Bounds for the quantifier depth in finite-variable logics: Alternation hierarchy

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Abstract

Given two structures G and H distinguishable in FO^k (first-order logic with k variables), let $A^k(G,H)$ denote the minimum alternation depth of a FO^k formula distinguishing G from H. Let $A^k(n)$ be the maximum value of $A^k(G,H)$ over n-element structures. We prove the strictness of the quantifier alternation hierarchy of FO^2 in a strong quantitative form, namely $A^2(n) \geq n/8-2$, which is tight up to a constant factor. For each $k \geq 2$, it holds that $A^k(n) > \log_{k+1} n - 2$ even over colored trees, which is also tight up to a constant factor if $k \geq 3$. For $k \geq 3$ the last lower bound holds also over uncolored trees, while the alternation hierarchy of FO^2 collapses even over all uncolored graphs.

We also show examples of colored graphs G and H on n vertices that can be distinguished in FO^2 much more succinctly if the alternation number is increased just by one: while in Σ_i it is possible to distinguish G from H with bounded quantifier depth, in Π_i this requires quantifier depth $\Omega(n^2)$. The quadratic lower bound is best possible here because, if G and H can be distinguished in FO^k with i quantifier alternations, this can be done with quantifier depth n^{2k-2}.

1998 ACM Subject Classification F.4.1 Finite model theory

Keywords and phrases Alternation hierarchy, finite-variable logic

Digital Object Identifier 10.4230/LIPIcs.CSL.2013.61

1 Introduction

Given structures G and H over vocabulary σ and a first-order formula Φ over the same vocabulary, we say that Φ *distinguishes* G from H if Φ is true on G but false on H. By *alternation depth* of Φ we mean the maximum length of a sequence of nested alternating quantifiers in Φ. Obviously, this parameter is bounded from above by the *quantifier depth* of Φ. We will examine the maximum alternation depth and quantifier depth needed to distinguish two structures for restrictions of first-order logic and particular classes of structures.

For a fragment $\mathcal{L}$ of first-order logic, by $A_{\mathcal{L}}(G,H)$ we denote the minimum alternation depth of a formula $\Phi \in \mathcal{L}$ distinguishing G from H. Similarly, we let $D_{\mathcal{L}}(G,H)$ denote the

* Supported by DFG grant VE 652/1–1. On leave from the Institute for Applied Problems of Mechanics and Mathematics, Lviv, Ukraine.

Computer Science Logic 2013 (CSL'13).
Editor: Simona Ronchi Della Rocca; pp. 61–80
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

minimum quantifier depth of such Φ. Obviously, $A_{\mathcal{L}}(G,H) \le D_{\mathcal{L}}(G,H)$. We define the *alternation function* $A_{\mathcal{L}}(n)$ to be equal to the maximum value of $A_{\mathcal{L}}(G,H)$ taken over all pairs of n-element structures G and H distinguishable in $\mathcal{L}$.

Our interest in this function is motivated by the observation that if the quantifier alternation hierarchy of $\mathcal{L}$ collapses, then $A_{\mathcal{L}}(n) = O(1)$. More specifically, $A_{\mathcal{L}}(n) \le a$ if the alternation hierarchy collapses to its a-th level $\Sigma_a \cup \Pi_a$. Thus, showing that

$$\lim_{n\to\infty} A_{\mathcal{L}}(n) = \infty \tag{1}$$

is a way of proving that the hierarchy is strict.

Note that Condition (1) is, in general, formally stronger than a hierarchy result. For example, while the alternation hierarchy of first-order logic FO is strict over colored directed trees by Chandra and Harel [3], we have $A_{\mathrm{FO}}(n) = 1$ for any class of structures over a fixed vocabulary.

An example of this nature also exists when we restrict our logic to two variables: While the alternation hierarchy of $\mathrm{FO}^2[<]$ is strict over words in an infinite alphabet by Immerman and Weis [10], we have $A_{\mathrm{FO}^2}(n) = 1$ for words in any alphabet.

Moreover, the rate of growth of $A_{\mathcal{L}}(n)$ can be naturally regarded as a quality of the strictness of the alternation hierarchy. Note that any pair of structures G and H with $A_{\mathcal{L}}(G,H) = a$ can serve as a certificate that the first a levels of the alternation hierarchy of $\mathcal{L}$ are distinct. Indeed, if G is distinguished from H by a formula $\Phi \in \mathcal{L}$ of the minimum alternation depth a, then the set of structures $L = \{\, S : S \models \Phi\}$ is not definable in $\mathcal{L}$ with less than a quantifier alternations. Thus, the larger the value of $A_{\mathcal{L}}(n)$ is, the more levels of the alternation hierarchy can be separated by a certificate of size n.

