Ramsey Theorem for Pairs As a Classical Principle in Intuitionistic Arithmetic*

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Abstract

We produce a first order proof of a famous combinatorial result, Ramsey Theorem for pairs and in two colors. Our goal is to find the minimal classical principle that implies the "miniature" version of Ramsey we may express in Heyting Arithmetic HA. We are going to prove that Ramsey Theorem for pairs with recursive assignments of two colors is equivalent in HA to the sub-classical principle Σ^0_3-LLPO. One possible application of our result could be to use classical realization to give constructive proofs of some combinatorial corollaries of Ramsey; another, a formalization of Ramsey in HA, using a proof assistant.

In order to compare Ramsey Theorem with first order classical principles, we express it as a schema in the first order language of arithmetic, instead of using quantification over sets and functions as it is more usual: all sets we deal with are explicitly defined as arithmetical predicates. In particular, we formally define the homogeneous set which is the witness of Ramsey theorem as a Δ^0_3-arithmetical predicate.

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1 Introduction

The purpose of this paper is to study, from the viewpoint of first order arithmetic, Ramsey Theorem [15] for pairs for recursive assignments of two colors, in order to find some principle of classical logic equivalent to it in Intuitionistic Arithmetic HA. Ramsey theorem is not intuitionistically provable, and a priori, it is not evident whether a classical principle expressing Ramsey in intuitionistic arithmetic exists. Our long-time research goal is to study the constructive content of corollaries in first order arithmetic of Ramsey Theorem using interactive realizability, and to this aim we want to find the statement and the proof of Ramsey in first order arithmetic requiring the minimum amount of classical logic. In the PhD thesis of Giovanni Birolo [4] there is an example of a constructive study of a classical proof obtained by interactive realizability. Birolo studied a geometric property that required

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the law of Excluded Middle of level one (EM_1); for Ramsey, the required principles are higher than EM_1 in the hierarchy of classical principles presented in [1].

Our study of Ramsey Theorem differs from the results in Classical Reverse Mathematics ([5], [6], [13], [8]) in many aspects. We already stressed that we formulate Ramsey in first order arithmetic, replacing set variables with explicit set definitions. Besides, Classical Reverse Mathematics is interested in the necessary set existence axioms needed to proof a theorem and investigates the minimum restriction of the induction schema required in a proof, while they assume the entire Excluded Middle schema. Our work may be considered a kind of Intuitionistic Reverse Mathematics: we assume the entire induction schema, and we investigate the minimum restriction of the Excluded Middle Schema and of some other classical schemas required in a classical proof. Therefore our approach is different from what Ishihara calls Constructive Reverse Mathematics in [9]. Ishihara works in Bishop's Constructive Mathematics which is an informal mathematics using intuitionistic logic and assuming some function existence axioms; instead, he does not study the level of classical principles used in the proof.

As regards the comprehension axiom, instead, there are some links with Classical Reverse Mathematics. Recall that the description axiom says that each arithmetic binary predicate that is fully and uniquely defined is a graph of some function: $\mathbb{N} \to \mathbb{N}$. If we add function variables and we assume the description axiom, the Excluded Middle for an arithmetic predicate and the comprehension axiom for the same predicate are equivalent in HA + functions.

We may stress the difference between the two approaches through an example. Let consider the Infinite Pigeonhole Principle. On the one hand, in reverse mathematics, this principle is equivalent to $B\Sigma^0_2$(the bounding principle for Σ^0_2-formulas, see [16]) which is equivalent to Δ^0_2-induction ([17]). On the other hand, in our setting, it is a consequence of the law of Excluded Middle of level two: EM_2. In [12] Liu considered the base system for reverse mathematics RCA_0, in which we assume the entire Excluded Middle, but only induction for Σ^0_1 formulas and recursive comprehension. Liu proved that Ramsey Theorem for pairs in two colors does not imply WKL_0, Weak König's Lemma for recursive trees, in RCA_0. Instead in [11] Kohlenbach and Kreuzer proved in $iRCA^*_0$, the intuitionistic system corresponding to RCA^* (Σ^0_0-induction, exponentiation axioms but no excluded middle), that Ramsey Theorem for pairs implies Π^0_2-LEM, which is more than WKL_0. In this work we drop function and set variables, and we consider Heyting Arithmetic HA, in which we have no Excluded Middle Schema but we have the full induction schema. Under these assumptions, we prove that recursive Ramsey Theorem for pairs in two colors is equivalent to Σ^0_3-LLPO (Lesser Limited Principle of Omniscience for Σ^0_3 predicates, a principle weaker than full Excluded Middle, but stronger than WKL_0, which we explain below).

Our study of Ramsey Theorem differs also from the no-counterexample [2], since we do not transform Ramsey Theorem into some weaker and constructively provable statement, but we study the minimum restriction of the Excluded Middle schema required to prove the original result in HA. We differ from the dialectica interpretation ([11], [14]), because it transforms RT^2_2 into a constructively provable, classically equivalent statement and deletes the non-constructive content leaving only the combinatorial core. Moreover the dialectica interpretation requires complex types and variables for each type, while we use the type of natural numbers and of functions over natural numbers only, and no function variable.

At the beginning of this work, in a private communication, Alexander Kreuzer conjectured that Erdős Rado proof of Ramsey Theorem may be formalized in $HA + EM_4$, Excluded Middle restricted to Σ^0_4 formulas. We prove that he was right. Moreover, by modifying

Jockusch's proof of Ramsey [10] (that is already a modified version of Erdős Rado proof of the same result) we prove that the classical principle Σ^0_3-LLPO is in fact equivalent to Ramsey Theorem in HA. Σ^0_3-LLPO (see [1]) is a classical principle weaker than Excluded Middle Schema for Σ^0_3 formulas, which may be restated as the conjunction of Excluded Middle for Σ^0_2 formulas and De Morgan Laws for Σ^0_3 formulas. If we add Choice to HA, Σ^0_3-LLPO is equivalent to WKL_3, Weak König's Lemma for Σ^0_2 trees.

We hope to apply, in future works, the method called interactive realizability to understand and explain the computational content of Ramsey Theorem, and to find new constructive proofs for some consequences of it. The interactive realizability is a realizability interpretation for first order classical arithmetic introduced in 2008 by Stefano Berardi and Ugo de' Liguoro [3]. If a corollary of Ramsey Theorem is a consequence of Intuitionistic Ramsey Theorem, an alternative method to prove it constructively could be to use the Coquand's work [7]. However his proof use the Brouwer's thesis, so this method does not guarantee a proof in HA.

This is the plan of the paper. In Section 2 we explain how to state Ramsey Theorem without using functions and set variables; in Section 3 we prove that Ramsey Theorem implies Σ^0_3-LLPO and in Section 4, by modifying Jockusch's proof, we prove the opposite implication. In the conclusions we discuss the interest of the equivalence with Σ^0_3-LLPO.

2 Ramsey Theorem and Classical Principles for Arithmetic

In this section we introduce some notations for Ramsey Theorem and for some classical principles. Any natural number n is identified with the set $\{0, \ldots, n-1\}$. We use $\mathbb{N}$ to denote the least infinite ordinal, which is identified with the set of natural numbers. For any set X and any natural number r,

$$[X]^r = \{Y \subseteq X \mid |Y| = r\}$$

denotes the set of subsets of X of cardinality r. If $r = 1$ then $[\mathbb{N}]^r$ is the set of singleton subsets of $\mathbb{N}$, and just another notation for $\mathbb{N}$. If $r = 2$ then $[\mathbb{N}]^2$ is the complete graph on $\mathbb{N}$: we think of any subset $\{x, y\}$ of $\mathbb{N}$ with $x \neq y$ as an edge of the graph. We will think that each edge $\{x, y\}$ has direction from $\min\{x, y\}$ to $\max\{x, y\}$. Let $n, m \in \mathbb{N}$, then a map $f : [\mathbb{N}]^r \to n$ is called a coloring of $[\mathbb{N}]^r$ with n colors. If $r = 2$ and $f(\{x, y\}) = c < n$, then we say that the edge $\{x, y\}$ has color c. If $f : [\mathbb{N}]^r \to n$ is a map then for all $X \subseteq \mathbb{N}$ we denote with $f''[X]^r$ the set of colors of hyper-edges of X, that is:

$$f''[X]^r = \{k \in \mathbb{N} \mid \exists e \in [X]^r \text{ such that } f(e) = k\}\,.$$

We say that $X \subseteq [\mathbb{N}]^r$ is homogeneous for f, or f is homogeneous on X, if all hyper-edges of X have the same color, that is, there exists $k < n$ such that $f''[X]^r = \{k\}$. We also say that X is homogeneous for f in color k. If $r = 1$ we can think of the function f as a point coloring map on natural numbers. In this case an homogeneous set X is any set of points of $\mathbb{N}$ which all have the same color. If $r = 2$ we can think of the function f as an edge coloring of a graph that has as its vertices the natural numbers. In this case an homogeneous set X is any set of points of $\mathbb{N}$ whose connecting edges all have the same color.

