# On the Pseudoperiodic Extension of $u^\ell = v^m w^n$ *

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## Abstract

We investigate the solution set of the pseudoperiodic extension of the classical Lyndon and Schützenberger word equations. Consider $u_1 \cdots u_\ell = v_1 \cdots v_m w_1 \cdots w_n$, where $u_i \in \{u, \theta(u)\}$ for all $1 \leq i \leq \ell$, $v_j \in \{v, \theta(v)\}$ for all $1 \leq j \leq m$, $w_k \in \{w, \theta(w)\}$ for all $1 \leq k \leq n$ and $u, v$ and $w$ are variables, and $\theta$ is an antimorphic involution. A solution is called pseudoperiodic, if $u, v, w \in \{t, \theta(t)\}^+$ for a word $t$. Czeizler et al. (2011) established that for small values of $\ell$, $m$, and $n$ non-periodic solutions exist, and that for large enough values all solutions are pseudoperiodic. However, they leave a gap between those bounds which we close for a number of cases. Namely, we show that for $\ell = 3$ and either $m, n \geq 12$ or $m, n \geq 5$ and either $m$ and $n$ are not both even or not all $u_i$'s are equal, all solutions are pseudoperiodic.

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## 1 Introduction

The study of the classical word equations $u^\ell = v^m w^n$ dates back to 1962. Lyndon and Schützenberger [6] showed that for $l, m, n \geq 2$, in all solutions of this equation in a free group, $u, v, w$ are necessarily powers of a common element. Their result holds canonically if $u, v$ and $w$ are elements of a free semigroup, but, for this case, simpler proofs exist [5].

Czeizler et al. [1] introduced a generalisation of Lyndon and Schützenberger's equations of the form $u_1 \cdots u_\ell = v_1 \cdots v_m w_1 \cdots w_n$, where $u_i \in \{u, \theta(u)\}$ for all $1 \leq i \leq \ell$, $v_j \in \{v, \theta(v)\}$ for all $1 \leq j \leq m$, and $w_k \in \{w, \theta(w)\}$ for all $1 \leq k \leq n$, and studied under which conditions $u, v, w \in \{t, \theta(t)\}^+$ for some word $t$. In other words, they studied the case when $u, v, w$ are generalised powers (more precisely, $\theta$-powers). Here, $\theta$ is a function on the letters of the alphabet, which acts as an antimorphism (i.e., $\theta(uv) = \theta(v)\theta(u)$ for all words $u, v$) and as an involution (i.e., $\theta(\theta(u)) = u$ for all words $u$). These so called *antimorphic involutions* are commonly used to formally model the Watson-Crick complement occurring in DNA structures; this connection sparked the interest towards studying the combinatorial properties of words that can be expressed as catenation of factors and their image under such antimorphic involutions (see, [1]). Apart from this initial bio-inspired motivation, there is a strong intrinsic mathematical motivation in studying such words. Indeed, one of the simplest and most studied operations on words is mirroring, the very basic antimorphic involution. It is, thus, natural to study equations on words in which not only powers of variables, but also repeated concatenations of a variable and its mirror image appear.

The results obtained in [1, 4] are summarised in Table 1. One can notice easily from this table that the more interesting cases in this generalised setting are those in which $\ell, m, n \geq 3$. Moreover, when $\ell = 3$ only several negative results were found. That is, there is a series

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**Table 1** The results known so far about the equation $u_1 \cdots u_\ell = v_1 \cdots v_m w_1 \cdots w_n$.

| $\ell$ | $m$ | $n$ | $u, v, w \in \{t, \theta(t)\}^+$? |
|---|---|---|---|
| $\geq 4$ | $\geq 3$ | $\geq 3$ | Yes |
| 3 | $\geq 5$ | $\geq 5$ | Open |
| 3 | 4 | $\geq 3$ and odd | Open |
| 3 | 4 | $\geq 4$ and even | No |
| 3 | 3 | $\geq 3$ | No |
| | one of $\{\ell, m, n\}$ equals 2 | | No |

of equations which have non-periodic solutions, but very little is known about those cases of such equations where the pseudoperiodicity of the solutions is forced, similarly to the classical Lyndon-Schützenberger equations (the only exception was the particular Lemma 12, see Prop. 51 in [4]). Finally, the case $\ell = 3$ seems to be especially intricate and interesting, because it separates the cases when the equation has only $\theta$-powers as solutions ($\ell \geq 4$) from the cases when it may have solutions which are not $\theta$-powers ($\ell \leq 2$). Accordingly, our work aims to add some relevant results regarding equations with $\ell = 3$, and solves some of the open cases presented in Table 1.

We show as a main result that for $\ell = 3$ and $m, n \geq 12$ the solutions of the equations $u_1 \cdots u_\ell = v_1 \cdots v_m w_1 \cdots w_n$ must be $\theta$-powers of a common word. The same holds if $m, n \geq 5$ and not both of the values are even. To the same end, we show that if the words $u_1, u_2$ and $u_3$ are not all equal, or if $|v_1 \cdots v_m| \geq 2|u|$, then the solutions of the aforementioned equation are, again, $\theta$-powers of a common word. Our results show the surprising fact that the case of $\ell = 3$ is the only one when we have both general equations $u_1 \cdots u_\ell = v_1 \cdots v_m w_1 \cdots w_n$ that have only solutions which are $\theta$-powers, and general equations that may have solutions which are not $\theta$-powers.

As expected (see the final remarks of [1]), we applied some arguments that have not been used in this context before, but an exhaustive case analysis on the alignments of parts of the equation seems unavoidable and these arguments must be adapted to every case separately. Due to space restrictions, some of the proofs (or parts thereof) had to be omitted.

**Basic concepts.** For more detailed definitions we refer to [5]. For a finite alphabet $\Sigma$, we denote by $\Sigma^*$ and $\Sigma^+$ the set of all words and the set of all non-empty words over $\Sigma$, respectively. The empty word is denoted by $\varepsilon$ and the length of a word $w$ is denoted by $|w|$. For a word $w = uvz$ we say that $u$ is a *prefix* of $w$, $v$ is a *factor* of $w$, and $z$ is a *suffix* of $w$. We denote that by $u \leq_p w$, $v \leq_f w$, and $v \leq_s w$, respectively. If $vz \neq \varepsilon$ we call $u$ a *proper prefix*, and we denote that by $u <_p w$, and symmetrically for suffixes. Similarly, $v$ is called a *proper factor* of $w$, denoted by $v <_f w$, if $u \neq \varepsilon, z \neq \varepsilon$. A word $w$ is called *primitive*, if $w = u^k$ implies $k = 1$ and $u = w$; otherwise, $w$ is called *power* or repetition. For a word $w$, we define the word $w^\omega$ as the infinite word whose prefix of length $n|w|$ is $w^n$, for all $n \in \mathbb{N}$. Primitive words are characterised as follows:

► **Proposition 1.** *If $w$ is primitive and $ww = xwy$, then either $x = \varepsilon$ or $y = \varepsilon$.*

A word $w$ is called *$\theta$-primitive*, if $w = u_1 \cdots u_k$ with $u_i \in \{u, \theta(u)\}$ for all $1 \leq i \leq k$ implies $k = 1$ and $u = w$. A $\theta$-primitive word is primitive, but the converse does not hold, as $w = abba$ is primitive but $w = ab\theta(ab)$, for $\theta$ being the mirror image. A word $w$ is a *$\theta$-palindrome* if $w = \theta(w)$. A word that is not $\theta$-primitive is called a *$\theta$-power*. Kari et al. [4] characterised $\theta$-primitive words similarly to Proposition 1:

▶ **Lemma 1.** *For a $\theta$-primitive word $x \in \Sigma^+$, neither $x\theta(x)$ nor $\theta(x)x$ can be a proper factor of a word in $\{x, \theta(x)\}^3$.*

The results of Proposition 2 and Theorem 2 are well known (see, e.g., [5]):

▶ **Proposition 2.** *If $xz = zy$ for some words $x, y, z \in \Sigma^*$, then there exist $p, q \in \Sigma^*$, such that $x = pq, y = qp$, and $z = (pq)^i p$ for some $i \geq 0$.*

The words $x, y$ from Proposition 2 are called *conjugates*, denoted by $x \sim y$.

