** (Arora, Lee, and Naor (2005), Theorem 3.1). *There exist constants $C \ge 1$ and $p \in (0, 1/4)$ such that for every n unit vectors x_u ($u \in V$), satisfying ℓ_2^2–triangle inequalities (6), and every $\Delta > 0$, the following holds. There exists a random subset U of V such that for every $u, v \in V$ with $\|x_u - x_v\|^2 \ge \Delta$, $\Pr\left(u \in U \text{ and } d(v,U) \ge \frac{\Delta}{C\sqrt{\log n}}\right) \ge p$, where $d(v,U) = \min_{u\in U}\|x_u - x_v\|^2$.*

First we describe an algorithm that randomly assigns each vertex u a symbol, either 0 or 1. Then we use this algorithm to generate an orthogonal separator.

▶ **Lemma 13.** *There is a randomized polynomial-time algorithm that given a finite set V, unit vectors $\varphi(\bar{u})$ for $u \in V$ satisfying ℓ_2^2-triangle inequalities and a parameter $\beta \in (0,1)$, returns a random assignment $\omega : V \to \{0,1\}$ that satisfies the following properties.*

- *For every u and v such that $\|\varphi(\bar{u}) - \varphi(\bar{v})\|^2 \ge \beta$, we have $\Pr(\omega(u) \ne \omega(v)) \ge 2p$, where $p > 0$ is the constant from Theorem 12.*
- *For every set $e \subset V$ of size at least 2,*

$$\Pr(\omega(u) \ne \omega(v) \text{ for some } u, v \in e) \le O(\beta^{-1}\sqrt{\log n}\max_{u,v\in e}\|\varphi(\bar{u}) - \varphi(\bar{v})\|^2).$$

Proof. Let U be the random set from Theorem 12 for vectors $x_u = \varphi(\bar{u})$ and $\Delta = \beta$. Choose $t \in (0, 1/(C\sqrt{\log n}))$ uniformly at random. Let

$$\omega(u) = \begin{cases} 0, & \text{if } d(U_i, u) \le t, \\ 1, & \text{otherwise.} \end{cases}$$

Consider first vertices u and v such that $\|\varphi(\bar{u}) - \varphi(\bar{v})\|^2 \geq \beta$. By Theorem 12,

$$\Pr\left(u \in U \text{ and } d(v,U) \geq \frac{\Delta}{C\sqrt{\log n}}\right) \geq p \quad \text{and} \quad \Pr\left(v \in U \text{ and } d(u,U) \geq \frac{\Delta}{C\sqrt{\log n}}\right) \geq p.$$

Note that in the former case, when $u \in U$ and $d(v,U) \geq \frac{\Delta}{C\sqrt{\log n}}$, we have $\omega(u) = 0$ and $\omega(v) = 1$; in the latter case, when $v \in U$ and $d(u,U) \geq \frac{\Delta}{C\sqrt{\log n}}$, we have $\omega(v) = 0$ and $\omega(u) = 1$. Therefore, the probability that $\omega(u) \neq \omega(v)$ is at least $2p$.

Now consider a set $e \subset V$ of size at least 2. Let $\tau_m = \min_{w\in e} d(U, \varphi(\bar{w}))$ and $\tau_M = \max_{w\in e} d(U, \varphi(\bar{w}))$. We have, $\tau_M - \tau_m \leq \max_{u,v\in e} \|\varphi(\bar{u}) - \varphi(\bar{v})\|^2$. Note that if $t < \tau_m$ then $\omega(u) = 1$ for all $u \in e$; if $t \geq \tau_M$ then $\omega(u) = 0$ for all $u \in e$. Thus $\omega(u) \neq \omega(v)$ for some $u, v \in e$ only if $t \in [\tau_m, \tau_M)$. Since the probability density of the random variable t is at most $C\sqrt{\log n}$, we get,

$$\Pr\left(\exists\, u, v \in e : \omega(u) \neq \omega(v)\right) \leq \Pr\left(t \in [\tau_m, \tau_M)\right) \leq \frac{C\sqrt{\log n}}{\beta} \max_{u,v\in e} \|\bar{u} - \bar{v}\|^2.$$

◀

We now amplify the result of Lemma 13.

▶ **Lemma 14.** *There is a randomized polynomial time algorithm that given V, vectors $\varphi(\bar{u})$ and $\beta \in (0,1)$ as in Lemma 13, and a parameter $m \geq 2$, returns a random assignment $\tilde{\omega} : V \to \{0,1\}$ such that:*

- *For every u and v such that $\|\varphi(\bar{u}) - \varphi(\bar{v})\|^2 \geq \beta$, $\Pr\left(\tilde{\omega}(u) \neq \tilde{\omega}(v)\right) \geq 1/2 - 1/\log_2 m$.*
- *For every set $e \subset V$ of size at least 2,*

$$\Pr\left(\tilde{\omega}(u) \neq \tilde{\omega}(v) \text{ for some } u, v \in e\right) \leq O(\beta^{-1}\sqrt{\log n} \cdot \log\log m \cdot \max_{u,v\in e} \|\varphi(\bar{u}) - \varphi(\bar{v})\|^2).$$

We independently sample $K = \max\left(\left\lceil \frac{\log_2 \log_2 m}{-\log_2(1-4p)} \right\rceil, 1\right)$ assignments $\omega_1, \ldots, \omega_K$, and let $\tilde{\omega}(u) = \omega_1(u) \oplus \cdots \oplus \omega_K(u)$ (where $\oplus$ denotes addition modulo 2). It is easy to see that the assignment $\tilde{\omega}$ satisfies the required properties. We defer the proof to the full version of the paper [10, Section E].