We denote Heyting Arithmetic, with one symbol and axioms for each primitive recursive map, with HA. We work in the language for Heyting Arithmetic with all primitive recursive maps, extended with the symbols $\{f_0, \ldots, f_n\}$, where n is a natural number and f_i denotes a total recursive function for all $i < n + 1$. These f_i will indicate an arbitrary coloring in the formulation of Ramsey Theorem below. If $P = \forall x_1 \exists x_2 \ldots p(x_1, x_2, \ldots)$, with p arithmetic atomic formula, and $Q = \exists x_1 \forall x_2 \dots \neg p(x_1, x_2 \dots)$, then we say that P, Q are dual each other and we write $P^\perp = Q$ and $Q^\perp = P$. Dual is defined only for prenex formulas as P, Q. We consider the classical principles as statement schemas as in [1]. A conjunctive schema is a set C of arithmetical formulas, expressing the second order statement "for all A in C, A holds" in a first order language. We prove a conjunctive schema C in HA if we prove any A in C in HA. A conjunctive schema C implies a formula A in HA if $s_1 \wedge \dots \wedge s_n \vdash A$ in HA for some $s_1, \dots, s_n \in C$. The conjunctive schema C implies another conjunctive schema C' in HA if C implies A in HA for any A in C'. In order to express Ramsey Theorem we also have to consider the dual concept of disjunctive schema D, expressing the second order statement "for some A in D, A holds" in a first order language. We prove a disjunctive schema D in HA if we prove $s_1 \vee \dots \vee s_n$ in HA for some $s_1, \dots, s_n \in D$. A disjunctive schema D implies a formula A in HA if $s \vdash A$ in HA for all $s \in D$.

The infinite Ramsey Theorem is a very important result for finite and infinite combinatorics. In this paper we study Ramsey Theorem in two colors, for singletons and for pairs. They are informally stated as follows:

▶ $\mathbf{RT^1_2(\Sigma^0_n)}$**.** For any coloring $c_a : \mathbb{N} \to 2$ of vertices with a parameter a, there exists an infinite subset of $\mathbb{N}$ homogeneous for the given coloring. ($c_a \in \Sigma^0_n$).

▶ $\mathbf{RT^2_2(\Sigma^0_n)}$**.** For any coloring $c_a : [\mathbb{N}]^2 \to 2$ of edges with a parameter a, there exists an infinite subset of $\mathbb{N}$ homogeneous for the given coloring. ($c_a \in \Sigma^0_n$).

$\mathrm{RT}^2_2(\Sigma^0_0)$ (respectively $\mathrm{RT}^1_2(\Sigma^0_0)$) says that given a family $\{c_a \mid a \in \mathbb{N}\}$ of recursive edge (vertex) colorings of a graph with $\mathbb{N}$ vertices, then for any coloring there exists a subgraph with $\mathbb{N}$ vertices such that each edge (vertex) of the subgraph has the same color.

In this work we formalize Ramsey Theorem for two colors, for pairs (respectively, for singletons) and for recursive colorings by the following disjunctive schema which we call Ramsey schema R:

$$R := \{\forall a(B(., c_a) \text{ infin. hom. black } \vee W(., c_a) \text{ infin. hom. white}) \mid B, W \text{ arithm. predic.}\}.$$

Here $c = \{c_a \mid a \in \mathbb{N}\}$ denotes any recursive family of recursive assignment of two colors, black and white. A sufficient condition to prove Ramsey schema is to find at least two predicates B, W and a proof of $\forall a(B(., c_a)$ infinite homogeneous black $\vee\ W(., c_a)$ infinite homogeneous white) in HA. For short we say that for each recursive family of recursive colorings there is an homogeneous set.

The conjunctive schemata for HA we consider, expressing classical principles and taken from [1], are the followings.

▶ $\mathbf{\Sigma^0_n}$**-LLPO.** Lesser Limited Principle of Omniscience. For any parameter a

$$\forall x, x' \ (P(x,a) \ \vee \ Q(x',a)) \implies \forall x P(x,a) \ \vee \ \forall x Q(x,a). \ (P,\ Q \in \Sigma^0_{n-1})$$

It is a kind of law for prenex formulas and if we assume the Axiom of Choice it is equivalent to Weak König's Lemma for Σ^0_{n-1} trees. We postpone the discussion about this principle at the conclusions of the paper.

▶ **Pigeonhole Principle for $\mathbf{\Pi^0_n}$.** The Pigeonhole Principle states that given a partition of infinitely many natural numbers in two classes, then at least one of these classes has infinitely many elements. For any parameter a

$$\forall x \ \exists z \ [z \geq x \ \wedge \ (P(z,a) \vee Q(z,a))] \implies$$

$$\forall x \ \exists z \ [z \geq x \ \wedge \ P(z,a)] \ \vee \ \forall x \ \exists z \ [z \geq x \ \wedge \ Q(z,a)]. \ (P, Q \in \Pi^0_n)$$

▶ $\mathbf{EM_n}$. Excluded Middle for Σ^0_n formulas. For any parameter a

$$\exists x\ P(x,a)\ \vee\ \neg\exists x\ P(x,a).\ (P \in \Pi^0_{n-1})$$

Recall that $P^\perp$ denotes the dual of P for any prenex P. As shown in [1, corollary 2.9] the law of Excluded Middle for Σ^0_n formulas is equivalent in HA to

$$\exists x\ P(x,a)\ \vee\ \forall x\ P(x,a)^\perp.\ (P \in \Pi^0_{n-1})$$

In all our schemata we use parameters. The parameter a is necessary since we need to use in HA statements with a free variable a, like

$$\forall a\ (\forall x\ P(x,a)\ \vee\ \exists x\ \neg P(x,a))$$

in our proof.

3 Ramsey Theorem for pairs and recursive coloring implies the Limited Lesser Principle of Omniscience for Σ^0_3 formulas

In this section we prove $\mathrm{RT}^2_2(\Sigma^0_0) \implies \Sigma^0_3$-LLPO in HA. From now on, all proofs are done in Intuitionistic Arithmetic HA. By definition of disjunctive schema, we have to prove that for each P in Σ^0_3-LLPO, there exist a finite number of recursive families of recursive colorings $c_{a,0},\dots,c_{a,j-1}$ such that, fixed any $W_i(.,c_{a,i})$ and $B_i(.,c_{a,i})$, if we assume

$$\{\forall a(W_i(.,c_{a,i}) \text{ infinite and homogeneous } \vee B_i(.,c_{a,i}) \text{ infinite and homogeneous}) \mid i<j\}$$

then we deduce P.

We say that a sequence is stationary if it is constant from a certain point on. In our proof we need some conjunctive schemata provable in Classical Arithmetic: that, in every primitive recursive family of monotone and bounded above sequences $s:\mathbb{N}\to\mathbb{N}$, each sequence is stationary and that, in every primitive recursive family of recursive sequences $t:\mathbb{N}\to\mathbb{N}$ for which there are at most k values of x such that $t(x)\neq t(x+1)$, each sequence is stationary. In order to obtain these results in HA from $\mathrm{RT}^2_2(\Sigma^0_0)$ we need to prove the EM_1 schema first, as shown by the following lemma (proved in HA, as all lemmas for now on).