▶ **Theorem 2.** *If $\alpha \in u\{u, v\}^*$ and $\beta \in v\{u, v\}^*$ have a common prefix of length at least $|u| + |v| - \gcd(|u|, |v|)$, then $u, v \in \{t\}^+$ for some word $t$.*

Czeizler et al. [2] established the following two generalisations of Theorem 2:

▶ **Theorem 3.** *Let $u, v \in \Sigma^+$ with $|u| \geq |v|$. If $\alpha \in \{u, \theta(u)\}^+$ and $\beta \in \{v, \theta(v)\}^+$ have a common prefix of length at least $2|u| + |v| - \gcd(|u|, |v|)$, then $u, v \in t\{t, \theta(t)\}^+$ for some $\theta$-primitive word $t \in \Sigma^+$.*

▶ **Theorem 4.** *Let $u, v \in \Sigma^+$ with $|u| \geq |v|$. If $\alpha \in \{u, \theta(u)\}^+$ and $\beta \in \{v, \theta(v)\}^+$ have a common prefix of length at least $\operatorname{lcm}(|u|, |v|)$, then $u, v \in t\{t, \theta(t)\}^+$ for some $\theta$-primitive word $t \in \Sigma^+$.*

Harju and Nowotka [3] investigated equations that are similar to the ones by Lyndon and Schützenberger with the following result, which we use in our proofs:

▶ **Theorem 5.** *Let $n \geq 2$ and $x, z_i \in \Sigma^*$ with $|x| \neq |z_i|$ and $k, k_i \geq 3$, for all $1 \leq i \leq n$. If $x^k = z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$ and $n \leq k$, then $x, z_i \in \{t\}^*$ for some word $t \in \Sigma^*$ and all $1 \leq i \leq n$.*

## 2 Overview

As mentioned in the Introduction, we are interested in solutions of the equation

$$u_1 u_2 u_3 = v_1 \cdots v_m w_1 \cdots w_n, \tag{1}$$

where $m, n \geq 5$, $u_1, u_2, u_3 \in \{u, \theta(u)\}$, $v_j \in \{v, \theta(v)\}$ for all $1 \leq j \leq m$ and $w_k \in \{w, \theta(w)\}$ for all $1 \leq k \leq n$.

Our main results are the following theorems. The first one shows that (1) has only pseudoperiodic solutions when the sequence $v_1 \cdots v_m$ is long enough.

▶ **Theorem 6.** *If $m|v| \geq 2|u|$, then (1) implies that $u, v, w \in \{t, \theta(t)\}^+$ for some word $t$.*

A similar result is obtained when not all of $u_1, u_2$, and $u_3$ are the same.

▶ **Theorem 7.** *If $\{u_1, u_2, u_3\} = \{u, \theta(u)\}$, then (1) implies that $u, v, w \in \{t, \theta(t)\}^+$ for some word $t$.*

Finally, if $m$ and $n$ are large enough, or at least one of these values is odd, (1) has only solutions which are $\theta$-powers of the same word, with no additional restrictions on $u$, $v$ or $w$:

▶ **Theorem 8.** *If $m, n \geq 12$, then (1) implies that $u, v, w \in \{t, \theta(t)\}^+$ for some word $t$.*

▶ **Theorem 9.** *If $m$ or $n$ is odd, then (1) implies that $u, v, w \in \{t, \theta(t)\}^+$ for some word $t$.*

The proofs of these theorems follow a common pattern. We first note that it is enough to prove the statements for the case when $v$ and $w$ are $\theta$-primitive. Then we assume for the sake of a contradiction that $\theta(v) \neq w \neq v \neq \theta(w)$. In this setting, $|v| = |w|$ leads easily to a contradiction, so we assume that $|v| \neq |w|$. Further, working under the particular assumptions of each of the above theorems, we try to find a long enough factor of $u_1u_2u_3$ that reflects an alignment between some $v$ factors and some $w$ factors, allowing us to apply periodicity results like Theorems 2 or 3. In some cases, this is already enough in order to reach a contradiction: the longer word appears as a $\theta$-power. However, sometimes we only get that a (well determined) conjugate of the longer word is a $\theta$-power of the shorter one. As a final step in our proofs we show that such a situation leads to a contradiction, as well. While the first steps of these proofs are based on a deep (and, maybe, finer compared to [1,4]) analysis of the alignments between the $v$'s and $w$'s and their consequences on the form of these words, several length-related arithmetic and combinatorial arguments (that enrich the toolbox developed in [1,4]) were needed to conclude them.

One of the drawbacks of our proofs is that, although they are based on the same strategy, we were not able to reorganise them as a collection of shorter general lemmas from which the final result of each case follows. Mainly, this was because each of the cases we analyse below leads to significantly different alignments between the $v$ and $w$ factors and using them to obtain the final result in the way described above requires some particular technicalities.

Note that this paper does not address the case of equations with $\ell = 3, m = 4$, and odd $n \geq 3$, left open in [1,4] (see Table 1). We conjecture, though, that our results and proofs can be adapted to that case as well. A general result in the line of Theorem 8 remains, however, to be found both for the case when $m, n \geq 6$ such that both $m$ and $n$ are even and at least one of them is less than 12, as well as for the case $m = 4$ and odd $n \geq 3$.

## 3 The Proofs

We always assume that $v_1 = v$ and often we assume that both $v$ and $w$ are $\theta$-primitive. Otherwise, if for instance $v \in \{v', \theta(v')\}^+$ for some word $v'$, we consider the equation $u_1u_2u_3 = v'_1 \cdots v'_{m'} w_1 \cdots w_n$ instead, where $v'_i \in \{v', \theta(v')\}^+$, for all $1 \leq i \leq m'$, with $m' > m$, and similarly if $w \in \{w', \theta(w')\}^+$ for some word $w'$. Moreover, if (1) holds and two of $u, v, w$ are in $\{t, \theta(t)\}^+$ for some word $t$, then so is the third.

We split the discussion into different sections depending on the length of $v_1 \cdots v_m$. One particularly easy case follows from Theorem 3.

▶ **Lemma 10.** *If $m|v| \geq 2|u| + |v|$ and (1) holds, then $u, v, w \in \{t, \theta(t)\}^+$ for some word $t$.*

**Proof.** By Theorem 3, we instantly get that $u, v \in \{t, \theta(t)\}^+$ for some $\theta$-primitive word $t$. From this, one can get easily that $w \in \{t, \theta(t)\}^+$, as well. ◀

### 3.1 The case $2|u| < m|v| < 2|u| + |v|$: Proof of Theorem 6.

In this section, we frequently use the following results from [1].

▶ **Proposition 3** (Prop. 20 and 21 in [1]). *Let $u, v \in \Sigma^+$ so that $v$ is $\theta$-primitive, $u_1, u_2, u_3 \in \{u, \theta(u)\}$ and $v_1, \ldots, v_m \in \{v, \theta(v)\}$ for $m \geq 3$. Assume that $v_1 \cdots v_m <_p u_1u_2u_3$ and $2|u| < m|v| < 2|u| + |v|$. If $m$ is odd, then $u_2 \neq u_1$ and $v_1 = \cdots = v_m$.*
*If $m$ is even, then one of the following holds:*
1. *$u_1 \neq u_2$ and $v_1 = \cdots = v_m$, or*
2. *$u_1 = u_2$, $v_1 = \cdots = v_{\frac{m}{2}}$ and $v_{\frac{m}{2}+1} = \cdots = v_m = \theta(v_1)$.*

We split the discussion further according to every valuation of $u_1, u_2$ and $u_3$.