We are now ready to present our algorithm.

1. Set $l = \lceil \log_2 m/(1 - \log_2(1 + 2/\log_2 m))\rceil = \log_2 m + O(1)$.
2. Sample l independent assignments $\tilde{\omega}_1, \ldots, \tilde{\omega}_l$ using Lemma 14.
3. For every vertex u, define word $W(u) = \tilde{\omega}_1(u) \ldots \tilde{\omega}_l(u) \in \{0,1\}^l$.
4. If $n \geq 2^l$, pick a word $W \in \{0,1\}^l$ uniformly at random. If $n < 2^l$, pick a random word $W \in \{0,1\}^l$ so that $\Pr_W(W = W(u)) = 1/n$ for every $u \in V$. This is possible since the number of distinct words constructed in step 3 is at most n (we may pick a word W not equal to any $W(u)$).
5. Pick $r \in (0,1)$ uniformly at random.
6. Let $S = \left\{u \in V : \|\bar{u}\|^2 \geq r \text{ and } W(u) = W\right\}$.

▶ **Theorem 15.** *Random set S is a hypergraph m-orthogonal separator with distortion $D = O\big(\sqrt{\log n} \times \frac{m \log m \log\log m}{\beta}\big)$, probability scale $\alpha \geq 1/n$ and separation threshold β.*

Proof. We verify that S satisfies properties 1–3 in the definition of a hypergraph m-orthogonal separator with $\alpha = \max(1/2^l, 1/n)$.

Property 1. We compute the probability that $u \in S$. Observe that $u \in S$ if and only if $W(u) = W$ and $r \le \|\bar{u}\|^2$ (these two events are independent). If $n \ge 2^l$, the probability that $W = W(u)$ is $1/2^l$ since we choose W uniformly at random from $\{0,1\}^l$; if $n < 2^l$ the probability is $1/n$. That is, $\Pr(W = W(u)) = \max(1/2^l, 1/n) = \alpha$. The probability that $r \le \|\bar{u}\|^2$ is $\|\bar{u}\|^2$. We conclude that property 1 holds.

Property 2. Consider two vertices u and v such that $\|\bar{u}-\bar{v}\|^2 \ge \beta \min(\|\bar{u}\|^2, \|\bar{v}\|^2)$. Assume without loss of generality that $\|\bar{u}\|^2 \le \|\bar{v}\|^2$. Note that $u, v \in S$ if and only if $r \le \|\bar{u}\|^2$ and $W = W(u) = W(v)$. We first upper bound the probability that $W(u) = W(v)$. We have, $2\langle \bar{u}, \bar{v}\rangle = \|\bar{u}\|^2 + \|\bar{v}\|^2 - \|\bar{u} - \bar{v}\|^2 \le (1-\beta)\|\bar{u}\|^2 + \|\bar{v}\|^2 \le (2-\beta)\|\bar{v}\|^2$. Therefore, $2\langle \bar{u}, \bar{v}\rangle / \|\bar{v}\|^2 \le 2-\beta$. Hence, $\|\varphi(\bar{u}) - \varphi(\bar{v})\|^2 = 2 - 2\langle \varphi(\bar{u}), \varphi(\bar{v})\rangle = 2 - \frac{2\langle \bar{u}, \bar{v}\rangle}{\max(\|\bar{u}\|^2, \|\bar{v}\|^2)} \ge \beta = \Delta$. From Lemma 14 we get that $\Pr(\tilde{\omega}_i(u) \ne \tilde{\omega}_i(v)) \ge \frac{1}{2} - \frac{1}{\log_2 m}$ for every i. The probability that $W(u) = W(v)$ is at most $(\frac{1}{2} + \frac{1}{\log_2 m})^l \le 1/m$. Therefore we have as required,

$$\begin{aligned}\Pr(u \in S, v \in S) = \Pr\left(r \le \min(\|\bar{u}\|^2, \|\bar{v}\|^2)\right) \times \Pr(W = W(u) = W(v) \mid W(u) = W(v)) \\ \times \Pr(W(u) = W(v)) \le \min(\|\bar{u}\|^2, \|\bar{v}\|^2) \times \alpha \times (1/m).\end{aligned}$$

Property 3. Let e be an arbitrary subset of V, $|e| \ge 2$. Let $\rho_m = \min_{w \in e} \|\bar{w}\|^2$ and $\rho_M = \max_{w \in e} \|\bar{w}\|^2$. Note that $\rho_M - \rho_m = \|\bar{w}_1\|^2 - \|\bar{w}_2\|^2 \le \|\bar{w}_1 - \bar{w}_2\|^2 \le \max_{u,v \in e} \|\bar{u} - \bar{v}\|^2$, for some $w_1, w_2 \in e$. Here we used that SDP constraint (7) implies that $\|\bar{w}_1\|^2 - \|\bar{w}_2\|^2 \le \|\bar{w}_1 - \bar{w}_2\|^2$.

Let $A = \{u \in e : \|\bar{u}\|^2 \ge r\}$. Note that $S \cap e = \{u \in A : W(u) = W\}$. Therefore, if e is cut by S then one of the following events happens.