▶ **Lemma 1.** **1.** $\mathrm{RT}^2_2(\Sigma^0_0)$ *implies* EM_1*;*
2. EM_1 *implies that, for any family* $F=\{s(n,\cdot)\mid n\in\mathbb{N}\}$ *of recursive monotone and bounded above sequences enumerated by a binary primitive recursive function* $s:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$*, each sequence in* F *is stationary;*
3. EM_1 *implies that, for any family* $G=\{t(n,\cdot)\mid n\in\mathbb{N}\}$ *of recursive sequences enumerated by a binary primitive recursive function* $t:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ *for which there are at most* k *values of* x *such that* $t(n,x)\neq t(n,x+1)$*, each sequence in* G *is stationary.*

Proof. **1.** $\mathrm{RT}^2_2(\Sigma^0_0)$ implies $\mathrm{RT}^1_2(\Sigma^0_0)$ that implies the infinite pigeonhole principle which implies EM_1.

a. For the first implication, given a coloring of the points $c_a:\mathbb{N}\to 2$ we consider a coloring of the edges

$$c^*_a:[\mathbb{N}]^2\to 2$$

that depends only on the smallest point of the edge, that is, for every $x<y$, $c^*_a(\{x,y\}):=c_a(x)$. The infinite homogeneous set for c^*_a, whose existence is guaranteed by $\mathrm{RT}^2_2(\Sigma^0_0)$, is such that it is homogeneous also for c_a. Then $\mathrm{RT}^2_2(\Sigma^0_0)$ implies $\mathrm{RT}^1_2(\Sigma^0_0)$.

b. The infinite pigeonhole principle can be stated as follows

$$\forall x\ \exists z\ [z \geq x\ \wedge\ (P(z,a) \vee Q(z,a))] \implies$$

$$\forall x\ \exists z\ [z \geq x\ \wedge\ P(z,a)]\ \vee\ \forall x\ \exists z\ [z \geq x\ \wedge\ Q(z,a)]\,,$$

with P and Q recursive predicates. Assuming the hypothesis of the principle, we define the following recursive coloring $c_a : \mathbb{N} \to 2$: for each $x \in \mathbb{N}$ $c_a(x) := 0$ if and only if the first witness z_x of

$$\exists z\ [z \geq x\ \wedge\ (P(z,a) \vee Q(z,a))]$$

is such that $P(z_x, a)$ is true. Thanks to $\mathrm{RT}^1_2(\Sigma^0_0)$ we have an infinite homogeneous set. If it is uniform in color 0 then P is true for infinitely many z, otherwise Q is true for infinitely many z.

c. By hypothesis we have the pigeonhole principle:

$$\forall x\ \exists z\ [z \geq x\ \wedge\ (P(z,a) \vee Q(z,a))] \implies$$

$$\forall x\ \exists z\ [z \geq x\ \wedge\ P(z,a)]\ \vee\ \forall x\ \exists z\ [z \geq x\ \wedge\ Q(z,a)]\,,$$

with P and Q recursive predicates. We want to show that

$$\exists x\ P(x,a)\ \vee\ \forall x\ \neg P(x,a)\,;$$

with P recursive predicate. To prove it, we apply the pigeonhole principle with

$$P^*(z,a) := \exists y \leq z\ P(y,a)$$

$$Q^*(z,a) := \forall y \leq z\ \neg P(y,a)\,.$$

The hypothesis of the pigeonhole principle holds for P^*, Q^* with $z = x$. For the same principle, we deduce that either

$$\forall x\ \exists z\ [z \geq x\ \wedge\ \exists y \leq z\ P(y,a)]$$

is true, from which it follows $\exists x\ P(x,a)$, or

$$\forall x\ \exists z\ [z \geq x\ \wedge\ \forall y \leq z\ \neg P(y,a)]$$

is true, from which it follows $\forall x\ \neg P(x,a)$.

2. Suppose that $n \in \mathbb{N}$ and $s(n,\cdot) \in F$. We assume that s is recursive and there is an $r \in \mathbb{N}$ such that for every $x,\ y \in \mathbb{N}$

$$x \leq y \implies s(n,x) \leq s(n,y) \leq r\,.$$

We prove that there exists m such that for every $y \geq m$ we have $s(n,m) = s(n,y)$. The proof is by induction on r. If $r = 0$ then $s(n,x) = 0$ for each x, hence we choose $m = 0$. Supposing the thesis holds for r, we prove the thesis for $r+1$ using EM_1. For EM_1, either there is m such that $s(n,m) = r+1$, or not. In the first case by monotonicity we have that for every x, $m \leq x$ implies $r+1 = s(n,m) \leq s(n,x) \leq r+1$, then $s(n,x) = r+1$ for every $x \geq m$. In the second case, we have $s(n,x) \leq r$ for each $x \in \mathbb{N}$, we apply the induction hypothesis and deduce the thesis. We need to use only one statement of EM_1

$$\forall n \forall r\ (\exists m(s(n,m) = r+1)\ \vee\ \forall m(s(n,m) \neq r+1))\,,$$

which implies all the formulas in EM_1 used in the proof.

3. Let $s(n,x)$ be the number of $y < x$ such that $t(n,y) \neq t(n,y+1)$. $s(n,\cdot)$ is monotone by construction. Since the number of changes of value of $t(n,\cdot)$ is bounded by some $r \in \mathbb{N}$ then $s(n,\cdot)$ is bounded by the same r. Moreover $\{s(n,\cdot) \mid n \in \mathbb{N}\}$ is enumerated by a primitive recursive function, since G has this property. So $s(n,\cdot)$ is stationary from a certain m onwards thanks to the second part of this Lemma. From the same point m even $t(n,\cdot)$ is stationary.

◀

We may now prove the main result of this section:

▶ **Theorem 2.** $\mathrm{RT}^2_2(\Sigma^0_0)$ *implies* Σ^0_3-LLPO.

Proof. Let a be a parameter, we assume the hypothesis of Σ^0_3-LLPO:

$$\forall x,\ x'\ (H_0(x,a) \vee H_1(x',a)),$$

where

$$H_0(x,a) := \exists y\ \forall z\ P_0(x,y,z,a)$$

$$H_1(x,a) := \exists y\ \forall z\ P_1(x,y,z,a)$$

for some $P_0,\ P_1$ primitive recursive predicates. In order to prove

$$\forall x\ H_0(x,a) \vee \forall x\ H_1(x,a)$$

we define a recursive 2-coloring such that:

- if there are infinitely many white (0) edges from x, then

$$H_0(0,a) \wedge \cdots \wedge H_0(x,a);$$

- if there are infinitely many black (1) edges from x, then

$$H_1(0,a) \wedge \cdots \wedge H_1(x,a).$$

Given x and m, where $m > x$, the color of $\{x,m\}$ expresses a conjecture based on a limited study of the predicates $H_i(x,a)$. White represents the hypothesis that $H_0(0,a)\wedge\cdots\wedge H_0(x,a)$ is true, after the analysis of the statements $H_0(0,a),\ \ldots,\ H_0(x,a)$ with quantifiers restricted to the set $[0,m]$. Vice versa, black represents the hypothesis that $H_1(0,a) \wedge \cdots \wedge H_1(x,a)$ is true, after the analysis of the statements $H_1(0,a),\ \ldots,\ H_1(x,a)$ with quantifiers restricted to the set $[0,m]$.

The coloring, and so the current hypothesis, is defined as follows. For every $n \in \mathbb{N}$ we define a primitive recursive function

$$y^a_n(m,c) : \mathbb{N} \times 2 \to m+1$$

that returns the minimum $y \leq m+1$ such that

$$\forall z \leq m\ P_c(n,y,z,a),$$

if such y exists. If such y does not exist then $y^a_n(m,c) = m$.

Note that $y^a_n(m,c)$ is weakly increasing: if $y^a_n(m+1,c) \leq m$, then by definition

$$\forall z \leq m+1\ P_c(n, y^a_n(m+1,c), z, a),$$

thus trivially

$$\forall z \leq m \; P_c(n, y_n^a(m+1,c), z, a)$$

follows and hence by construction $y_n^a(m,c) \leq y_n^a(m+1,c) \leq m$; on the other hand if $y_n^a(m+1,c) = m+1$ we obtain $y_n^a(m+1,c) > m \geq y_n^a(m,c)$.

For all $x \in \mathbb{N}$ define a sequence $C_x : \mathbb{N} \to 2$, where, for all $m > x$, $C_x^a(m)$ will be the color of the edge $\{x, m\}$.