**Equations of the form $u\theta(u)u = v_1 \cdots v_m w_1 \cdots w_n$.** The following holds:

▶ **Lemma 11.** *If $2|u| < m|v| < 2|u| + |v|$ and $u_1u_2u_3 = u\theta(u)u$ and (1) holds, then $u, v, w \in \{t, \theta(t)\}^+$ for some word $t$.*

**Proof.** By Proposition 3 we get $v_1 \cdots v_m = v^m$. By the explanations given above, we assume that $v$ and $w$ are $\theta$-primitive.

If $2n|w| < 2|v| + |w|$, we get that $|w| < \frac{2|v|}{2n-1} < \frac{|v|}{4}$. Now, if $m = 5$, we see that $u = v^2y$ with $|y| < \frac{|v|}{2}$ and $y\theta(y) \leq_p v$. Furthermore, the part of $v_5$ that overlaps with $u_3$ is of length $|v| - 2|y|$. Hence, $v = (y\theta(y))^k v'$ for some $k \geq 1$ and $v' \leq_p y\theta(y)$. From the length of this overlap we also get that $|w_1 \cdots w_n| = (2|v| + |y|) - (|v| - 2|y|) = |v| + 3|y|$ and, as $2n|w| < 2|v| + |w|$, we have $2(|v| + 3|y|) < 2|v| + |w|$. It follows that $6|y| < |w|$, and thus $|v| > 4|w| > 24|y|$. Therefore we actually have $v = (y\theta(y))^k v'$ with $k \geq 12$. As a consequence, $\theta(y)(y\theta(y))^{k-1}v'$ is a prefix of $\theta(u)$. As $\theta(w_n) \cdots \theta(w_1)$ also is a prefix of $\theta(u)$, it has a common prefix with $\theta(y)(y\theta(y))^{k-1}v'$ of length at least $\frac{|v|}{2} + |y| > 2|w| + |y|$. So we can apply Theorem 3, and get that $w$ is not $\theta$-primitive, a contradiction. In the case $m \geq 6$ we see that $|u| \geq \frac{5|v|}{2}$ must hold, so $|w_1 \cdots w_n| \geq |u| - |v| \geq \frac{3|v|}{2}$, and thus $2n|w| \geq 3|v| > 2|v| + |w|$, a contradiction.

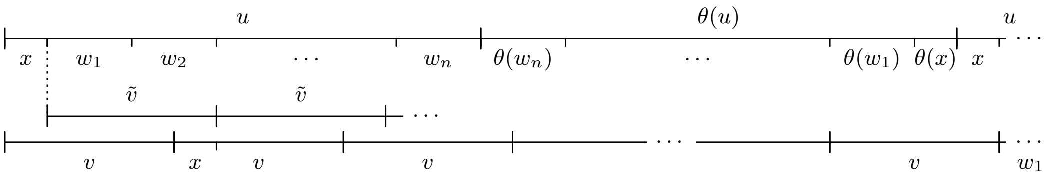


**Figure 1** The alignment of $\tilde{v}^{m-1}$ with $w_1 \cdots w_n\theta(w_n) \cdots \theta(w_1)$.

Consequently, we have $2n|w| \geq 2|v| + |w|$. Then we can apply Theorem 3 to the factor $w_1w_2 \cdots w_n\theta(w_n)\theta(w_{n-1}) \cdots \theta(w_1)$, centred on the border between $u$ and $\theta(u)$, and the factor $\tilde{v}^{m-1}$, where $\tilde{v} \sim v$ and $\tilde{v}$ appears after the prefix of length $|u| - n|w|$ in $u$. We get that $\tilde{v} \in \{w, \theta(w)\}^+$, as we assumed $w$ to be $\theta$-primitive. Clearly, $|w|$ divides $|v|$.

As $\tilde{v} \sim v$, it follows that the prefix of length $|u| - n|w|$ of $v^m$ has the form $x\{w, \theta(w)\}^*$, with $|x| < |w|$. So $u$ has the same form and furthermore the factor $v' \sim v$ occurring in $u$ after the prefix $x$ is in $\{w, \theta(w)\}^+$ as well (note that in Figure 1, we have $\tilde{v} = v'$, but this is so just to simplify the figure, and not the case in general, when $v'$ is obtained as explained). Moreover, exchanging $w$ and $\theta(w)$ if necessary, we can assume that $u \in xw\{w, \theta(w)\}^+$. If $x = \varepsilon$, we have $u \in \{w, \theta(w)\}^+$ and $v \in \{w, \theta(w)\}^+$ and since we assumed that $v$ is $\theta$-primitive, it follows that $v \in \{w, \theta(w)\}$, and the statement holds with $t = w$. Thus, assume $|x| > 0$. Since $|w|$ divides $3|u| \equiv 3|x| \mod 3|w|$, it follows that $|w|$ divides $3|x|$. But $|x| < |w|$, so either $3|x| = 2|w|$ or $3|x| = |w|$. In both cases, 3 divides $|w|$.

If $|w| = 3|x|$, we have $u\theta(u) \in x\{w, \theta(w)\}^+\theta(x)$. Since $m|v| > |u\theta(u)|$ and $3|x|$ divides $|v|$ we get that $m|v| = 2|u| + \ell|w| + |x|$, for some integer $\ell \geq 0$. Thus, a prefix of length $2|x|$ of $w$ or of $\theta(w)$ occurs after the prefix of length $2|u| - |x|$ in $u\theta(u)u$. We have $w, \theta(w) \notin \{x, \theta(x)\}^3$, as $w$ is $\theta$-primitive. Hence, if $\theta(x)x \leq_p w$ then $w = \theta(x)xy$ for some word $y$ with $|y| = |x|$ and $y \notin \{x, \theta(x)\}$, and if $\theta(x)x \leq_p \theta(w)$ then $w = \theta(y)\theta(x)x$ with $y$ as above.

Further, we analyse what values $m|v|$ might have. For $\ell = 0$, i.e., $m|v| = 2|u| + |x|$, we have that $v^m = u\theta(u)x$. As $\theta(u)$ ends with $\theta(x)$, we have $\theta(x)x \leq_s v$. Thus, $\theta(x)xx \leq_s v'$, and it follows that $w \in \{x, \theta(x)\}^3$, a contradiction. So, $m|v| > 2|u| + |x|$. However, because one of $w$ or $\theta(w)$ occurs as a factor of $v$ after the prefix of length $2|u|-|x|$ in $u\theta(u)u$, we get that the factor of length $|x|$ starting after a prefix of length $2|u|+|x|$ in $u\theta(u)u$ is neither $x$ nor $\theta(x)$. Thus, it can only be $y$. Now, one of $w$ or $\theta(w)$ occurs after the prefix of length $2|u|+|x|$ in $u\theta(u)u$ as well, by the fact that there exists a sequence of $w$'s and $\theta(w)$'s that starts there and is a suffix of $u\theta(u)u$. In both cases, unless $x \in \{y, \theta(y)\}$, it follows immediately that $y = \theta(y)$ and $y\theta(x)x$ appears after a prefix of length $2|u|+|x|$ in $u\theta(u)u$. However, the prefix of length $2|u|+|w|+|x|$ of $u\theta(u)u$ ends with $\theta(x)x$. By the same reasoning as above we get that $v^m$ cannot end here and so $m|v| > 2|u|+|w|+|x|$. We repeat this reasoning to see that, actually, $m|v| \neq 2|u|+\ell|w|+|x|$ for all $\ell \geq 0$. However, $m|v|$ should have this form. Therefore, we reached a contradiction in this case.

The reasoning for the case $3|x| = 2|w|$ is similar and omitted here. ◀

■ **Equations of the form** $uu\theta(u) = v_1 \cdots v_m w_1 \cdots w_n$**.** These are the only equations of the form (1) with $\ell = 3$ that were already investigated (see [4]), with the following result:

▶ **Lemma 12** (Proposition 51 in [4])**.** *If* $2|u| < m|v| < 2|u|+|v|$ *and* $u_1u_2u_3 = uu\theta(u)$ *and (1) holds, then* $u, v, w \in \{t, \theta(t)\}^+$ *for some word* $t$*.*

■ **Equations of the form** $u\theta(u)\theta(u) = v_1 \cdots v_m w_1 \cdots w_n$**.** In this case we were able to establish the following result.