- Event $\mathcal{E}_1$: $A \ne e$ and $S \cap e \ne \varnothing$.
- Event $\mathcal{E}_2$: $A = e$ and $A \cap S \ne \varnothing$, $A \cap S \ne A$.

If $\mathcal{E}_1$ happens then $r \in [\rho_m, \rho_M]$ since $A \ne e$ and $A \ne \varnothing$. We have,

$$\Pr(\mathcal{E}_1) \le \Pr(r \in (\rho_m, \rho_M]) \le |\rho_M - \rho_m| \le \max_{u,v \in e} \|\bar{u} - \bar{v}\|^2.$$

If $\mathcal{E}_2$ happens then (1) $r \le \rho_m$ (since $A = e$) and (2) $W(u) \ne W(v)$ for some $u, v \in e$. The probability that $r \le \rho_m$ is ρ_m. We now upper bound the probability that $W(u) \ne W(v)$ for some $u, v \in e$. For each $i \in \{1, \ldots, l\}$,

$$\begin{aligned}\Pr(\tilde{\omega}_i(u) \ne \tilde{\omega}_i(v) \text{ for some } u, v \in e) &\le O(\beta^{-1}\sqrt{\log n} \cdot \log\log m) \max_{u,v \in e} \|\varphi(\bar{u}) - \varphi(\bar{v})\|^2 \\ &\le O(\beta^{-1}\sqrt{\log n} \cdot \log\log m) \max_{u,v \in e} \frac{2\|\bar{u} - \bar{v}\|^2}{\min(\|\bar{u}\|^2, \|\bar{v}\|^2)} \\ &\le O(\beta^{-1}\sqrt{\log n} \cdot \log\log m) \times \rho_m^{-1} \times \max_{u,v \in e} \|\bar{u} - \bar{v}\|^2.\end{aligned}$$

By the union bound over $i \in \{1, \ldots, l\}$, the probability that $W(u) \ne W(v)$ for some $u, v \in e$ is at most $O(l \times \beta^{-1}\sqrt{\log n} \cdot \log\log m) \times \rho_m^{-1} \times \max_{u,v \in e} \|\bar{u} - \bar{v}\|^2$. Therefore,

$$\begin{aligned}\Pr(\mathcal{E}_2) &\le \rho_m \times O(l \times \beta^{-1}\sqrt{\log n}\, \log\log m) \times \rho_m^{-1} \times \max_{u,v \in e} \|\bar{u} - \bar{v}\|^2 \\ &\le O(\beta^{-1}\sqrt{\log n}\, \log m \log\log m) \times \max_{u,v \in e} \|\bar{u} - \bar{v}\|^2.\end{aligned}$$

We get that the probability that e is cut by S is at most

$$\Pr(\mathcal{E}_1) + \Pr(\mathcal{E}_2) \le O(\beta^{-1}\sqrt{\log n}\, \log m \log\log m) \times \max_{u,v \in e} \|\bar{u} - \bar{v}\|^2.$$

For $D = O(\beta^{-1}\sqrt{\log n}\, \log m \log\log m)/\alpha$ we get $\Pr(e \text{ is cut by } S) \le \alpha D \max_{u,v \in e} \|\bar{u} - \bar{v}\|^2$. Note that $\alpha \ge 1/2^l \ge \Omega(1/m)$. Thus $D \le O(\beta^{-1}\sqrt{\log n}\, m \log m \log\log m)$. ◀

5 ℓ_2–ℓ_2^2 Hypergraph Orthogonal Separators

In this section, we present another variant of hypergraph orthogonal separators, which we call ℓ_2–ℓ_2^2 hypergraph orthogonal separators. The advantage of ℓ_2–ℓ_2^2 hypergraph orthogonal separators is that their distortions do not depend on n (the number of vertices). Then in Section 6, we use ℓ_2–ℓ_2^2 hypergraph orthogonal separators to prove Theorem 21 (which, in turn, implies Theorem 4).

▶ **Definition 16** (ℓ_2–ℓ_2^2 Hypergraph Orthogonal Separator)**.** Let $\{\bar{u} : u \in V\}$ be a set of vectors in the unit ball. We say that a random set $S \subset V$ is *a ℓ_2–ℓ_2^2 hypergraph m-orthogonal separator* with ℓ_2–distortion $D_{\ell_2} : \mathbb{N} \to \mathbb{R}$, ℓ_2^2–distortion $D_{\ell_2^2}$, probability scale $\alpha > 0$, and separation threshold $\beta \in (0,1)$ if it satisfies the following properties.

1. For every $u \in V$, $\Pr(u \in S) = \alpha\|\bar{u}\|^2$.
2. For every u and v such that $\|\bar{u} - \bar{v}\|^2 \geq \beta \min(\|\bar{u}\|^2, \|\bar{v}\|^2)$

$$\Pr(u \in S \text{ and } v \in S) \leq \alpha \frac{\min(\|\bar{u}\|^2, \|\bar{v}\|^2)}{m}.$$

3. For every $e \subset V$,

$$\Pr(e \text{ is cut by } S) \leq \alpha D_{\ell_2^2} \cdot \max_{u,v\in e} \|\bar{u} - \bar{v}\|^2 + \alpha D_{\ell_2}(|e|) \cdot \min_{w\in e} \|barw\| \cdot \max_{u,v\in e} \|\bar{u} - \bar{v}\|.$$

(This definition differs from Definition 9 only in item 3.)