$C_x^a(m) = c$ expresses that, analysing the interval $[0, m]$, $H_c(0,a) \wedge \cdots \wedge H_c(x,a)$ is believed to be true. The definition of $C_x^a(m)$ is given by induction on m.

- $C_x^a(0) = 0$;
- if for all $n \leq x$ $y_n^a(m, C_x^a(m)) = y_n^a(m+1, C_x^a(m))$ then $C_x^a(m+1) = C_x^a(m)$, else $C_x^a(m+1) = 1 - C_x^a(m)$.

We paint the edge $\{x, m\}$ with color $C_x^a(m)$. Now we want to prove that for some m_0 and for all $m \geq m_0$, that $C_x^a(m)$ is stationary, that $y_n^a(m,c)$ is stationary for every $n \leq x$, and that $y = y_n^a(m,c)$ is a witness of

$$H_c(n,a) := \exists y \; \forall z P_c(n,y,z,a) \, .$$

As a matter of fact we supposed:

$$\forall n, \; n' \leq x(H_0(n,a) \vee H_1(n',a)) \, .$$

Hence we can constructively prove that witnesses exist either for $H_0(0,a) \wedge \cdots \wedge H_0(x,a)$ or for $H_1(0,a) \wedge \cdots \wedge H_1(x,a)$, so there exist $d_1, \; d_2, \; \ldots, \; d_x$ such that either for all $n = 0, \ldots, x$

$$\forall z \; P_0(n, d_n, z, a)$$

or for $n = 0, \; \ldots, \; x$

$$\forall z \; P_1(n, d_n, z, a).$$

In the first case we have

$$y_0^a(m,0) \leq d_0, \; \ldots, \; y_x^a(m,0) \leq d_x$$

for each m, so, thanks to the first and the second part of Lemma 1, the recursive sequences $(y_0^a(m,0), \; \ldots, \; y_x^a(m,0))$ are stationary. In the other case we have

$$y_0^a(m,1) \leq d_0, \; \ldots, \; y_x^a(m,1) \leq d_x$$

for each m, so, as above, the recursive sequence $(y_0^a(m,1), \; \ldots, \; y_x^a(m,1))$ are stationary. Moreover the sequences $(y_0^a(m,c), \; \ldots, \; y_x^a(m,c))_{m \in \mathbb{N}}$ with $c < 2$ increase in at least one component every second change of color. Since one of these is stationary, from a point onwards there could be only one change of color, so the number of change of values of $C_x^a(m)$ is bounded above. Thanks to the first and the third part of Lemma 1 the sequence $C_x^a(m)$ is stationary, for each $x \in \mathbb{N}$.

Now we need to prove that if there exists m_0 such that for all $m \geq m_0$ $C_x^a(m) = c$, then $H_c(0,a) \wedge \cdots \wedge H_c(x,a)$. In this case, by definition of $y_n^a(\cdot, c)$, there exist $e_0, \; \ldots, \; e_x$ such that $y_n^a(m,c) = e_n$ for all $n = 0, \; \ldots, \; x$. It follows that

$$\forall z \leq m \; P_c(n, e_n, z, a)$$

for each $n \leq x$, $m \geq m_0$, hence

$$\forall z\ P_c(n, e_n, z, a)$$

for every $n \leq x$, and thus $H_c(n, a)$ for all $n \leq x$.

Applying $\mathrm{RT}^2_2(\Sigma^0_0)$, there exists an infinite homogeneous set X. Hence if X is homogeneous in color c, and $x \in X$, then by stationarity of $C^a_x(m)$ every edge $\{x, m\}$ is of color c, except for a finite number of cases. Thus $H_c(0, a) \wedge \cdots \wedge H_c(x, a)$ for each $x \in X$ and so for infinitely many x. We obtain

$$\forall x\ H_c(x, a)\,.$$

In order to obtain an implication between schemata, observe that only three finite sets of statements in $\mathrm{RT}^2_2(\Sigma^0_0)$ are required in the proof: the statement that corresponds to the coloring of the edges and finitely many statements which corresponds to the two uses of Lemma 1 in the previous page. ◀

4 The Limited Lesser Principle of Omniscience for Σ^0_3 formulas implies Ramsey Theorem for pairs and recursive coloring

In this section we modify Jockusch's proof of Ramsey Theorem [10] in order to obtain a proof in HA of Σ^0_3-LLPO $\implies$ $\mathrm{RT}^2_2(\Sigma^0_0)$. It is enough to prove that if $\{c_a \mid a \in \mathbb{N}\}$ is a recursive family of recursive colorings, a finite number of statement in Σ^0_3-LLPO imply that there are predicates $W(., c)$ and $B(., c)$ such that,

$$\forall a(W(., c_a) \text{ infinite and homogeneous } \vee B(., c_a) \text{ infinite and homogeneous })\,.$$

We first sketch Jockusch's proof of RT^2_2 (which is itself a modification of Erdős Rado proof of RT^2_2): it consists in defining a suitable infinite binary tree J. We first remark that RT^1_2 (Ramsey Theorem for colors and points of $\mathbb{N}$) is nothing but the Pigeonhole Principle: indeed, if we have a partition of $\mathbb{N}$ into two colors, then one of the two classes is infinite. We informally prove now RT^2_2 from RT^1_2. Fix any coloring $f : [\mathbb{N}]^2 \to 2$ of all edges of the complete graph having support $\mathbb{N}$. If X is any subset of $\mathbb{N}$, we say that X defines a 1-coloring of X if for all $x \in X$, any two edges from x to some y, z in X have the same color. If X is infinite and defines a 1-coloring, then, by applying RT^1_2 to X we produce an infinite subset Y of X whose points all have the same color c, that is, such that all edges from all points of X all have the color c. Thus, a sufficient condition for RT^2_2 is the existence of an infinite set defining a 1-coloring. In fact we need even less. We say that a tree V included in the graph $\mathbb{N}$ defines a 1-coloring w.r.t. V if for all $x \in V$, for any two proper descendants y, z of x in V, the edges x to y, z have the same color. Assume there exists some infinite binary tree V defining a 1-coloring w.r.t. V. Then V has some infinite branch B by König's Lemma. B is a total order in V, therefore B is a complete subgraph of $\mathbb{N}$. Thus, B defines an infinite 1-coloring over the points of B, and proves RT^2_2. Therefore a sufficient condition for RT^2_2 is the existence of an infinite binary tree V defining a 1-coloring w.r.t. V. Erdős Rado proof, Jockusch's proof and our proof differ in the definition of V, even if the general idea is similar.

▶ **Theorem 3.** Σ^0_3-LLPO *implies* $\mathrm{RT}^2_2(\Sigma^0_0)$ *in* HA.

Proof. We consider Jockusch's version of Erdős Rado proof of RT^2_2 and we modify it in order to do not use classical principles stronger than Σ^0_3-LLPO. Erdős and Rado introduce an ordering relation $\prec_E$ on $\mathbb{N}$ which defines the proper ancestor relation of a binary tree E

structure on $\mathbb{N}$. The 2-coloring on edges of $\mathbb{N}$, restricted to the set of pairs $x \prec_E y$, gives the same color to any two edges $x \prec_E y$ and $x \prec_E z$ with the same origin x. This defines a canonical 1-coloring over the nodes of E. Jockusch defines a relativization $\prec_J$ to an infinite set J included in $\mathbb{N}$ of the relation $\prec_E$, that still defines a binary tree and a 1-coloring over the nodes of J. In both proofs, an infinite homogeneous set is obtained from an infinite set of nodes of the same color in an infinite branch of the tree. In Erdős-Rado and Jockusch's proofs, the pigeonhole principle is applied to a Δ^0_3-branch obtained by König's Lemma. To formalize this proof in HA we would have to use the classical principle Σ^0_4-LLPO. Our goal is to prove $\mathrm{RT}^2_2(\Sigma^0_0)$ using the weaker principle Σ^0_3-LLPO. We will define an infinite binary tree T with order relation $\prec_T$ such that T is Π^0_1 and has exactly one infinite branch, the rightmost. T is a variant of J such that we may prove that there are infinitely many nodes of the same color in the infinite branch using only Σ^0_3-LLPO. An infinite set totally ordered for $\prec_T$ and painted on the same color will be the monochromatic set for the original graph. Moreover our proof recursively defines two monochromatic Δ^0_3-sets, one of each color, that can not be both finite, even if we can not decide which of these is the infinite one.