▶ **Lemma 13.** *If* $2|u| < m|v| < 2|u|+|v|$ *and* $u_1u_2u_3 = u\theta(u)\theta(u)$ *and (1) holds, then* $u, v, w \in \{t, \theta(t)\}^+$ *for some word* $t$*.*

**Proof.** By Proposition 3, we know that $v_1 = \ldots = v_m = v$. We assume that $v$ and $w$ are $\theta$-primitive. We analyse first the case of $m$ being even.

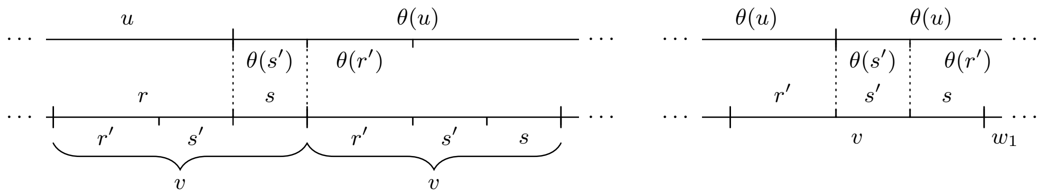


**Figure 2** The situation at the two borders $u_1u_2$ and $u_2u_3$.

**Case** $m = 6$**.** We have $u_1 = vvr$, where $v = rs$ and $|r| \geq |s|$. As $v$ is $\theta$-primitive we can assume that $|r| > |s|$, as otherwise, we had $v = r\theta(r)$. Hence, let $r'$ be the prefix of length $|r|-|s|$ of $r$ and $s'$ be the suffix of $r$ such that $r = r's'$ (see Figure 2).

From the border between $u_1 = u$ and $u_2 = \theta(u)$, we can see that $r's'sr' = r'\theta(s)\theta(s')\theta(r')$. It follows that $\theta(s) = s'$ and $\theta(r') = r'$. Now, looking at the border between $u_2 = \theta(u)$ and $u_3 = \theta(u)$, we get that $\theta(s')r' \leq_p \theta(u)$ and $s's \leq_p \theta(u)$. It follows that $s = \theta(s) = s'$.

**Subcase** $|r'| < |s|$**.** In this case $r' \leq_p s$. Therefore, $s = \theta(s)$ ends with $\theta(r') = r'$. As a consequence, we get that $w_1 \cdots w_n = r'sr'ssr'$, and so $u_3 = ssr'sr'ssr'$. However, also $u_3 = sr'ssr'ssr'$ holds, thus $r's = sr'$ and $v$ is not primitive.

**Subcase** $|r'| \geq |s|$**.** In this case, let $r' = sp$. As $v$ is primitive, $|p| > 0$ must hold. We have $w_1 \cdots w_n = ps^3ps^3p$. Hence, $ps^3p = w_1 \cdots w_k w' = w'' w_{n-k+1} \cdots w_n$, with $k \geq 2$. By Lemma 1 we get that $w_1 = \ldots = w_k$ and $w_{n-k+1} = \ldots = w_n$. Since $w_1$ is primitive, $w_1 \neq w_{n-k+1}$ must hold, thus $w_n = \theta(w_1)$.

If $|p| \leq |w|$, we get that $p \leq_p w_1$, so $\theta(p) \leq_s w_n = \theta(w_1)$. However, $p \leq_s w_n$ holds as well, and so $p = \theta(p)$. Yet, we have $sp = r' = \theta(r') = \theta(p)\theta(s) = ps$. This shows that $p, s \in \{t\}^+$ for some word $t$, so $v \in t\{t\}^+$, a contradiction.

If $|p| > |w|$ we can apply Theorem 3 to $ps^3ps^3p$ and $w_1 \cdots w_n$ and obtain that $ps^3, p \in \{w, \theta(w)\}^+$. It follows that $s^3 \in \{w, \theta(w)\}^+$. It is not hard to see that this leads again to a contradiction with the primitivity of $v$ or of $w$.

**Case** $m \geq 8$**.** We follow the exact same steps as above before splitting the discussion into the cases $|r'| < |s|$ and $|r'| \geq |s|$.

If $|r'| < |s|$, we get that $w_1 \cdots w_n = r'sr'(ssr')^k$ with $k \geq 2$. As $r'sr'$ is a suffix of $ssr'$ of length at least $\frac{|ssr'|}{2}$ and $n \geq 5$, we can apply Theorem 3 and obtain that $ssr' \in \{w, \theta(w)\}^+$ and thus also $r'sr' \in \{w, \theta(w)\}^+$. As $|r'| < |s|$, the word $ssr'$ is not $\theta$-primitive, but $ssr' = \theta(v)$, a contradiction. If $|r'| \geq |s|$, we get that $w_1 \cdots w_n = p(s^3p)^k$ with $k \geq 3$. By Theorem 3, it follows that $s^3p \in \{w, \theta(w)\}^+$ and $p \in \{w, \theta(w)\}^+$, thus $s^3p$ is not $\theta$-primitive. However, $s^3p = ssr' = \theta(v)$, again a contradiction.

This concludes the analysis for even $m$. The case when $m$ is odd had to be omitted due to the space restrictions. ◀

## ■ Equations of the form $uuu = v_1 \cdots v_m w_1 \cdots w_n$. We show the following:

▶ **Lemma 14.** *If $2|u| < m|v| < 2|u| + |v|$ and $u_1u_2u_3 = uuu$ and (1) holds, then $u, v, w \in \{t, \theta(t)\}^+$ for some word $t$.*

**Proof.** Assume that $v$ and $w$ are $\theta$-primitive. By Proposition 3, $m$ is even, $v_1 = \ldots = v_k = v$, and $v_{k+1} = \ldots = v_m = \theta(v)$, where $k$ is such that $(k-1)|v| < |u| < k|v|$. Furthermore, as $m|v| < 2|u| + |v|$, the prefix of $v_k$ occurring as a suffix of $u$ is longer than $\frac{|v|}{2}$. From the border between $u_1$ and $u_2$ we get that $v = rpr$ with $r = \theta(r)$ and $pr = r\theta(p)$. The solutions of this equation are, clearly, $r = (\alpha\beta)^i\alpha, p = (\alpha\beta)^j, \theta(p) = (\beta\alpha)^j$ for some $\theta$-palindromes $\alpha$ and $\beta$, $i \geq 0, j \geq 1$, and $\alpha\beta$ primitive.

Furthermore, if $w = \theta(w)$, then (1) is actually $u^3 = v^{\frac{m}{2}}\theta(v)^{\frac{m}{2}}w^n$. As $m, n \geq 5$ and $m$ is even, we can apply Theorem 5 to get that $u, v, \theta(v), w \in \{t\}^+$ for some word $t$, and the statement of this lemma holds. Therefore, we also assume $w \neq \theta(w)$ in the following.

We have $u_1 = v^{\frac{m}{2}-1}rp$ and $u_2 = r\theta(v)^{\frac{m}{2}-1}v'$ where $v' \leq_p \theta(v)$ and $|v'| = |p|$. Since $v = rpr$, the suffix of $v_m = \theta(v)$ that is a prefix of $u_3$ is of length $|rr|$. Furthermore, since $u_3 = u$ starts with $v = rpr = rr\theta(p)$ (as $pr = r\theta(p)$), we get that $w_1 \cdots w_n = \tilde{v}^{\frac{m}{2}-2}\theta(p)rp$, where $\tilde{v} = \theta(p)rr \sim v$. We will show that for $m \geq 8$, this equation leads to a contradiction:

First of all, $\theta(p)rp \leq_p \theta(p)rpr = \theta(p)rr\theta(p)$ and thus $w_1 \cdots w_n \leq_p \tilde{v}^{\frac{m}{2}}$.