Results that we now know about the function $A_{\mathcal{L}}(n)$ are displayed in Figure 1. The upper bound $A_{\mathrm{FO}^k}(n) \le n^{k-1} + 1$ holds true even for the quantifier depth. It follows from the relationship of the distinguishability in FO^k to the $(k-1)$-dimensional color refinement (Weisfeiler-Lehman) procedure discovered in [6, 2] and the standard color stabilization argument; see [8]. The logarithmic upper bound for trees (Theorem 3.4) holds true also for the quantifier depth.

Class of structures	Logic	Bounds for $A_{\mathcal{L}}(n)$	
uncolored trees	$\mathcal{L} = \mathrm{FO}^2$	≤ 2	Theorem 3.3
	$\mathcal{L} = \mathrm{FO}^k$, $k \ge 3$	$> \log_{k+1} n - 2$	Theorem 3.2
		$< (k+3)\log_2 n$	Theorem 3.4
colored trees	$\mathcal{L} = \mathrm{FO}^k$, $k \ge 2$	$> \log_{k+1} n - 2$	Theorems 3.1 and 3.2
	$\mathcal{L} = \mathrm{FO}^k$, $k \ge 3$	$< (k+3)\log_2 n$	Theorem 3.4
uncolored graphs	$\mathcal{L} = \mathrm{FO}^2$	≤ 2	Theorem 3.3
	$\mathcal{L} = \mathrm{FO}^k$, $k \ge 3$	$> \log_{k+1} n - 2$	Theorem 3.2
		$\le n^{k-1} + 1$	cf. [8]
colored graphs	$\mathcal{L} = \mathrm{FO}^2$	$> n/8 - 2$	Theorem 4.1
		$\le n + 1$	cf. [6]
	$\mathcal{L} = \mathrm{FO}^k$, $k \ge 3$	$> \log_{k+1} n - 2$	Theorem 3.2
		$\le n^{k-1} + 1$	cf. [8]

Figure 1 Results about $A_{\mathcal{L}}(n)$.

Additionally, in Section 5 we show that the Σ_i fragment of FO^2 is not only strictly more expressive than the Σ_{i-1} fragment but also more succinct in the following sense: There are colored graphs G and H on n vertices such that they can be distinguished in $\Sigma_{i-1} \cap \mathrm{FO}^2$ and, moreover, this is possible with bounded quantifier depth in $\Sigma_i \cap \mathrm{FO}^2$ while in $\Pi_i \cap \mathrm{FO}^2$ this requires quantifier depth $\Omega(n^2)$. The quadratic lower bound is best possible here because, if G and H can be distinguished in FO^k with i quantifier alternations, this can be done with quantifier depth n^{2k-2}.

2 Preliminaries

We consider first-order formulas only in the negation normal form (i.e., any negation stands in front of a relation symbol and otherwise only monotone Boolean connectives are used). For each $i \geq 1$, let Σ_i (resp. Π_i) denote the set of (not necessary prenex) formulas where any sequence of nested quantifiers has at most $i-1$ quantifier alternations and begins with $\exists$ (resp. $\forall$). In particular, *existential logic* Σ_1 consists of formulas without universal quantification. Up to logical equivalence, $\Sigma_i \cup \Pi_i \subset \Sigma_{i+1} \cap \Pi_{i+1}$. By the *quantifier alternation hierarchy* we mean the interlacing chains $\Sigma_1 \subset \Sigma_2 \subset \ldots$ and $\Pi_1 \subset \Pi_2 \subset \ldots$. We are interested in the corresponding fragments of a finite-variable logic.

As a short notation we use $D^k_{\mathcal{L}}(G,H) = D_{\mathcal{L}\cap\mathrm{FO}^k}(G,H)$ and $A^k_{\mathcal{L}}(G,H) = A_{\mathcal{L}\cap\mathrm{FO}^k}(G,H)$. The subscript FO can be dropped; for example, $D^k(G,H) = D^k_{\mathrm{FO}}(G,H)$ and $A^k(n) = A^k_{\mathrm{FO}}(n)$. Sometimes we will write $D^k_{\exists}(G,H)$ in place of $D^k_{\Sigma_1}(G,H)$.

The universe of a structure G will be denoted by $V(G)$, and the number of elements in $V(G)$ will be denoted by $v(G)$. Since binary structures can be regarded as vertex- and edge-colored directed graphs, the elements of $V(G)$ will also be called vertices. A vertex in a simple undirected graph is *universal* if it is adjacent to all other vertices.