Let V be a subset of $\mathbb{N}$ such that $0 \in V$. Firstly define, for each subset V of $\mathbb{N}$ such that $0 \in V$, a tree structure $\prec_V$ for V, then we choose a certain set for V. More precisely, we define a relation $x \prec_V y$ for each $x \in V$ and $y \in \mathbb{N}$, that restricted to $V \times V$ will define a tree with root 0. The definition of $x \prec_V y$ is given by induction on x: at each step we use only the subset $V \cap (x+1)$ of V.

- $0 \prec_V 1$.
- $x \prec_V y$ if and only if $x \in V$ and $y \in \mathbb{N}$ and $x < y$ and for every z such that $z \prec_V x$: $\{z, x\}$ and $\{z, y\}$ have the same color.

We define a tree T choosing an infinite sequence of points $x_0,\ x_1,\ \ldots$ of $\mathbb{N}$. The Jockusch relation $\prec_J$ restricted from $J \times \mathbb{N}$ to $J \times J$ in general is different from the Erdős Rado relation $\prec_E$ restricted from $\mathbb{N} \times \mathbb{N}$ to $J \times J$, but both relations have the same properties, which hold also for our relation $\prec_V$, no matter what is $V \subseteq \mathbb{N}$. Let us briefly state them.

▶ **Lemma 4.** *Let $V \subseteq \mathbb{N}$ be any predicate of* HA*, $0 \in V$, and $\prec_V$ defined as above.*

1. $\prec_V \subseteq <$.
2. $0 \prec_V x$ *for every* $x \in \mathbb{N} \setminus \{0\}$.
3. *If* $x,\ y \in \mathbb{N}$ *and* $V \cap (x+1) = U \cap (x+1)$ *then*

$$x \prec_V y \iff x \prec_U y\,.$$

4. $\prec_V$ *is transitive.*
5. *If* $x < y \prec_V z$ *and* $x \prec_V z$ *then* $x \prec_V y$.
6. *Let* $z \in \mathbb{N}$. *The relations* $<$ *and* $\prec_V$ *describe the same order on*

$$\mathrm{pd}_V(z) := \{x \in V \mid x \prec_V z\}\,,$$

i.e. for each $x, y \in \mathrm{pd}_V(z)$

$$x < y \iff x \prec_V y\,.$$

Proof. 1. It follows from the definition of $\prec_V$.
2. It follows from definition of $\prec_V$ and from the fact that does not exist a natural number z such that $z \prec_V 0$, since for the first point we should have $z < 0$.

3. Prove by induction on x. For $x = 0$ it follows from the second point. Suppose that it is true for each $z < x$. Prove $\Rightarrow$. Assume $x \prec_V y$, then by definition

$$x \in V \ \wedge \ y \in \mathbb{N} \ \wedge \ \forall z \prec_V x \ c_a(\{z, x\}) = c_a(\{z, y\}) .$$

By hypothesis it follows that $x \in U$, since

$$V \cap (x + 1) = U \cap (x + 1) ,$$

and thus, by induction hypothesis on $z < x$ and by $V \cap (z + 1) = U \cap (z + 1)$, we obtain

$$z \prec_V x \iff z \prec_U x ,$$

hence

$$x \in U \ \wedge \ y \in \mathbb{N} \ \wedge \ \forall z \prec_U x \ c_a(\{z, x\}) = c_a(\{z, y\}) ;$$

i.e. $x \prec_U y$. The proof of the vice versa is analogous.
4. $(x \prec_V y) \wedge (y \prec_V z) \implies x \prec_V z$.
By induction on z. For $z = 0$ it is true since $x, \ y \prec_V 0$ is false. Assume that the transitivity holds for all $z' < z$ and that

$$x \prec_V y \wedge y \prec_V z ,$$

then, by definition and by inductive hypothesis on $y < z$,

$$\forall w \prec_V x \ (w \prec_V y \ \wedge \ c_a(\{w, x\}) = c_a(\{w, y\}) = c_a(\{w, z\})) ,$$

we conclude $x \prec_V z$ by the definition of V.
5. By induction on x. If $x = 0$ it is trivial. Assume that it is true for each $t < x$ and we prove it for x. Observe that $x \in V$, $y \in V$ and $z \in \mathbb{N}$. Since $x \prec_V z$, we have that

$$\forall t \prec_V x \ c_a(\{t, x\}) = c_a(\{t, z\}) ,$$

and since $y \prec_V z$ we obtain

$$\forall t' \prec_V y \ c_a(\{t', y\}) = c_a(\{t', z\}) .$$

Since $x < y$, in order to prove $x \prec_V y$ it suffices to show that

$$\forall t \prec_V x (t \prec_V y) .$$

Let $t \prec_V x$, then $t \prec_V x \prec_V z$ and so, thanks to transitivity, we obtain $t \prec_V z$. Since we have $t < x < y \prec_V z$ and $t \prec_V z$, then $t \prec_V y$ by induction hypothesis. Therefore $x \prec_V y$.
6. $(\Leftarrow)$ follows from the first property. $(\Rightarrow)$. Let x, y be such that $x, y \prec_V z$ and $x < y$. Then, thanks to point 5 and since $x < y \prec_V z$ and $x \prec_V z$, we have $x \prec_V y$.

◀

By the sixth point of Lemma 4, the relation $\prec_V$ defines a total order on $\mathrm{pd}_V(z)$ for each $z \in V$; by the second point of Lemma 4 we have $0 \in \mathrm{pd}_V(z)$ if $z > 0$. Hence $\prec_V$ defines a tree with root 0 (we say that $\prec_V$ is the father/child relation).

It remains to choose a particular tree T definable by a predicate of HA, to use it in the proof of Ramsey Theorem. Define, by induction on n, the set of the first $n + 1$ nodes of T:

$$T_n := \{x_0, \ \ldots, \ x_n\} .$$

As auxiliary parameter we define a color c_n in $\{0,1\}$ as follows: if $n=0$ then $c_n=0$ and if $n>0$ then $c_n = c(\{\text{Father}(x_n), x_n\})$. The next edge added to T_n, if possible, should come from x_n and have color c_n. The proof of correctness of the definition of T requires the law of Excluded Middle of level 1, which is a consequence of Σ^0_3-LLPO (see [1]). T is a finite conjunction of decidable statements or simply universal statements and so it is Π^0_1.

The next node x_{n+1} of T is the first natural number z which satisfies the predicate we call "First Choice", or, if none exists, the first which satisfies the predicate we call "Second Choice".

- z is a first choice node after T_n if z is greater than x_n in the relation defined by T_n, and the edge from x_n to z has color c_n;

  $$\text{FirstChoice}(z, T_n) := z \succ_{T_n} x_n \wedge c(\{z, x_n\}) = c_n\,.$$

  $\text{FirstChoice}(z, T_n)$ is decidable.
- z is a second choice node after T_n if z is the first node greater than some ancestor x_p of x_n in the relation defined by T_n, and for no proper descendant of x_p and ascendant of x_n there is such a z.

  $$\text{SecondChoice}(z, T_n) := \exists p < n+1\{[z \succ_{T_n} x_p \wedge \forall y < z(y > x_n \Rightarrow y \not\succ_{T_n} x_p)] \\ \wedge \forall h \leq n[(h \geq p+1 \wedge x_h \succ_{T_n} x_p \wedge x_n \not\succeq_{T_n} x_h) \Rightarrow \forall w(w > x_n \Rightarrow w \not\succ_{T_n} x_h)]\}.$$

  $\text{SecondChoice}(z, T_n)$ is Π^0_1.

Formally, z is the chosen node after T_n either if z is the minimal first choice node, or if there are not first choice nodes and z is the unique second choice node;

$$\text{Chosen}(\{z, T_n\}) := [\text{FirstChoice}(z, T_n) \wedge \forall y < z \neg\, \text{FirstChoice}(y, T_n)] \\ \vee [\forall y \neg\, \text{FirstChoice}(y, T_n) \wedge \text{SecondChoice}(z, T_n)].$$

$\text{Chosen}(z, T_n)$ is Π^0_1. We informally define the tree T, then we translate its definition in HA.