If $|r| < |p|$, then $\tilde{v}^{\frac{m}{2}-2}\theta(p)rp = \tilde{v}^{\frac{m}{2}-1}p'$ for some $p' \leq_s p$ with $|p'| < |p|$. Since $m \geq 8$, this word is of length at least $3|v|$. Thus, if $|v| \geq |w|$, Theorem 3 is applicable and we get $\tilde{v}, w \in \{t, \theta(t)\}^+$ for some word $t$. On the other hand, if $|v| < |w|$, then as $|p'| < |p| < |v|$, the word $w_1 \cdots w_{n-1}$ is a prefix of $\tilde{v}^\omega$. As $m \geq 5$, this prefix is of length at least $2|w| + |v|$, and by Theorem 3, we get $\tilde{v}, w \in \{t, \theta(t)\}^+$ in this case as well. Since $w$ is $\theta$-primitive, $\tilde{v} \in \{w, \theta(w)\}^+$ must hold. By the assumption that $|r| < |p|$, and because $pr = r\theta(p)$, we can write $p = rs$ for some word $s$. Then, since $\theta(p)rr \in \{w, \theta(w)\}^+$ and $w_1 \cdots w_n = (\theta(p)rr)^{\frac{m}{2}-2}\theta(p)rp$, also $\theta(p)rrs = \theta(p)rp \in \{w, \theta(w)\}^+$ holds. Combining these last two results, we see that $s \in \{w, \theta(w)\}^+$ and thus also $\theta(s) \in \{w, \theta(w)\}^+$. However, as $\theta(p)rr = \theta(s)rrr \in \{w, \theta(w)\}^+$, by Theorem 4, also $r \in \{w, \theta(w)\}^+$. As a consequence, $p = rs \in \{w, \theta(w)\}^+$, and so $v = rpr \in \{w, \theta(w)\}^+$, contradicting the $\theta$-primitivity of $v$.

If $|r| \geq |p|$, then $\theta(p)rp \leq_p \theta(p)rr$, so the words $w_1 \cdots w_n$ and $\tilde{v}^\omega$ have a common prefix of length at least $\max\{(\frac{m}{2}-1)|v|, 5|w|\}$. If $m \geq 10$, this is at least $\max\{3|v|, 5|w|\}$ which is always long enough to apply Theorem 3 to get that $\tilde{v}, w \in \{w, \theta(w)\}^+$. In the case $m = 8$, we have $w_1 \cdots w_n = \tilde{v}^2\theta(p)rp$. If $|w| > |\theta(p)rp|$, then $n|w| > |\tilde{v}^2\theta(p)rp|$, as $|\theta(p)rp| > \frac{|v|}{2}$, a contradiction. Thus, $|w| \leq |\theta(p)rp|$, and so we have a common prefix of $\tilde{v}^\omega$ and $w_1 \cdots w_n$ of length $2|v| + |w|$. By Theorem 3, once again, we get $\tilde{v} \in \{w, \tilde{w}\}^+$, as $w$ is

$\theta$-primitive. Now, dually to the previous case, we write $r = ps'$. As $\theta(p)rr = \theta(p)rps'$ and $\theta(p)rp$ are both in $\{w, \theta(w)\}^+$, so is $s'$. Furthermore, as $\theta(p)rp = \theta(p)ps'p \in \{w, \theta(w)\}^+$, if $\theta(p)p \in \{w, \theta(w)\}^+$, then by Theorem 4, also $p \in \{w, \theta(w)\}^+$, and so $r = ps' \in \{w, \theta(w)\}^+$. This is a contradiction, since $v = rpr$ is $\theta$-primitive. Therefore, $p\theta(p) \notin \{w, \theta(w)\}^+$, which means that $s' \in \{w, \theta(w)\}^+$ is a proper factor of some word in $\{w, \theta(w)\}^+$. By Lemma 1, we must have that $s' \in \{w\}^+$ or $s' \in \{\theta(w)\}^+$, as $w$ is $\theta$-primitive. However, $pps' = pr = r\theta(p) = ps'\theta(p)$, so $ps' = s'\theta(p)$, and we saw before that this means that $s'$ is a $\theta$-palindrome. In conclusion, $w = \theta(w)$ in both cases, and we get a contradiction.

Therefore, as $m$ must be even, the only case left is when $m = 6$, in which (1) is of the form $uuu = vvv\theta(v)\theta(v)\theta(v)w_1 \cdots w_n$.

We shift our attention to the factor $w_1 \cdots w_n$. As $m = 6$, we know that $u = rpr^2pr^2p = r^2\theta(p)rpr^2p$ and $w_1 \cdots w_n$ starts after a prefix of length $2|r|$ in $u$, so $w_1 \cdots w_n = \theta(p)rpr^2p = (\beta\alpha)^j(\alpha\beta)^i\alpha(\alpha\beta)^{i+j}\alpha\alpha(\beta\alpha)^i(\alpha\beta)^j$. Since $\alpha$ and $\beta$ are $\theta$-palindromes, so is $w_1 \cdots w_n$.

If $n$ is odd, from $w_1 \cdots w_n = \theta(w_1 \cdots w_n)$ we immediately get that $w_{\frac{n+1}{2}} = \theta(w_{\frac{n+1}{2}})$. It follows that $w = \theta(w)$, which contradicts the assumption $w \neq \theta(w)$ we made at the beginning.

So we can further assume $n$ to be even and so $n \geq 6$.

If $(\beta\alpha)^j(\alpha\beta)^i\alpha(\alpha\beta)^j \in \{w, \theta(w)\}^+$, then also $(\beta\alpha)^j(\alpha\beta)^i\alpha(\alpha\beta)^i\alpha \in \{w, \theta(w)\}^+$. Thus, if $i \geq j$, then $(\alpha\beta)^{i-j}\alpha \in \{w, \theta(w)\}^+$, and if $i < j$, then $(\beta\alpha)^{j-i-1}\beta \in \{w, \theta(w)\}^+$. In both cases, those words are $\theta$-palindromes, so since $w \neq \theta(w)$, either $w\theta(w)$ or $\theta(w)w$ occurs as a factor in them.

If $i \geq j$, the factor $(\alpha\beta)^{i-j}\alpha$ appears in $w_1 \cdots w_n$ after the prefix $(\beta\alpha)^j$. By Lemma 1, we must have $(\beta\alpha)^j \in \{w, \theta(w)\}^+$ and by Theorem 4 thus $\beta\alpha \in \{w, \theta(w)\}^+$. Together with $(\alpha\beta)^{i-j}\alpha \in \{w, \theta(w)\}^+$, this leads to $\alpha, \beta \in \{w, \theta(w)\}^+$, which contradicts the $\theta$-primitivity of $v$.

If $j > i$ and $i > 0$, then $(\beta\alpha)^{j-i-1}\beta$ appears inside the factor $(\beta\alpha)^{i+j}$ both as a prefix and after the prefix $\beta\alpha$. Thus, in this case $\beta\alpha \in \{w, \theta(w)\}^+$ as well, which again leads to $\alpha, \beta \in \{w, \theta(w)\}^+$. If $j > i$ and $i = 0$, then $(\beta\alpha)^j(\alpha\beta)^i\alpha(\alpha\beta)^j = (\beta\alpha)^j\alpha(\alpha\beta)^j$, and $(\beta\alpha)^j(\alpha\beta)^i\alpha(\alpha\beta)^i\alpha = (\beta\alpha)^j\alpha\alpha$. So we immediately get that $(\beta\alpha)^j \in \{w, \theta(w)\}^+$, which leads to the same contradiction as above.

By the previous paragraphs, we can assume that $(\beta\alpha)^j(\alpha\beta)^i\alpha(\alpha\beta)^j = w_1 \cdots w_\ell w'$ for some $\ell$, and some nonempty $w' \leq_p w_{\ell+1}$. As $(\beta\alpha)^j(\alpha\beta)^i\alpha(\alpha\beta)^j$ appears also as a suffix of $w_1 \cdots w_n$, we have $w_1 = \cdots = w_\ell$ by Lemma 1. Without loss of generality, let $w_1 = w$. Since $|(\beta\alpha)^j(\alpha\beta)^i\alpha| = \frac{n}{3}|w|$, and $n \geq 6$, we get that $ww \leq_p (\beta\alpha)^j(\alpha\beta)^i\alpha$.