The *k-pebble Ehrenfeucht-Fraïssé game on structures G and H*, is played by two players, Spoiler and Duplicator, to whom we will refer as he and she respectively. The players have equal sets of k pairwise different pebbles. A *round* consists of a move of Spoiler followed by a move of Duplicator. Spoiler takes a pebble and puts it on a vertex in G or in H. Then Duplicator has to put her copy of this pebble on a vertex of the other graph. Duplicator's objective is to keep the following condition true after each round: the pebbling should determine a partial isomorphism between G and H. The variant of the game where Spoiler starts playing in G and is allowed to jump from one graph to the other less than i times during the game will be referred to as the *Σ_i game*. In the *Π_i game* Spoiler starts in H.

For each positive integer r, the r-round Ehrenfeucht-Fraïssé game (as well as its Σ_i and Π_i variants) is a two-person game of perfect information with a finite number of positions. Therefore, either Spoiler or Duplicator has a *winning strategy* in this game, that is, a strategy winning against every strategy of the opponent.

▶ **Lemma 2.1** (e.g., [10]). *$D^k_{\Sigma_i}(G,H) \leq r$ if and only if Spoiler has a winning strategy in the r-round k-pebble Σ_i game on G and H.*

The lifting construction

Note that separation of the ground floor of the alternation hierarchy for FO^2 costs nothing. We can take graphs G and H with three isolated vertices each, color one vertex of G in red, and color the other vertices of G and all vertices of H in blue. Obviously, $D^2_{\exists}(G,H) = 1$ while $D^2_{\forall}(G,H) = \infty$. It turns out that any separation example can be lifted to higher floors in a rather general way.

The lifting gadget provided by Lemma 2.2 below is a reminiscence of the classical construction designed by Chandra and Harel to prove the strictness of the first-order alternation hierarchy. The Chandra-Harel construction is applicable to other logics (see, e.g., [5, Section 8.6.3]) and can be used as a general scheme for obtaining hierarchy results. This approach was also used by Oleg Pikhurko (personal communication, 2007) to construct, for each i, a sequence of pairs of trees G_n and H_n such that $D_{\Sigma_i}(G_n, H_n) = O(1)$ while $D_{\Pi_i}(G_n, H_n) \to \infty$ as $n \to \infty$.

Given colored graphs G_0 and H_0, we recursively construct graphs G_i and H_i as shown in Fig. 2. H_1 consists of three disjoint copies of H_0 and an extra universal vertex, that will be referred to as the root vertex of H_1. The root vertex is colored in a new color absent in G_0 and H_0, say, in gray. The graph G_1 is constructed similarly but, instead of three H_0-branches, it has two H_0-branches and one G_0-branch. Suppose that $i \geq 1$ and the rooted graphs G_i and H_i are already constructed. The graph H_{i+1} consists of three disjoint copies of G_i and the gray root vertex adjacent to the root of each G_i-part. The graph G_{i+1} is constructed similarly but, instead of three G_i-branches, it has two G_i-branches and one H_i-branch.

We will say that Spoiler plays *continuously* if, after each of his moves, the two pebbled vertices are adjacent.

▶ **Lemma 2.2.** *Assume that Spoiler has a continuous strategy allowing him to win the 2-pebble Σ_1 game on G_0 and H_0 in r moves. Then, for each $i \geq 1$,*

1. $D^2_{\Sigma_i}(G_i, H_i) < r + i$*;*
2. $D^2_{\Sigma_i}(G_i, H_i) \geq D^2_{\Pi_{i+1}}(G_i, H_i) \geq D^2_{\exists}(G_0, H_0)$*;*
3. $D^2_{\Pi_i}(G_i, H_i) = \infty$*;*
4. *If, moreover, Spoiler has a continuous strategy allowing him to win the 2-pebble Σ_2 game on G_0 and H_0 in s moves, then $D^2_{\Sigma_{i+1}}(G_i, H_i) < s + i$.*

Proof. 1. In the base case of $i = 1$ Spoiler is able to win the Σ_1 game on G_1 and H_1 in r moves. He forces the Σ_1 game on G_0 and H_0 by playing continuously inside the G_0-part of G_1 and wins by assumption. Furthermore, we recursively describe a strategy for Spoiler in the Σ_{i+1} game on G_{i+1} and H_{i+1} and inductively prove that it is winning. For each i, the strategy will be continuous, and the vertex pebbled in the first round will be adjacent to the root. Note that this is true in the base case.