▶ **Definition 5** (Informal definition of T)**.** We informally define T_n by induction on n.
- If $n = 0$ then $T_0 = x_0 := 0$.
- For $n+1$, if $\text{Chosen}(x_{n+1}, T_n)$, then $T_{n+1} = T_n \cup \{x_{n+1}\}$.
  $T = \bigcup_{n \in \mathbb{N}} T_n\,.$

The definition 5 of T (which is not yet a definition in HA) uses EM_1, in other words an oracle for the properties Σ^0_1, hence T is a Δ^0_2 tree. We may represent in HA by some Π^0_1 predicates: "$x_0, \ldots, x_n$ are the first n nodes of T" and $x \in T$.

▶ **Definition 6** (Formal definition of T)**.**
- "$x_0, \ldots, x_n$ are the first n nodes of T" is the predicate of HA:

  $$(x_0 = 0) \wedge \forall i < n\, \text{Chosen}(x_{i+1}, \{x_0, \ldots, x_i\})$$

- "x is a node of T" is the predicate of HA:

  $$\text{Node}(x) := \exists n < x \exists x_0, \ldots, x_n < x(\text{Chosen}(x, \{x_0, \ldots, x_n\}) \wedge \\ \text{“}x_0, \ldots, x_n \text{ are the first } n \text{ nodes of } T\text{”})\,;$$

Both predicates are Π^0_1. Now, we are going to prove that T of definition 6 satisfies the requirements of definition 5.

▶ **Lemma 7.** *If T is the tree defined by definition 5, every occurrence of the relation $\prec_{T_n}$ in* FirstChoice *and* SecondChoice *can be replaced by an occurrence of the relation $\prec_T$.*

Proof. Just see that the definition guarantees that for each n

$$T_n \cap (x+1) = T \cap (x+1)\,,$$

for each $x \in T_n$. Thus, applying the third point of Lemma 4, for every $x \in T_n$ and for every $y \in \mathbb{N}$

$$x \prec_{T_n} y \iff x \prec_T y.$$ ◀

The fact that T of definition 6 satisfies the requirements of definition 5 is a consequence of the uniqueness of the chosen node.

▶ **Lemma 8.** *For each n there exists a unique z such that* Chosen(z, T_n).

Proof. The uniqueness follows since we choose either the minimal first choice node, or, if it does not exist, the unique second choice node. The existence is a consequence of the EM_1 statement:

$$\forall z \neg\, \mathrm{FirstChoice}(z, T_n) \vee \exists z \mathrm{FirstChoice}(x, T_n)\,.$$

If there exists z which satisfies FirstChoice(z, T_n) then z is the chosen node, otherwise we prove that the second choice node exists. As a matter of fact, thanks to Σ^0_3-LLPO, EM_1 holds; and, by EM_1, we may prove in HA that either there is a first z such that $z \succ_T x_n$, a statement we may write as $\phi(x_n)$:

$$\phi(x) := \exists z((z \succ_T x) \wedge \forall y < z(y > x \implies y \not\succ_T x)$$

or for all z, $z \succ_T x_n$ is false, a statement we may write as $\psi(x_n)$, where:

$$\psi(x) := \forall z(z \not\succ_T x)\,.$$

Informally, if $\phi(x_n)$, i.e. if x_n has a first child z greater than x_n, we chose z. On the other hand, if $\psi(x_n)$, i.e., if x_n has no child z greater than x_n, we can decide if the father x_p of x_n has got a child greater than x_n or not, and so on. In the worst case we arrive at the root 0, which has at least the child $x_n + 1$, which is $> x_n$.

Formally, we have to prove the following formula:

$$\exists x \le x_n(\forall y \le x_n((y > x \wedge y \prec_{T_n} x_n) \implies \psi(y)) \wedge (x \preceq_{T_n} x_n) \wedge \phi(x));$$

which follows by the maximalization principle applied to the list $0 = x_{n_0}, \ldots, x_{n_p} = x_n$ of ancestors of x_n, and by $\phi(x_{n_0})$ and $\forall x.\phi(x) \vee \psi(x)$. ◀

Observe that the construction of the tree required one instance of two formulas of the EM_1 schema with different parameters. Each formula in EM_1 used in the proof above of Lemma 8 is an instance of one of the following formulas:

$$\forall n \forall \langle x_0, \ldots, x_n, c_n\rangle(\forall x \neg\, \mathrm{FirstChoice}(x, T_n) \vee \exists x \mathrm{FirstChoice}(x, T_n))\,,$$

and

$$\forall x(\exists z((z \succ_T x) \wedge \forall y < z(y > x \implies y \not\succ_T x) \vee \forall z(z \not\succ_T x)\,.$$

So only two statements of Σ^0_3-LLPO (the ones that imply the above formulas in EM_1) are sufficient in order to prove the existence of the tree.

Let r_n the branch in T_n that ends with x_n.

$$r_n = \{x_{i_0}, \ldots, x_{i_m}\},$$

where $x_{i_0} = 0$ and $x_{i_m} = x_n$. We describe how r_n grows. If the z which satisfies $\mathrm{Chosen}(z, T_n)$ is such that $\mathrm{FirstChoice}(z, T_n)$ then $r_{n+1} = r_n \cup z$, while if it satisfies $\mathrm{SecondChoice}(z, T_n)$ then there exists $x_p \in T_n$ such that $z \succ_{T_n} x_p$ moreover for every $y > x_n$ and for each $h > p$ such that x_h is in r_n between x_{p+1} and x_n, $y \succ_{T_n} x_h$ does not hold. Observe that since $x_n \succcurlyeq_{T_n} x_p$, we have $x_p \in r_n$. From this characterization of r_n we deduce:

▶ **Lemma 9.** *Let T be the tree defined above, and $x,\ y,\ z \in \mathbb{N}$.*
1. *All nodes of T having descendants after x_n are in r_n: if $x_i \in T_n$, $z > x_n$, and $z \succ_{T_n} x_i$, then $x_i \in r_n$.*
2. *If $x \in T$ has two children $y,\ z \in T$, with $y < z$ then y has no descendants in T which are $> z$.*

Proof. **1.** We prove the statement for all z, i by induction on n. If $n = 0$ it is trivial. Now suppose that the thesis is true for n and prove it for $n + 1$. Let r_{n+1} be the branch of T_{n+1} that ends with x_{n+1}. We have to check that for each $x_k \in T_{n+1} \smallsetminus r_{n+1}$, there are no $y \succ_T x_k$ such that $y > x_{n+1}$. By definition of T, we have $r_{n+1} \cap x_n = \{x_{i_0}, \ldots, x_{i_q}\}$, where x_{i_q} is the x_p of the predicate SecondChoice. Thus, if $x_k \in T_{n+1} \smallsetminus r_{n+1}$, there are two possibilities left: either $x_k \in \{x_{i_{q+1}}, \ldots, x_{i_m}\}$, or $x_k \in T_n \smallsetminus r_n$. In the first case, by the choice of x_p there is not any $y > x_n$ such that

$$y \succ_T x_{i_m} \ \vee \ \ldots \ \vee \ y \succ_T x_{i_{q+1}}.$$

Even more so, there is not any $y > x_{n+1} > x_n$ such that

$$y \succ_T x_{i_m} \ \vee \ \ldots \ \vee \ y \succ_T x_{i_{q+1}}.$$

In the second case, by induction hypothesis, for every $x_k \in T_n \smallsetminus r_n$ there do not exist any $y \succ_T x_k$ for which $y > x_n$, hence there are not any $y \succ_T x_k$ for which $y > x_{n+1} > x_n$.

2. Assume $z = x_{n+1}$ is the node chosen by some $T_n = \{x_0,\ \ldots,\ x_n\}$. x has a child $y < z$ in T, therefore some child $y \in T_n$, hence $x \neq x_n$ because x_n is a leaf in T_n. z is a child of x in T, therefore, by definition of Chosen, z is a second choice node with $x_p = x$ for some $p < n$. By definition of $\mathrm{SecondChoice}(z, T_n)$ we have

$$y \succ_T x \wedge x_n \succ_T y \Rightarrow \forall w(w > x_n \Rightarrow w \not\succ_T y).$$

Since $z > x_n$ we obtain

$$\forall w(w > z \Rightarrow w \not\succ_T y).$$ ◀

Moreover we need to prove that the tree T is a binary tree: each node has at most two children.