Now, if $i \geq j$, we can write $w^\ell w' = (\beta\alpha)^j(\alpha\beta)^j(\alpha\beta)^{i-j}\alpha(\alpha\beta)^j$. We observe that $w \leq_p (\beta\alpha)^j(\alpha\beta)^j$ must hold: Assume towards a contradiction, that $|w| > 2j|\alpha\beta|$. Then, the second $w$ of the prefix $ww$ of $(\beta\alpha)^j(\alpha\beta)^i\alpha$ begins inside the factor $(\alpha\beta)^{i-j}\alpha$. Since $w$ starts with $\beta\alpha$ and this is primitive, we deduce that $w = (\beta\alpha)^j(\alpha\beta)^j(\alpha\beta)^k$ for some $k$. However, then the second occurrence of $w$ that follows immediately afterwards is a prefix of $(\beta\alpha)^{i-j-k}(\alpha\beta)^j$. This is only possible if $\alpha\beta = \beta\alpha$, which is a contradiction to the primitivity of $\alpha\beta$. Thus we can safely assume that $w \leq_p (\beta\alpha)^j(\alpha\beta)^j$. This word $(\beta\alpha)^j(\alpha\beta)^j$ is a suffix of $w^\ell w' = (\beta\alpha)^j(\alpha\beta)^{i-j}(\alpha\beta)^j\alpha(\alpha\beta)^j$. Since $w$ is assumed to be primitive, by Lemma 1, we must have $(\beta\alpha)^j\alpha(\alpha\beta)^{i-j} \in \{w\}^+$. Let $y = (\beta\alpha)^j\alpha(\alpha\beta)^{i-j}$. Then $w_1 \ldots w_n = yy(\beta\alpha)^{2j}(\alpha\beta)^j\theta(y)$, from which we conclude that $(\beta\alpha)^{2j}(\alpha\beta)^j \in \{w, \theta(w)\}^+$. Applying Lemma 4 now gives us $\alpha\beta \in \{w, \theta(w)\}^+$, from which we deduce the contradiction $\alpha, \beta \in \{w, \theta(w)\}^+$ as before.

On the other hand, if $i < j$, then $|w| < |(\beta\alpha)^j|$, since $n \geq 6$. Furthermore $w_1 \cdots w_\ell w' = (\beta\alpha)^j(\alpha\beta)^i\alpha(\alpha\beta)^j$ is then the rest of $w_1 \cdots w_n$, so $\ell \geq \frac{n}{2}$. Therefore, we got $w_1 \cdots w_n = w^{\frac{n}{2}}\theta(w)^{\frac{n}{2}}$. If $|w| < |(\beta\alpha)^{j-1}|$, we would have $|w|$ occurring as a prefix of $w_1 \cdots w_n$ and after the prefix $\beta\alpha$. Thus $w = \beta\alpha$ by Lemma 1. However, then $w_{j+1} = w = \alpha\beta$, contradicting the primitivity of $\alpha\beta$. Hence, $|(\beta\alpha)^{j-1}| < |w| < |(\beta\alpha)^j|$.

If $j \geq 2$, then $(\beta\alpha)^{j-1} < |w|$ and $i < j$ imply that $|(\beta\alpha)^j(\alpha\beta)^i\alpha| < 3|w|$. Therefore $n < 9$, and as $n$ is even, either $n = 8$ or $n = 6$. If $n = 8$, then $w_4w_5$ must be a factor of $(\alpha\beta)^{i+j}\alpha$, and since $j \geq 2$, the word $\beta\alpha$ is a prefix of $w_4 = w$. Using Lemma 1, this $\beta\alpha$ must be aligned with some $\beta\alpha$ inside $(\alpha\beta)^{i+j}\alpha$. This allows us to deduce that $j = i+1$, and that $w_4 = w = (\beta\alpha)^{j-1}\beta'$, where $\beta = \beta'\theta(\beta')$. Then, $w_2 = w \leq_p \theta(\beta')\alpha(\alpha\beta)^{j-1}$. Now if $j \geq 3$, the factor $\alpha\beta$ appears as a proper factor inside $(\alpha\beta)^2$, unless $\beta = \theta(\beta')\alpha$. However, if $\beta = \theta(\beta')\alpha$, then $\alpha = \theta(\beta')$, and thus $\alpha\beta$ is not $\theta$-primitive. Therefore $j = 2$ must hold, in which case we get that $\beta\alpha\beta' \leq_p \theta(\beta')\alpha\alpha\beta$. From this it immediately follows that $\alpha$ is not primitive, and furthermore that $\alpha \in \{\theta(\beta')\}^+$, again a contradiction to $\alpha\beta$ being $\theta$-primitive. If $n = 6$, then $w_1w_2 = w^2 = (\beta\alpha)^j(\alpha\beta)^i\alpha$ and $w_3w_4 = w\theta(w) = (\alpha\beta)^{i+j}\alpha$. As $|w| \geq |\beta\alpha|$, we get the contradiction $\beta\alpha = \alpha\beta$.

Thus the only possibility that remains is $j = 1$ and thus $i = 0$. This means that $w^{\frac{n}{2}}\theta(w)^{\frac{n}{2}} = \beta\alpha^3\beta\alpha^3\beta$. By concatenating $\alpha^3$ to the left on both sides, we get $\alpha^3w^{\frac{n}{2}}\theta(w)^{\frac{n}{2}} = (\alpha^3\beta)^3$, to which we can apply Theorem 5 to get $w = \theta(w)$, a contradiction. ◀

All the other valuations of $u_1u_2u_3$ follow from the ones considered in the last three paragraphs by the fact that $\theta$ is an involution. Hence, Theorem 6 follows.

## 3.2 The case $m|v| < 2|u|$: Proofs of Theorems 7, 8 and 9.

We continue with the case when the border between $v_m$ and $w_1$ lies inside $u_2$.

■ **Equations of the form $uu_2\theta(u) = v_1 \cdots v_mw_1 \cdots w_n$, with $u_2 \in \{u, \theta(u)\}$.** For both possible values of $u_2$ the following lemma holds:

▶ **Lemma 15.** *If $u_1 = u$, $u_3 = \theta(u)$ and (1) holds, then $u, v, w \in \{t, \theta(t)\}^+$ for some word $t$.*

**Proof.** We can assume that $|v| \geq |w|$, otherwise we just change the roles of $v$ and $w$ in the following reasoning. Actually, if $|v| = |w|$, we get that $v_1 = \theta(w_n)$, and that $v, w \in \{v, \theta(v)\}$, so in this case the statement holds.

Therefore we can assume that $|v| > |w|$. Now, if $|u| \geq 3|v|$, then we have $u \leq_p v_1 \cdots v_m$ and $u \leq_p \theta(w_n) \cdots \theta(w_1)$, and $|u| \geq 2|v| + |w|$, so by Theorem 3 we get that $v, w \in \{t, \theta(t)\}^+$ for some word $t$, so the statement also holds in this case.

Since $m|v| < 2|u|$ and $m \geq 5$, it follows that $m = 5$ and $u = v_1v_2r$ for $v_3 = rs$. Furthermore, again from the facts that $m|v| < 2|u|$ and $m \geq 5$, it follows immediately that $|r| > |s|$. If $|w| \leq |r|$, then $u$ would still be a prefix of $v_1 \cdots v_m$ and $\theta(w_n) \cdots \theta(w_1)$, long enough to apply Theorem 3, so we assume $|w| > |r|$.

As $|u| = 2|v| + |r| = 3|r| + 2|s|$, we get that $|w_1 \cdots w_n| = 3|u| - 5|v| = 3(3|r| + 2|s|) - 5(|r| + |s|) = 4|r| + |s|$, and so as $|r|, |s| < |w|$, this contradicts the fact that $n \geq 5$. ◀

■ **Equations of the form $u\theta(u)u = v_1 \cdots v_mw_1 \cdots w_n$.** We start this paragraph with two simple lemmas, that we use in our proofs. Their proofs are left out here.