▶ **Theorem 17.** *There is a polynomial-time randomized algorithm that given a set of vertices V, a set of vectors $\{\bar{u}\}$ satisfying ℓ_2^2–triangle inequalities, and parameters m and β generates an ℓ_2–ℓ_2^2 hypergraph m-orthogonal separator with probability scale $\alpha \geq 1/n$ and distortions:*

$$D_{\ell_2^2} = O(m) \qquad \text{and} \qquad D_{\ell_2}(r) = O(\beta^{-1/2}\sqrt{\log r}\, m \log m \log\log m).$$

Note that distortions $D_{\ell_2^2}$ and D_{ℓ_2} do not depend on n.

The algorithm and its analysis are very similar to those in the proof of Theorem 10. The only difference is that we use another procedure to generate random assignments $\omega : V \to \{0,1\}$. The following lemma is an analog of Lemma 13.

▶ **Lemma 18.** *There is a randomized polynomial time algorithm that given a finite set V, vectors $\varphi(\bar{u})$ for $u \in V$, satisfying ℓ_2^2 triangle inequalities, and a parameter $\beta \in (0,1)$, returns a random assignment $\omega : V \to \{0,1\}$ that satisfies the following properties.*

- *For every set $e \subset V$ of size at least 2,*

$$\Pr(\omega(u) \neq \omega(v) \text{ for some } u, v \in e) \leq O(\beta^{-1/2}\sqrt{\log|e|}) \times \max_{u,v\in e} \|\varphi(\bar{u}) - \varphi(\bar{v})\|.$$

- *For every u and v such that $\|\varphi(\bar{u}) - \varphi(\bar{v})\|^2 \geq \beta$, $\Pr(\omega(u) \neq \omega(v)) \geq 0.3$.*

Proof. We sample a random Gaussian vector $g \sim \mathcal{N}(0, I_n)$ (each component g_i of g is distributed as $\mathcal{N}(0,1)$, all random variables g_i are mutually independent). Let N be a Poisson process on $\mathbb{R}$ with rate $1/\sqrt{\beta}$. Let $w(u) = 1$ if $N(\langle g, u\rangle)$ is even, and $w(u) = 0$ if $N(\langle g, \varphi(\bar{u})\rangle)$ is odd. Note that $\omega(u) = \omega(v)$ if and only if $N(\langle g, \varphi(\bar{u})\rangle) - N(\langle g, \varphi(\bar{v})\rangle)$ is even.

Consider a set $e \subset V$ of size at least 2. Denote $\operatorname{diam}(e) = \max_{u,v\in e} \|\varphi(\bar{u}) - \varphi(\bar{v})\|$. Let $\tau_m = \min_{w\in e}\langle g, \varphi(\bar{w})\rangle$ and $\tau_M = \max_{w\in e}\langle g, \varphi(\bar{w})\rangle$. Note that

$$N(\tau_m) = \min_{w\in e} N(\langle g, \varphi(\bar{w})\rangle) \qquad \text{and} \qquad N(\tau_M) = \max_{w\in e} N(\langle g, \varphi(\bar{w})\rangle).$$

If all numbers $N(\langle g, \varphi(\bar{u})\rangle)$ are equal then $\omega(u) = \omega(v)$ for all $u, v \in e$. Thus if $\omega(u) \neq \omega(v)$ for some $u, v \in e$ then $N(\langle g, \varphi(\bar{u})\rangle) \neq N(\langle g, \varphi(\bar{v})\rangle)$ for some $u, v \in e$. In particular, then $N(\tau_M) - N(\tau_m) > 0$. Given g, $N(\tau_M) - N(\tau_m)$ is a Poisson random variable with rate $(\tau_M - \tau_m)/\sqrt{\beta}$. We have,

$$\begin{aligned}\Pr\left(\omega(u) \neq \omega(v) \text{ for some } u, v \in e \mid g\right) &\leq \Pr\left(N(\tau_M) - N(\tau_m) > 0 \mid g\right) \\ &= 1 - e^{-(\tau_M - \tau_m)/\sqrt{\beta}} \leq \beta^{-1/2}(\tau_M - \tau_m).\end{aligned}$$

Let $\xi_{uv} = \langle g, \varphi(\bar{u})\rangle - \langle g, \varphi(\bar{v})\rangle$ for $u, v \in e$ $(u \neq v)$. Note that ξ_{uv} are Gaussian random variables with mean 0, and

$$\operatorname{Var}[\xi_{uv}] = \operatorname{Var}[\langle g, \varphi(\bar{u})\rangle - \langle g, \varphi(\bar{v})\rangle] = \|\varphi(\bar{u}) - \varphi(\bar{v})\|^2 \leq \operatorname{diam}(e)^2$$

Note that the expectation of the maximum of (not necessarily independent) N Gaussian random variables with standard deviation bounded by σ is $O(\sqrt{\log N}\sigma)$. We have,

$$\mathbb{E}\left[\tau_M - \tau_m\right] = \mathbb{E}\left[\max_{u,v \in e}(\xi_{uv})\right] = O(\sqrt{\log |e|}\operatorname{diam}(e))$$

since the total number of random variables ξ_{uv} is $|e|(|e| - 1)$. Therefore,

$$\begin{aligned}\Pr\left(\omega(u) \neq \omega(v) \text{ for some } u, v \in e\right) &\leq \beta^{-1/2}\,\mathbb{E}\left[\tau_M - \tau_m\right] \\ &= O(\beta^{-1/2}\sqrt{\log |e|}\max_{u,v \in e}\|\varphi(\bar{u}) - \varphi(\bar{v})\|).\end{aligned}$$