▶ **Lemma 10.** *Let T be the predicate from definition 6, defining a tree.*
1. *The following is a sufficient condition for $x \prec_T y$. If $i,\ x \in T$ and $y \in \mathbb{N}$ are such that x is an immediate successor of i with respect to the relation $\prec_T$, $i \prec_T y$, $x < y$ and $c_a(\{i, x\}) = c_a(\{i, y\})$, then $x \prec_T y$.*
2. *Each node i of T has at most one child x such that $\{i, x\}$ is black, and at most one child y such that $\{i, y\}$ is white.*

Proof. 1. By hypothesis we have that

$$\forall t \prec_T i \; c_a(\{t,i\}) = c_a(\{t,x\})$$

and

$$\forall t \prec_T i \; c_a(\{t,i\}) = c_a(\{t,y\}),$$

so we have

$$\forall t \prec_T i \; c_a(\{t,x\}) = c_a(\{t,y\}). \tag{1}$$

Since x is an immediate successor of i,

$$t \prec_T x \iff t \prec_T i \lor t = i$$

by formula 1 and by the hypothesis $c_a(\{i,x\}) = c_a(\{i,y\})$, we obtain the thesis $x \prec_T y$.

2. Let $i \in T$ and let x and y be two children of i. Then we have that $x \prec_T y$ and $y \prec_T x$ are false, otherwise we should have $i \prec_T x \prec_T y$ and $i \prec_T y \prec_T x$. By point 1 above, since $x < y$ or $y < x$, it follows that $c(\{i,x\}) \neq c(\{i,y\})$. Therefore the number of children must be lesser than the number of colors, i.e. two. ◀

The tree T is infinite by construction and is binary by Lemma 10.2. We are going to prove, using EM_2 (that is a consequence of Σ^0_3-LLPO, [1]), that each node with infinitely many descendants has at least one child with infinitely many descendants, then that each node with infinitely many descendants has exactly one child with infinitely many descendants. This implies that T has exactly one infinite branch, which, to be accurate, is the rightmost branch of T, if we order children according to their integer value.

Observe that, by the definition of the tree, we have that, given a node t with infinitely many descendants, his first child has infinitely many descendants if and only if the first child is also the unique child (see Lemma 9.2). We define the uniqueness of the children of x as follows:

$$\mathrm{Unique}(x) := \forall x \forall z((\mathrm{Child}(x,t) \land \mathrm{Child}(z,t)) \implies x = z),$$

where

$$\mathrm{Child}(x,t) := \exists n < t \exists x_0, \ldots, x_n < t$$

$$(\text{“}x_0, \ldots, x_n, t, x \text{ are the first } n+2 \text{ nodes of } T\text{”}).$$

This is an assertion Π^0_2, since Child is Π^0_1. Indeed, using EM_1, we can transform the occurrence of $\mathrm{Child}(x,t)$ in $\mathrm{Unique}(x)$ in a Σ^0_1 formula and the whole predicate $\mathrm{Unique}(x)$ in a Π^0_2 formula. If we apply EM_2 to $\mathrm{Unique}(x)$ we deduce that either that t has at most one child, or there exist two different children x and z of t. In the first case the first node x_{n+1} chosen after $t = x_n$ in T is a child of t, otherwise, by definition of T_{n+1}, t would not belong to the rightmost branch r_{n+1} of T_{n+1}, and by Lemma 9.1, t would not have descendants. So the node x is the unique child of t, and the infinitely many descendants of t are descendants of x. In the second case if $x < z$ are two children of t then z is the second child of t. Since we proved that a node has at most two children and by the definition of T, every descendant of t grater of z is descendant also of z, otherwise from a point onward t would not have descendants. Hence the second child of t, z, has infinitely many descendants. Observe that only one statement of Σ^0_3-LLPO is sufficient in order to prove that “t has only one child or not" for every $t \in T$; as a matter of fact we need the formula in Σ^0_3-LLPO that implies the following formula in EM_2

$$\forall t(\forall x \forall z((\mathrm{Child}(x,t) \wedge \mathrm{Child}(z,t)) \implies x = z) \vee$$

$$\neg(\forall x \forall z((\mathrm{Child}(x,t) \wedge \mathrm{Child}(z,t)) \implies x = z)))\,.$$

We prove now that the infinite branch exists, is unique and define two monochromatic sets, where at least one is infinite. Now define r as follows; we say that $x \in r$ if and only if

$$\mathrm{InfiniteBranch}(x) \iff \forall y > x(\mathrm{Node}(y) \Rightarrow x \prec_T y).$$

▶ **Lemma 11.** *Let T be the tree defined above.*
1. *T has a unique infinite branch, r, the rightmost branch, which consists of all and only the nodes with infinitely many descendants.*
2. *If T has infinitely many edges with color c, then r has infinitely many edges with color c.*

Proof. **1.** Thanks to the second part of Lemma 9, if a node has two children the first child has not got descendants greater than the second one, and therefore each node of T has at most one immediate infinite subtree. Since we have just proved the existence of the infinite subtree, it follows that each node of T that has infinitely many descendants is a root of a infinite subtree that has exactly one infinite subtree. Then the set of nodes with infinite children in T, which includes the root because T is infinite, has exactly one child for each node, and then defines the only infinite branch r of T.

2. Let $r = \{x_{i_0}, \ldots, x_{i_n}, \ldots\}$ be the unique infinite branch of T. Suppose that T has infinitely many edges of color c and prove that r has infinitely many edges of color c. Consider any node x_{i_p} of r, we want to prove that r has an edge of color c below x_{i_p}. If $\{x_{i_p}, x_{i_{p+1}}\}$ has color c we are done. Suppose it has color $1-c$: then $c_{i_{p+1}} = 1-c$. By hypothesis, there exists n such that $n \geq i_{p+1}$ and there exists $m < n$ such that $\{x_m, x_n\}$ has color c. Since r is infinite, there exists q such that $i_q \geq n+1 > n \geq i_{p+1}$. We prove that at least one of the edges

$$\{x_{i_{p+1}}, x_{i_{p+2}}\},\ \ldots,\ \{x_{i_{q-1}}, x_{i_q}\}$$

has color c. Suppose by contradiction that they all have color $1-c$ (we are using Excluded Middle over a decidable statement about the colors of finitely many edges). In this case, for every $k \in [p+1, q-1]$ there exists $y > x_{i_{k+1}-1} \geq x_{i_k}$ such that $y \succ_T x_{i_k}$ and $\{y, x_{i_k}\}$ has color $1-c$, since $\{x_{i_k}, x_{i_{k+1}}\}$ has color $1-c$; so there exists a first choice node. Since for each such k there is a first choice node (with color $1-c$), it follows that between i_{p+1} and i_q the tree T grows keeping $c_{i_k} = 1-c$ and only along the branch r. So we do not add the edge $\{x_m, x_n\}$ of color c between i_{p+1} e i_q, contradiction. ◀

We have still to prove that, indeed, the infinite branch of T has infinitely many pairs $x \prec_T y$ of color c. By Lemma 11.2, it is enough to prove that T has infinitely many pairs $x \prec_T y$ of color c, for some c. $\prec_T$ is a Π^0_1 predicate. Thus, if we apply the infinite pigeonhole principle for Π^0_1 predicates, we deduce that T either has infinite white edges, or has infinitely many black edges. However, the pigeonhole principle for Π^0_1 predicates is a classical principle, therefore we have to derive the particular instance we use from Σ^0_3-LLPO.