▶ **Lemma 16.** *Let $p$ and $r$ be $\theta$-palindromes such that $r \leq_p p$ and $\frac{|p|}{2} < |r| < |p|$. If $p$ and $r$ are not primitive, then neither is $pr$.*

▶ **Lemma 17.** *If $u\theta(u)u = v^mw_1 \cdots w_n$, $|w| > |v|$, $|u| > 3|w|$, and $|u| < m|v| < 2|u|$, then $w$ is not $\theta$-primitive.*

We analyse (1) for all possible relations between $|v|$ and $|w|$:

▶ **Lemma 18.** *If $u\theta(u)u = v_1 \cdots v_m w_1 \cdots w_n$, $|w| > |v|$ and $\frac{3}{2}|u| \leq n|w| < 2|u|$, then either $v$ or $w$ is not $\theta$-primitive.*

**Proof.** We show this by contradiction, and assume that $v$ and $w$ are $\theta$-primitive. By the length-restrictions, we have $u\theta(u) = v_1 \cdots v_m w_1 \cdots w_{i-1}s$, where $w_i$ is the word overlapping with the border between $\theta(u)$ and the second $u$. We have two cases, depending on the position of $w_i$ on the border between $\theta(u)$ and $u$.

**Case $u = \theta(s)rw_{i+1} \cdots w_n$ and $w_i = s\theta(s)r$.** We can assume $r \neq \varepsilon$, as otherwise $w$ would not be $\theta$-primitive, so $|s| < \frac{|w|}{2}$. Hence $i \geq 3$, as otherwise $n|w| \geq \frac{3}{2}|u|$ would not hold.

Furthermore, we can see that $w_{j-1}w_j <_f \theta(w_{2i-j+1})\theta(w_{2i-j})\theta(w_{2i-j-1})$ for all $j$ with $i \geq j \geq 2$. Therefore, by Lemma 1 we have that $w_1 = \ldots = w_i$. By the same arguments and the fact that $u\theta(u)$ is a $\theta$-palindrome, we also get that $w_i = \ldots = w_{2i-2}$ and that $s\theta(s) <_p w_{2i-1}$. Let $N = |w_1 \cdots w_{i-1}s| = (i-1)|w| + |s|$ and $M = N \mod |v|$. That is, $M$ is the difference between the length of $w_1 \cdots w_{i-1}s$ and the longest $\theta$-power of $v$ that occurs as a suffix thereof. Now let $\tilde{w}$ be the conjugate of $w_i$ occurring in $w_1 \cdots w_{i-1}s$ after the prefix of length $M$. The length of the prefix of $\tilde{w}^\omega$ that starts there is at least $(2i-2)|w| - M + 2|s| \geq 4|w| - M + 2|s| \geq 2|w| + |v|$. Therefore, we can apply Theorem 3 to this prefix and the $\theta$-power of $v$, that occurs there and is at least as long, to get that $\tilde{w} \in \{v, \theta(v)\}^+$. Thus, $|v|$ divides $|w|$. Since we assume that $|v|$ does not divide $|u|$ (as otherwise the statement would trivially hold), and we have that $|v|$ divides $m|v| + n|w| = 3|u|$, it follows that $|v| = 3d$ for some $d$ with $d \mid |u|$. We let $k$ be such that $(k-1)|v| < |u| < k|v|$, and write $v_k = x_1x_2x_3$. By our previous divisibility reasoning, we have that $|u| = 3(k-1)d + d$ or $|u| = 3(k-1)d + 2d$. We only treat the first case explicitly here. In this case we get that $x_1 \leq_s u$ and $x_2x_3 \leq_p \theta(u)$, so $\theta(x_1) = x_2$. If $v_{k-1} = \theta(v_k)$, then $\theta(x_2) = x_3$ holds, and so $v$ is not $\theta$-primitive. Therefore, $v_{k-1} = v_k$ and, by the same reasoning, $v_{k+1} = v_k$. Repeating this process we get that $v_k = v_{k+1} = \ldots = v_m$ and that $x_1x_2 \leq_p w_1$. Hence, $x_1\theta(x_1) \leq_p w_1$. As $M = 2$, it follows that $x_3 \leq_s w_1$, as otherwise $v$ would not be $\theta$-primitive. Now $x_3x_1 \leq_s u$ and so if $w_n = w_1$, we have that $x_1 = x_3$, a contradiction to the $\theta$-primitivity of $v$. Similarly, if $w_n = \theta(w_1)$, we have $x_3x_1 = \theta(x_2)\theta(x_1) = x_1\theta(x_1)$, so $x_1 = x_3$ and $v$ is again not $\theta$-primitive, a contradiction. The other case, $|u| = 3(k-1)d + 2d$, leads to the same result in an identical fashion, and is left to the reader. Therefore, when $u = \theta(s)rw_{i+1} \cdots w_n$ and $w_i = s\theta(s)r$, one of $v, w$ is not $\theta$-primitive.

**Case $u = \theta(s)w_{i+1} \cdots w_n$ and $w_i = rs\theta(s)$.** This case can be analysed in a somewhat similar manner. However, due to the page limit, we have to omit this. ◀

The next case follows in a similar way.

▶ **Lemma 19.** *If $u\theta(u)u = v_1 \cdots v_m w_1 \cdots w_n$, $|w| \leq |v|$ and $\frac{3}{2}|u| \leq n|w| < 2|u|$, then either it is the case that $v$ or $w$ is not $\theta$-primitive or $v \in \{w, \theta(w)\}$.*

As a consequence of the previous two lemmas we get the main result of this paragraph, which, together with Theorem 6 and Lemma 15, proves Theorem 7:

▶ **Lemma 20.** *If $|u| < m|v| < 2|u|$ and $u_1u_2u_3 = u\theta(u)u$ and (1) holds, then $u, v, w \in \{t, \theta(t)\}^+$ for some word $t$.*

■ **Equations of the form $uuu = v_1 \cdots v_m w_1 \cdots w_n$.**

▶ **Lemma 21.** *If $uuu = v_1 \cdots v_m w_1 \cdots w_n$, $|u| < m|v| < 2|u|$, at least one of $m$ and $n$ is odd, and (1) holds, then $u, v, w \in \{t, \theta(t)\}^+$ for some word $t$.*