We proved that ω satisfies the first property. Now we verify that ω satisfies the second condition. Consider two vertices u and v with $\|\varphi(\bar{u}) - \varphi(\bar{v})\|^2 \geq \beta$. Given g, the random variable $Z = N(\langle g, \varphi(\bar{u})\rangle) - N(\langle g, \varphi(\bar{v})\rangle)$ has Poisson distribution with rate $\lambda = |\langle g, \varphi(\bar{u})\rangle) - \langle g, \varphi(\bar{v})\rangle|/\sqrt{\beta}$. We have,

$$\Pr\left(Z \text{ is even} \mid g\right) = \sum_{k=0}^{\infty}\Pr\left(Z = 2k \mid g\right) = \sum_{k=0}^{\infty}\frac{e^{-\lambda}\lambda^{2k}}{(2k)!} = \frac{1 + e^{-2\lambda}}{2}.$$

Note that λ is the absolute value of a Gaussian random variable with mean 0 and standard deviation $\sigma = \|\varphi(\bar{u}) - \varphi(\bar{v})\|/\sqrt{\beta} \geq 1$. Thus $\Pr\left(Z \text{ is even}\right) = \mathbb{E}\left[1 + e^{-2\sigma|\gamma|}\right]/2$, where γ is a standard Gaussian random variable, $\gamma \sim \mathcal{N}(0, 1)$. We have,

$$\Pr\left(\omega(u) \neq \omega(v)\right) = \mathbb{E}\left[\frac{1 - e^{-2\sigma|\gamma|}}{2}\right] \geq \mathbb{E}\left[\frac{1 - e^{-2|\gamma|}}{2}\right] \geq 0.3.$$

◀

Now we use the algorithm from Theorem 10 to obtain ℓ_2–ℓ_2^2 hypergraph orthogonal separators. The only difference is that we use the procedure from Lemma 18 rather than from Lemma 13 to generate assignments ω. We obtain a ℓ_2–ℓ_2^2 hypergraph orthogonal separator.

▶ **Theorem 19.** *Random set S is a hypergraph m-orthogonal separator with distortion*

$$D_{\ell_2^2} = O(m) \qquad \text{and} \qquad D_{\ell_2}(r) = O(\beta^{-1/2}\sqrt{\log r}\, m \log m \log\log m),$$

probability scale $\alpha \geq 1/n$ and separation threshold $\beta \in (0, 1)$.

Proof. The proof of the theorem is almost identical to that of Theorem 15. We first check conditions 1 and 2 of ℓ_2–ℓ_2^2 hypergraph orthogonal separators in the same way as we checked conditions 1 and 2 of hypergraph orthogonal separators in Theorem 15. When we verify that property 3 holds, we use bounds from Lemma 18. The only difference is how we upper bound the probability of the event $\mathcal{E}_2$.

If $\mathcal{E}_2$ happens then (1) $r \le \rho_m$ (since $A = e$) and (2) $W(u) \neq W(v)$ for some $u, v \in e$. The probability that $r \le \rho_m$ is ρ_m. We upper bound the probability that $W(u) \neq W(v)$ for some $u, v \in e$. For each $i \in \{1, \dots, l\}$,

$$\Pr\left(\tilde{\omega}_i(u) \neq \tilde{\omega}_i(v) \text{ for some } u, v \in e\right) \le O(\beta^{-1/2}\sqrt{\log|e|}\,\log\log m) \max_{u,v\in e} \|\varphi(\bar{u}) - \varphi(\bar{v})\|$$

$$\le O(\beta^{-1/2}\sqrt{\log|e|}\,\log\log m) \max_{u,v\in e} \frac{\|\bar{u}-\bar{v}\|}{\min(\|\bar{u}\|, \|\bar{v}\|)}$$

$$\le O(\beta^{-1/2}\sqrt{\log|e|}\,\log\log m) \times \rho_m^{-1/2} \times \max_{u,v\in e} \|\bar{u}-\bar{v}\|.$$

By the union bound over $i \in \{1, \dots, l\}$, the probability that $W(u) \neq W(v)$ for some $u, v \in e$ is at most $O(l \times \beta^{-1/2}\sqrt{\log|e|}\,\log\log m) \times \rho_m^{-1/2} \times \max_{u,v\in e} \|\bar{u}-\bar{v}\|$. Therefore,

$$\Pr(\mathcal{E}_2) \le \rho_m \times O(l \times \beta^{-1/2}\sqrt{\log|e|}\,\log\log m) \times \rho_m^{-1/2} \times \max_{u,v\in e} \|\bar{u}-\bar{v}\|$$

$$\le O(l \times \beta^{-1/2}\sqrt{\log|e|}\,\log\log m) \times \rho_m^{1/2} \times \max_{u,v\in e} \|\bar{u}-\bar{v}\|.$$

We get that the probability that e is cut by S is at most

$$\Pr(\mathcal{E}_1) + \Pr(\mathcal{E}_2) \le \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2 + O(l \times \beta^{-1/2}\sqrt{\log|e|}\,\log\log m)\,\rho_m^{1/2} \max_{u,v\in e} \|\bar{u}-\bar{v}\|$$

$$\le \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2 + O(l \times \beta^{-1/2}\sqrt{\log|e|}\,\log\log m) \min_{w\in e} \|\bar{w}\| \max_{u,v\in e} \|\bar{u}-\bar{v}\|.$$