▶ **Lemma 12.** Σ^0_3-LLPO *implies the infinite pigeonhole principle for Π^0_1 predicates.*

Proof Lemma 12. The infinite pigeonhole principle for Π^0_1 predicates can be stated as follows:

$$\forall x\ \exists z\ [z \geq x\ \wedge\ (P(z,a) \vee Q(z,a))]$$

$$\implies \forall x\ \exists z\ [z \geq x\ \wedge\ P(z,a)]\ \vee\ \forall x\ \exists z[z \geq x \wedge Q(z,a)]\,,$$

with P and Q Π^0_1 predicates. We prove that the formula above is equivalent in HA to some formula of Σ^0_3-LLPO. Let

$$H(x,a) := \exists z\ [z \geq x \wedge P(z,a)]$$
$$K(x,a) := \exists z\ [z \geq x \wedge Q(z,a)].$$

In fact both H and K are equivalent in HA to Σ^0_2 formulas $H',\ K'$. By intuitionistic prenex properties (see [1])

$$\exists z[z \geq x\ \wedge\ (P(z,a) \vee Q(z,a))]$$

is equivalent to

$$\exists z[z \geq x\ \wedge\ P(z,a)]\ \vee\ \exists z[z \geq x\ \wedge\ Q(z,a)]\,.$$

The formula above is equivalent to $H' \vee K'$. Thus, any formula of pigeonhole principle for Π^0_1 with $H,\ K$ is equivalent to the instance of Σ^0_3-LLPO with $H',\ K'$. ◀

Thus, there exist infinitely many edges of r in color c. Their smaller nodes define a monochromatic set for the original graph, since given an infinite branch r and $x \in r$, if there exists $y \in r$ such that $x \prec_T y$ and $\{x,y\}$ has color c, then for every $z \in r$ such that $x \prec_T z$, the edge $\{x,z\}$ has color c. Thus we can devise a coloring on r, given color c to x if $\{x,y\}$ has color c, with y child of x in r. After that, every infinite set of points with the same color in r defines an infinite set with all edges of the same color, and then it proves Ramsey Theorem in HA starting from the assumption of Σ^0_3-LLPO. ◀

Observe that the infinite branch r is Π^0_2. Moreover r can not be Δ^0_2. Here we prove it classically for short. Suppose by contradiction that r is Δ^0_2. In this hypothesis we will prove that for each recursive coloring there exists an infinite homogeneous set Δ^0_2. Indeed, using the fact that all edges from the same point of r to another point of r have the same color, we may describe the homogeneous set of color $c = 0, 1$ as the set of points whose edges to any other point of r all have color c:

$$\text{HomSet}(y) \iff y \in r \wedge \forall z > y(\text{InfiniteBranch}(z) \implies c(\{y,z\}) = c)$$

and also as the set of points having some edge to another point of r of color c:

$$\text{HomSet}(y) \iff y \in r \wedge \exists z > y(\text{InfiniteBranch}(z) \wedge c(\{y,z\}) = c)\,.$$

Therefore, if r is Δ^0_2 then the first formula is Π^0_2 and the second one is Σ^0_2. So for any $c = 0, 1$ the homogeneous set is Δ^0_2. Since at least one of these sets is infinite and since Jockusch proved that exists a coloring of $[\mathbb{N}]^2$ that has no infinite homogeneous set Σ^0_2, we obtain a contradiction. So $r \notin \Delta^0_2$ in general.

In Jockusch's proof he shows that one of the homogeneous sets (the red one in his notation) is Π^0_2, since at the beginning of each step he looks for red edges; while the second one is Δ^0_3. In our proof we can see that both the homogeneous sets are Δ^0_3, since our construction is symmetric with respect to the two colors. As a matter of fact, since r is Π^0_2, the previous two formulas are respectively Π^0_3 and Σ^0_3. This is enough in order to prove that both the homogeneous sets are Δ^0_3. There always is an infinite homogeneous set Π^0_2, but apparently the proof is purely classical and cannot compute the integer code of such Π^0_2 predicate. Again we refer to Jockusch [10] for details.

5 Conclusions

Σ^0_3-LLPO is a principle of uncommon use, but it is equivalent to König's Lemma, given function variables and choice axiom [1]. The first goal of this section is to present the equivalence between Σ^0_3-LLPO and two more common principles: EM_2 and $\mathrm{DeMorgan}(\Sigma^0_3)$. After that we present some possible future developments.

First of all we want to prove that Σ^0_n-LLPO is equivalent to the union of $\mathrm{DeMorgan}(\Sigma^0_\mathrm{n})$ and EM_{n-1}, where

$$\mathrm{DeMorgan}(\Sigma^0_\mathrm{n}) := \neg(P \wedge Q) \implies \neg P \vee \neg Q.\ (P, Q \in \Sigma^0_n)$$

$\mathrm{DeMorgan}(\Sigma^0_3)$ is a principle outside the hierarchy considered in [1] and incomparable with EM_1.

In order to prove the equivalence claimed above we need the following statements; their proof are shown in [1].

▶ **Lemma 13.** *Let* Σ^0_n-LLPO$^* := \neg(P \wedge Q) \implies P^\perp \vee Q^\perp$ *where* $P, Q \in \Sigma^0_n$*, then:*
1. Σ^0_n-LLPO *is equivalent to* Σ^0_n-LLPO**;*
2. Σ^0_n-LLPO *implies* EM_{n-1}.

Now, we can prove the equivalence. This equivalence helps us to analyse the proof of Theorem 3. Observing it, we can see that the most of the proof uses only EM_2 and that $\mathrm{DeMorgan}(\Sigma^0_3)$ (and so Σ^0_3-LLPO) is used only in the last part (Lemma 12).

▶ **Theorem 14.** Σ^0_n-LLPO $\iff \mathrm{DeMorgan}(\Sigma^0_\mathrm{n}) + \mathrm{EM}_{n-1}$.

Proof. Denote with P, Q any two Σ^0_3 formulas.

⇒. Thanks to Lemma 13 we have Σ^0_n-LLPO $\implies \mathrm{EM}_{n-1}$. We have to prove $\mathrm{DeMorgan}(\Sigma^0_\mathrm{n})$. By the first part of Lemma 13, it suffices to prove that Σ^0_n-LLPO* implies $\mathrm{DeMorgan}(\Sigma^0_\mathrm{n})$. In HA holds $P^\perp \implies \neg P$, so we obtain

$$\neg(P \wedge Q) \implies P^\perp \vee Q^\perp \implies \neg P \vee \neg Q\,.$$

⇐. Thanks to De Morgan we have $\neg(P \wedge Q) \implies \neg P \vee \neg Q$. Moreover, by EM_{n-1}, we obtain $\neg P \implies P^\perp$ [1, corollary 2.9]. So, it follows Σ^0_n-LLPO* (that is equivalent to Σ^0_n-LLPO):

$$\neg(P \wedge Q) \implies P^\perp \vee Q^\perp\,.$$ ◀

The first question that raises after this work is what is the minimal classical principle that implies $\mathrm{RT}^2_2(\Sigma^0_n)$, Ramsey Theorem for pairs in two colors, but with any Σ^0_n family of colorings. We conjecture that, modifying conveniently the proofs of Theorem 2 and Theorem 3, we should obtain

$$\Sigma^0_{n+3}\text{-LLPO} \iff \mathrm{RT}^2_2(\Sigma^0_n). \tag{2}$$

A first development of this paper might be to check of the equivalence 2, for each $n \in \mathbb{N}$.

We conjecture that the result $\mathrm{RT}^2_2(\Sigma^0_0)$ may be generalized from 2 colors to any finite number of colors, that is, to the theorem $\mathrm{RT}^2_n(\Sigma^0_0)$, for any $n \in \mathbb{N}$. Apparently, however, the proof of Theorem 3 requires non-trivial changes in the case of n colors.

In this paper we consider Ramsey Theorem as schema in order to work with first order statements. Now our idea is to study Ramsey Theorem working in HA + functions + description axiom (that is a conservative extension of HA, see [1]), in order to use only one statement to express Ramsey Theorem for pairs in two colors. It seems to us that this unique statement is still equivalent to Σ^0_3-LLPO.

As we said in the introduction, in the future we hope to apply the interactive realizability [3] in order to study the computational content of Ramsey Theorem, and to find new constructive proofs for some consequences of it. Since the use of EM_n corresponds to n nested limits in this interpretation, thanks to our results, we may state that only three nested limits suffice to formalize this proof.

A further development would be to use this equivalence in order to find the minimal classical principles which imply a given corollary of Ramsey Theorem in HA.

Moreover we may observe that our proofs are semi-formal in HA, so it could be formalized using proof assistant software, like Coq.

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