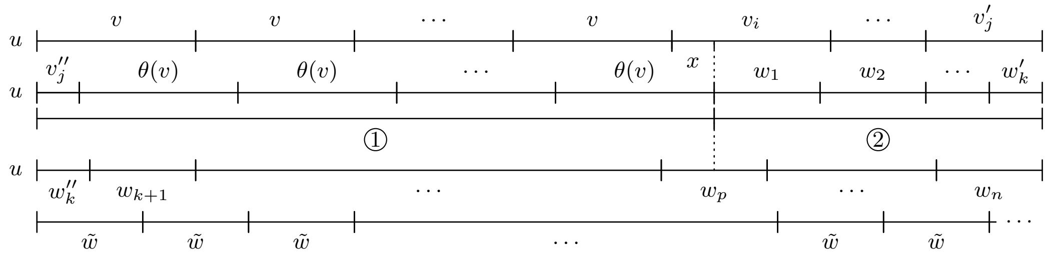


**Figure 3** The situation in the case $uuu = v_1 \cdots v_m w_1 \cdots w_n$ with $|u| < m|v| < 2|u|$.

**Proof.** The situation of this case is depicted in Figure 3 (with $v_j = v'_j v''_j$ and $w_k = w'_k w''_k$).

As $m, n \geq 5$, either $v_m$ or $w_1$ has to be a proper factor of $u_2$. We assume without loss of generality, that $v_m$ is a factor of $u_2$, therefore $m > j$ (see Figure 3). We now show that the factor ① in Figure 3 is a $\theta$-palindrome; this part of the proof holds also for the case when both $m$ and $n$ are even. To streamline the presentation we assume $v$ to be $\theta$-primitive. Otherwise $v \in \{v', \theta(v')\}^+$ for some $\theta$-primitive word $v'$, and we apply the reasoning below to $v'$, reaching the same conclusion. Therefore, if $m - j \geq 2$, we have $v_1 = v_2 = \ldots = v_{i-1}$ and $v_{j+1} = \ldots = v_m = \theta(v)$ by Lemma 1. On the other hand, if $m = j+1$, we use another result by Kari et al. [4], stating that if $x_1 x_2 y = z x_3 x_4$ holds, where $x_i \in \{t, \theta(t)\}$ for all $1 \leq i \leq 4$ with $t$ a $\theta$-primitive word, then $x_2 \neq x_3$. Applying this to $x_1 = v_j, x_2 = v_m, x_3 = v_1, x_4 = v_2$, and $y$ and $z$ chosen accordingly, we get that $v_m = \theta(v_1)$. If $v_i = \theta(v_j)$, then ① is clearly a $\theta$-palindrome. If $v_i = v_j = v$, then $x$ (from Figure 3) is a prefix of $v$ and we see that $x = \theta(x)$, and thus $\theta(v^{i-1}x) = x\theta(v)^{i-1} = v^{i-1}x$. The same reasoning applies if $v_i = v_j = \theta(v)$.

Furthermore, the factor ② is a $\theta$-palindrome by the same arguments. Here, there is another case to be considered, though, namely when ② is shorter than $|w|$. If $w_1 = \theta(w_n)$, then ② is obviously a $\theta$-palindrome. If $w_1 = w_n$ we get that $w_1 = x'y$, where ② $= x'$ is the suffix of $u_2$ and $w_n = zx'$. As ① is a $\theta$-palindrome, it holds that $z = \theta(y)$. Thus $x'y = \theta(y)x'$, and the solution of this equation is given by $y = (\alpha\beta)^i, \theta(y) = (\beta\alpha)^i, x' = (\beta\alpha)^j\beta$ for some $i \geq 1$, $j \geq 0$ and $\theta$-palindromes $\alpha$ and $\beta$. Consequently, $x' =$ ② is a $\theta$-palindrome.

As the factors ① and ② are $\theta$-palindromes, so are $v_1 \cdots v_m$ and $w_1 \cdots w_n$. Now, if $m$ is odd, we get that $v_{\frac{m+1}{2}} = \theta(v_{\frac{m+1}{2}})$ and therefore $v = \theta(v)$. Similarly, $w = \theta(w)$ if $n$ is odd.

Hence, if both $m$ and $n$ are odd, we have the equation $u^3 = v^m w^n$, and as $m, n \geq 5$, we get that $u, v, w \in \{t\}^+$ for some word $t$ by Lyndon and Schützenberger's original result.

Therefore, assume that only $n$ is odd, while $m$ is even (the other case works analogously). Thus we have the equation $u^3 = v_1 \cdots v_m w^n$, with $m \geq 6$ and $w = \theta(w)$. Furthermore, we can assume $v$ to be $\theta$-primitive in this case, as otherwise we would consider the same equation with $v$ replaced by its $\theta$-primitive root, and as $m$ is even, this would not change the parity.

First we show the statement for $|v| > |w|$. Since $v_1 \cdots v_m$ has a common prefix with $\tilde{w}^\omega$ (where $\tilde{w} \sim w$, see Figure 3) of length $|u|$, and $|u| \geq 2|v| + |w|$ (if this was not true, we had $6|v| + 3|w| > 3|u|$, a contradiction), we can apply Theorem 3 and get that $v, \tilde{w} \in \{t, \theta(t)\}^+$ for some $t$. If $|v| > |w|$ then $v$ is not $\theta$-primitive, a contradiction. Thus, we assume $|v| \leq |w|$ in the following. Also, if $|②| \geq |w|$, we can apply Theorem 2 to get that $u, w \in \{t\}^+$ for some $t$; as $w = \theta(w)$ we get that $t = \theta(t)$ and the conclusion follows easily.

Therefore we can assume $|v| \leq |w|$ and also $|②| < |w|$. As $n \geq 5$, we have $4|w| < |u|$ and consequently $|w| < \frac{|u|}{4}$ and also $|v| < \frac{|u|}{4}$. It follows that the length of $v_1 \cdots v_{i-1} = v^{i-1}$ is at least $|u| - |②| - |v|$, thus $|v^{i-1}| \geq \frac{|u|}{2} \geq |v| + |w|$. Thus we can apply Theorem 2 to $v^{i-1}$ and $\tilde{w}^\omega$, to get that $v, \tilde{w} \in \{t\}^+$ for some word $t$. As we assumed $v$ to be $\theta$-primitive and thus primitive, we get $\tilde{w} \in \{v\}^+$. Therefore, as $u_1$ is completely covered by $\tilde{w}^\omega$, $u_1$ must be of the form $v^{j-1}x$, where $|x| < |v|$ and $x$ is a prefix of $v_j$. As $|v| \leq |w| < \frac{|u|}{4}$, we have the equation $u^3 = v^{j-1}v_j\theta(v)^{m-j}w^n$ where both $j-1 \geq 3$ and $m-j \geq 3$. Hence, whatever value $v_j \in \{v, \theta(v)\}$ has, we can always apply Theorem 5 to get the claimed result. ◀

Theorem 9 now follows directly by combining Lemma 15, 18, 19 and 21.

The techniques developed so far allow us to establish two lemmas, which formalise, for the current case, the two meta-steps of our general approach, described in Section 2. These technical results are used in the proof of Lemma 24.

▶ **Lemma 22.** *In the setting of (1), assume that $v$ is $\theta$-primitive and $|v|$ does not divide $|u|$. Let $j$ be so that $(j-1)|v| < |u| < j|v|$. Then $v_{j+1} = \ldots = v_m = \theta(v_1)$ and $v_1 = \ldots = v_{m-j}$. If $j|v| - |u| \geq \frac{|v|}{2}$, then $v_1 = v_{m-j+1}$ and $v_j = \theta(v_1)$ also.*

▶ **Lemma 23.** *In the setting of (1), assume that $|u| < m|v| < 2|u|$, $w$ is $\theta$-primitive, $|v| > |w|$, and there exists a word $\tilde{v} \sim v$, such that $\tilde{v}$ occurs in $vv$ after the prefix of length $i = |u| \bmod |w|$, and $\tilde{v} \in \{w, \theta(w)\}^+$. Then $v = \tilde{v}$.*

The following lemma states the final result of this section. Alongside Theorems 6 and 7, it establishes the central result of our paper, namely Theorem 8.

▶ **Lemma 24.** *If $|u| < m|v| < 2|u|$, $u_1u_2u_3 = uuu$, and $m, n \geq 12$, then (1) implies that $u, v, w \in \{t, \theta(t)\}^+$ for some word $t$.*

**Proof.** We refer again to the notation used in Figure 3 and, as usual, we assume that $v$ and $w$ are $\theta$-primitive.

First of all, without loss of generality, we assume that $|①| \geq |②|$. Then $|①| \geq 4|v|$, otherwise $|②| > 4|v|$ had to hold, but $|①| \geq |②|$. By the same reasoning $|①| \geq 4|w|$, so $p \geq k+5$ in Figure 3.

If $|w| \geq |v|$, then ① is long enough to apply Theorem 3 and we obtain $\tilde{v} \in \{w, \theta(w)\}^+$. On the other hand, if $|w| < |v|$, then $w'' \leq_p v_1$ and $w''w_{k+1} \leq_p v_1v_2$. Thus, $|①| - |w''_k| \geq 2|v| + |w|$ and as $x \leq_p v$, we can again apply Theorem 3 to get $\tilde{v} \in \{w, \theta(w)\}^+$. Now, using Lemma 23 we get that $v = \tilde{v} \in \{w, \theta(w)\}^+$, a contradiction. ◀

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