For $D_{\ell_2^2} = 1/\alpha$ and $D_{\ell_2}(r) = O(\beta^{-1/2}\sqrt{\log r}\,\log m \log\log m)/\alpha$, we get

$$\Pr(e \text{ is cut by } S) \le \alpha D_{\ell_2^2} \cdot \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2 + \alpha D_{\ell_2}(|e|) \cdot \min_{w\in e} \|\bar{w}\| \cdot \max_{u,v\in e} \|\bar{u}-\bar{v}\|.$$

Note that $\alpha \ge 1/2^l \ge \Omega(1/m)$. Thus

$$D_{\ell_2^2} = O(m) \qquad \text{and} \qquad D_{\ell_2}(r) = O(\beta^{-1/2}\sqrt{\log r}\, m \log m \log\log m).$$

◀

6 Algorithm for Hypergraph Small Set Expansion via ℓ_2–ℓ_2^2 Hypergraph Orthogonal Separators

In this section, we present another algorithm for Hypergraph Small Set Expansion. The algorithm finds a set with expansion proportional to $\sqrt{\phi^*_{G,\delta}}$. The proportionality constant depends on degrees of vertices and hyperedge size but not on the graph size. Here, we present our result for arbitrary hypergraphs. The result for uniform hypergraphs (Theorem 4) stated in the introduction follows from our general result. In order to state our result for arbitrary graphs, we need the following definition.

▶ **Definition 20.** Consider a hypergraph $H = (V, E)$. Suppose that for every edge e we are given a non-empty subset $e^\circ \subseteq e$. Let

$$\eta(u) = \sum_{e: u\in e^\circ} \frac{\log_2|e|}{|e^\circ|} \qquad \text{and} \qquad \eta_{max} = \max_{u\in V} \eta(u).$$

Finally, let $\eta^H_{\max}$ be the minimum of η_{max} over all possible choices of subsets e°.

▶ **Claim 6.1.** 1. $\eta^H_{max} \le \max_{u\in V} \sum_{e:u\in e}(\log_2 |e|)/|e|$.
2. *If H is a r-uniform graph with maximum degree d_{max} then $\eta^H_{max} \le (d_{max}\log_2 r)/r$.*
3. *Suppose that we can choose one vertex in every edge so that no vertex is chosen more than once. Then $\eta^H_{max} \le \log_2 r_{max}$, where r_{max} is the size of the largest hyperedge in H.*

Proof.

1. Let $e^\circ = e$ for every $e \in E$. We have, $\eta^H_{\max} \le \max_{u\in V} \sum_{e:u\in e}(\log_2 |e|)/|e|$.
2. By **1**, $\eta^H_{\max} \le \max_{u\in V} \sum_{e:u\in e}(\log_2 |e|)/|e| = \max_{u\in V} \sum_{e:u\in e}(\log_2 r)/r = (d_{\max}\log_2 r)/r$.
3. For every edge $e \in E$, let e° be the set that contains the vertex chosen for e. Then $|e^\circ| = 1$ and $|\{e : u \in e^\circ\}| \le 1$ for every u. We have,

$$\eta^H_{\max} \le \max_{u\in V} \sum_{e:u\in e^\circ} \frac{\log_2 |e|}{|e^\circ|} \le \max_{u\in V} \sum_{e:u\in e^\circ} \frac{\log_2 r_{\max}}{1} = \log_2 r_{\max}.$$

◀

▶ **Theorem 21.** *There is a randomized polynomial-time algorithm for the Hypergraph Small Set Expansion problem that given a hypergraph $H = (V, E)$, and parameters $\varepsilon \in (0, 1)$ and $\delta \in (0, 1/2]$, finds a set $S \subset V$ of size at most $(1 + \varepsilon)\delta n$ such that*

$$\begin{aligned}\phi(S) &\le O_\varepsilon\left(\delta^{-1}\log\delta^{-1}\log\log\delta^{-1}\sqrt{\eta^H_{max}\cdot\phi^*_{H,\delta}} + \delta^{-1}\phi^*_{H,\delta}\right)\\ &= \tilde{O}_\varepsilon\left(\delta^{-1}\left(\sqrt{\eta^H_{max}\phi^*_{H,\delta}} + \phi^*_{H,\delta}\right)\right),\end{aligned}$$

In particular, if H is an r-uniform hypergraph with maximum degree d_{max}, then we have,

$$\phi(S) \le \tilde{O}_\varepsilon\left(\delta^{-1}\left(\sqrt{d_{max}\frac{\log_2 r}{r}\phi^*_{H,\delta}} + \phi^*_{H,\delta}\right)\right).$$

Proof. The proof is similar to that of Theorem 11. We solve the SDP relaxation for H-SSE and obtain an SDP solution $\{\bar{u}\}$. Denote the SDP value by sdp-cost. Consider an ℓ_2–ℓ_2^2 hypergraph orthogonal separator S with $m = 4/(\varepsilon\delta)$ and $\beta = \varepsilon/4$. Define a set S':

$$S' = \begin{cases} S, \text{ if } |S| \le (1+\varepsilon)\delta n,\\ \varnothing, \text{ otherwise.}\end{cases}$$

Clearly, $|S'| \le (1+\varepsilon)\delta n$. As in the proof of Theorem 11, $\Pr(u \in S') \in \left[\frac{\alpha}{2}\|\bar{u}\|^2, \alpha\|\bar{u}\|^2\right]$. Note that

$$\Pr(S' \text{ cuts edge } e) \le \Pr(S \text{ cuts edge } e) \le \alpha D_{\ell_2}\max_{u,v\in e}\|\bar{u}-\bar{v}\|^2 + \alpha D_{\ell_2}(r)\min_{w\in e}\|\bar{w}\|\max_{u,v\in e}\|\bar{u}-\bar{v}\|.$$

Let $\mathcal{C} = \alpha^{-1}\mathbb{E}[|E_{cut}(S')|]$. Let $Z = |S'| - \frac{|E_{cut}(S')|}{4\mathcal{C}}$. We have,

$$\mathbb{E}[Z] = \mathbb{E}[|S'|] - \mathbb{E}\left[\frac{|E_{cut}(S')|}{4\mathcal{C}}\right] \ge \sum_{u\in V}\frac{\alpha}{2}\cdot\|\bar{u}\|^2 = \frac{\alpha}{2} - \frac{\alpha}{4} = \frac{\alpha}{4}.$$

Now we upper bound $\mathcal{C}$. Consider the optimal choice of e° for H in the definition of $\eta^H_{\max}$.

$$\begin{aligned}
\mathcal{C} &= \alpha^{-1}\mathbb{E}\left[|E_{cut}(S')|\right] \le \alpha^{-1}\sum_{e\in E}\Pr\left(e \text{ is cut by } S\right)\\
&\le D_{\ell_2^2}\sum_{e\in E}\max\|\bar{u}-\bar{v}\|^2 + \sum_{e\in E}D_{\ell_2}(|e|)\min_{w\in e}\|\bar{w}\|\max_{u,v\in e}\|\bar{u}-\bar{v}\|\\
&\le D_{\ell_2^2}\cdot\text{sdp-cost} + \sum_{e\in E}D_{\ell_2}(|e|)\sum_{w\in e^\circ}\left(\frac{\|\bar{w}\|}{|e^\circ|}\right)\times\max_{u,v\in e}\|\bar{u}-\bar{v}\|\\
&\le D_{\ell_2^2}\cdot\text{sdp-cost} + \sum_{e\in E}\sum_{w\in e^\circ}\frac{D_{\ell_2}(|e|)\|\bar{w}\|}{\sqrt{|e^\circ|}}\times\frac{\max_{u,v\in e}\|\bar{u}-\bar{v}\|}{\sqrt{|e^\circ|}}\\
&\le D_{\ell_2^2}\cdot\text{sdp-cost} + \sqrt{\sum_{e\in E}\sum_{w\in e^\circ}\frac{D_{\ell_2}(|e|)^2\|\bar{w}\|^2}{|e^\circ|}}\sqrt{\sum_{e\in E}\sum_{w\in e^\circ}\frac{\max_{u,v\in e}\|\bar{u}-\bar{v}\|^2}{|e^\circ|}}\\
&\le D_{\ell_2^2}\cdot\text{sdp-cost} + \sqrt{\sum_{w\in V}\sum_{e:w\in e^\circ}\frac{D_{\ell_2}(|e|)^2}{|e^\circ|}\|\bar{w}\|^2}\,\sqrt{\text{sdp-cost}}.
\end{aligned}$$

For every vertex w,

$$\sum_{e:w\in e^\circ}\frac{D_{\ell_2}(|e|)^2}{|e^\circ|} \le O_\beta(m\log m\log\log m)^2\sum_{e:w\in e^\circ}\frac{\log_2|e|}{|e^\circ|} \le O_\beta(m\log m\log\log m)^2\times\eta^H_{\max}.$$

and $\sum_{w\in V}\|\bar{w}\|^2 = 1$. Therefore, $\mathcal{C} \le O_\beta\left(m\,\text{sdp-cost} + m\log m\log\log m\,\sqrt{\eta^H_{\max}\cdot\text{sdp-cost}}\right)$. By the argument from Theorem 11, we get that if we sample S' sufficiently many times (i.e., $(4n^2/\alpha)$ times), we will find a set S' such that

$$\frac{|E_{cut}(S')|}{|S'|} \le 4\mathcal{C} \le O_\beta\left(\delta^{-1}\log\delta^{-1}\log\log\delta^{-1}\sqrt{\eta^H_{\max}\cdot\text{sdp-cost}} + \delta^{-1}\,\text{sdp-cost}\right)$$

with probability exponentially close to 1. ◀

7 SDP Intgrality Gap

In this section, we present an integrality gap for the SDP relaxation for H-SSE. We also give a lower bound on the distortion of a hypergraph m-orthogonal separator.

▶ **Theorem 22.** *For $\delta = 1/r$, the integrality gap of the SDP for H-SSE is at least $1/(2\delta) = r/2$.*

Proof. Consider a hypergraph $H = (V, E)$ on $n = r$ vertices with one hyperedge $e = V$ (e contains all vertices). Note that the expansion of every set of size $\delta n = 1$ is 1. Thus $\phi^*_{H,\delta} = 1$.

Consider an SDP solution that assigns vertices mutually orthogonal vectors of length $1/\sqrt{r}$. It is easy to see this is a feasible SDP solution. Its value is $\max_{u,v\in e}\|\bar{u}-\bar{v}\|^2 = 2/r$. Therefore, the SDP integrality gap is at least $r/2$. ◀

Now we give a lower bound on the distortion of hypergraph m-orthogonal separators.

▶ **Lemma 23.** *For every $m > 4$, there is an SDP solution such that every hypergraph m-orthogonal separator with separation threshold $\beta \ge 0$ has distortion at least $\lceil m\rceil/4$.*

Proof. Consider the SDP solution from Theorem 22 for $n = r = \lceil m\rceil$. Consider a hypergraph m-orthogonal separator S for this solution. Let D be its distortion. Note that condition (2) from the definition of hypergraph orthogonal separators applies to any pair of distinct vertices (u, v) since $\langle\bar{u},\bar{v}\rangle = 0$.

By the inclusion–exclusion principle, we have,

$$\begin{aligned}\Pr(|S|=1) &\geq \sum_{u\in S}\Pr(u\in S) - \frac{1}{2}\sum_{u,v\in S, u\neq v}\Pr(u\in S, v\in S)\\ &\geq \sum_{u\in S}\alpha\|\bar{u}\|^2 - \frac{1}{2}\sum_{u,v\in S,u\neq v}\frac{\alpha\min(\|\bar{u}\|^2,\|\bar{v}\|^2)}{m} = \alpha - \frac{\alpha n(n-1)}{2mr} \geq \alpha/2.\end{aligned}$$

On the other hand, if $|S|=1$ then S cuts e. We have,

$$\Pr(|S|=1) \leq \Pr(S \text{ cuts } e) \leq \alpha D \max_{u,v\in e}\|\bar{u}-\bar{v}\|^2 = 2\alpha D/r.$$

We get that $\alpha/2 \leq 2\alpha D/r$ and thus $D \geq r/4 = \lceil m\rceil/4$. ◀

8 Reduction from Vertex Expansion to Hypergraph Expansion

In the reduction from vertex expansion to hypergraph expansion, we will use the notion of *Symmetric Vertex Expansion*. For a graph $G=(V,E)$, and for a set $S\subset V$, we define its outer neighborhood $N(S)$ as follows.

$$N(S) = \{u\in\bar{S} : \exists\, v\in S \text{ such that } \{u,v\}\in E\}.$$

The symmetric vertex expansion of a set, denoted by $\Phi^V(S)$, is defined as

$$\Phi^V(S) = \frac{|N(\bar{S})\cup N(S)|}{\min(|S|,|\bar{S}|)} \qquad \text{and} \qquad \Phi^V_{G,\delta} = \min_{\substack{S\subset V\\ 0<|S|\leq\delta n}}\Phi^V(S).$$

We will use the following reduction from vertex expansion to symmetric vertex expansion.

▶ **Theorem 24** (Louis, Raghavendra and Vempala [13]). *Given a graph G, there exists a graph G' such that $c_1\phi^V_{G,\delta} \leq \Phi^V_{G',\delta} \leq c_2\phi^V_{G,\delta}$. where $c_1, c_2 > 0$ are absolute constants, and the maximum degree of graph G' is equal to the maximum degree of graph G. Moreover, there exists a polynomial time algorithm to compute such graph G'.*

Proof of Theorem 6. Starting with graph G, we use Theorem 24 to obtain a graph $G'=(V',E')$ such that $c_1\phi^V_{G,\delta} \leq \Phi^V_{G',\delta} \leq c_2\phi^V_{G,\delta}$. Next we construct hypergraph $H=(V',E'')$ as follows. For every vertex $v\in V'$, we add the hyperedge $\{v\}\cup N(\{v\})$ to E'' (note that $N(\{v\})$ is the set of neighbors of v in G). Fix an arbitrary set $S\subset V$.

We first show that $\Phi^V(S)\leq\phi_H(S)$. Consider the vertices $N(\bar{S})$. Each vertex in $v\in N(\bar{S})$ has a neighbor, say u, in $\bar{S}$. Therefore the hyperedge $\{v\}\cup N(\{v\})$ is cut by S in H. Similarly, for each vertex $v\in N(S)$, the hyperedge $\{v\}\cup N(\{v\})$ is cut by S in H. All these hyperedges are disjoint by construction. Therefore, $\Phi^V(S) = \frac{|N(\bar{S})|+|N(S)|}{|S|} \leq \frac{|E_{cut}(S)|}{|S|} \leq \phi_H(S)$.

Now we verify that $\phi_H(S)\leq\Phi^V(S)$. For any hyperedge $(\{v\}\cup N(\{v\}))\in E_{cut}(S)$, the vertex v has to belong to either $N(\bar{S})$ or $N(S)$. Therefore, $\phi_H(S)\leq\frac{|E_{cut}(S)|}{|S|}\leq\frac{|N(\bar{S})|+|N(S)|}{|S|} = \Phi^V(S)$. Thus, we get that $\phi_H(S)=\Phi^V(S)$ for every $S\subset V$, and hence $\phi^*_{H,\delta} = \Phi^V_{G',\delta}$. Therefore, $c_1\phi^V_{G,\delta}\leq\phi^*_{H,\delta}\leq c_2\phi^V_{G,\delta}$.

Finally, we upper bound $\eta^H_{\max}$. We use part 3 of Claim 6.1. We choose vertex v in the hyperedge $\{v\}\cup N(\{v\})$. By Claim 6.1, $\eta^H_{\max}\leq\log_2 r_{\max}$, where $r_{\max}$ is the size of the largest hyperedge. Note that $|\{v\}\cup N(\{v\})| = \deg v+1$. Thus $\eta^H_{\max}\leq\log_2 r_{\max}\leq\log_2(d_{\max}+1)$ ◀

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