In the first round Spoiler pebbles the root of the H_i-branch of G_{i+1}. Duplicator is forced to pebble the root of one of the G_i-branches of H_{i+1}. Indeed, if she pebbles a gray vertex at the different distance from the root of H_{i+1}, then Spoiler pebbles a shortest possible path upwards in G_{i+1} or H_{i+1} and wins once he reaches a non-gray vertex. In the second round

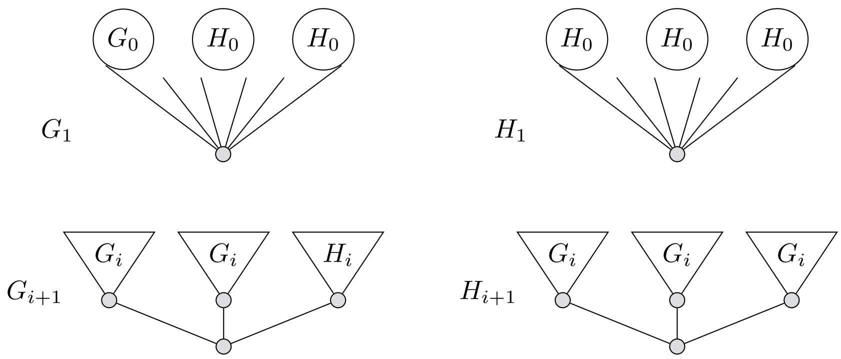


Figure 2 The lifting construction.

Spoiler jumps to this G_i-branch and, starting from this point, forces the Σ_i game on G_i and H_i by playing recursively and, hence, continuously. The only possibility for Duplicator to avoid the recursive play and not to lose immediately is to pebble a gray vertex below. In this case Spoiler wins in altogether $i+1$ moves by pebbling a path upwards in the graph where he stays, as already explained. If the game goes recursively, then by the induction assumption Spoiler needs less than $1+r+i$ moves to win.

2. In the base case of $i=1$ we have to design a strategy for Duplicator in the Π_2 game on G_1 and H_1. First of all, Duplicator pebbles the gray vertex always when Spoiler does so. Furthermore, whenever Spoiler pebbles a vertex in an H_0-branch of G_1 or H_1, Duplicator pebbles the same vertex in an H_0-branch of the other graph. It is important that, if the pebbles are in two different H_0-branches of G_1 or H_1, Duplicator has a possibility to pebble different H_0-branches in the other graph. It remains to describe Duplicator's strategy in the case that Spoiler moves in the G_0-branch of G_1. Note that once Spoiler does so, he cannot change the graph any more. In this case, Duplicator chooses a free H_0-branch in H_1 and follows her optimal strategy in the Σ_1 game on G_0 and H_0. Since the gray vertex is universal in both graphs and the G_0- and H_0-branches are isolated from each other, Spoiler wins only when he wins the Σ_1 game on G_0 and H_0, which is possible in $D^2_\exists(G_0,H_0)$ moves at the earliest.

In the Π_{i+2} game on G_{i+1} and H_{i+1} Duplicator plays similarly. She always respects the root vertex, the G_i-branches, and takes care that the pebbled vertices are either in the same or in distinct G_i-branches in both graphs. Once Spoiler moves in the H_i-branch of G_{i+1}, Duplicator invokes her optimal strategy in the Σ_{i+1} game on H_i and G_i, what is the same as the Π_{i+1} game on G_i and H_i. There is no other way for Spoiler to win than to win this subgame. By the induction assumption, this takes at least $D^2_\exists(G_0,H_0)$ moves.

3. By induction on i, we show that Duplicator has a strategy allowing her to resist arbitrarily long in the Π_i game on G_i and H_i. An important feature of the strategy is that Duplicator will always respect the distance of a pebbled gray vertex from the root. In the base case of $i=1$, such a strategy exists because in G_1 there are two copies of H_0, where Duplicator can mirror Spoiler's moves. In the Π_{i+1} game on G_{i+1} and H_{i+1}, Duplicator makes use of the existence of two copies of G_i in both graphs. Whenever Spoiler pebbles the root vertex or moves in a G_i-part in any of G_{i+1} and H_{i+1}, Duplicator mirrors this move in the other graph. Whenever Spoiler moves for the first time in the H_i-part of G_{i+1}, Duplicator responds in a free G_i-part of H_{i+1} according to her level-preserving strategy for the Π_i game on G_i and H_i, that exists by the induction assumption. When Spoiler moves in the H_i-part also with the other pebble, Duplicator continues playing in the same G_i-part of H_{i+1} following the same strategy.

4. Spoiler has a recursive winning strategy for the Σ_{i+1} game on G_i and H_i similarly to the proof of part 1. ◀

3 Alternation function for FO^k over trees

▶ **Theorem 3.1.** *$A^2(n) > \log_3 n - 2$ over colored trees.*

Proof. Applying the lifting construction described in Section 2 to a pair of single-vertex, differently colored graphs G_0 and H_0, we obtain the sequence of pairs of colored trees G_i and H_i with $v(G_i)=v(H_i)$ as shown in Fig. 3. For $i \ge 1$, we have $D^2_{\Sigma_i}(G_i,H_i) \le i$ by part 1 of Lemma 2.2 and $D^2_{\Pi_i}(G_i,H_i)=\infty$ by part 3 of this lemma. It follows that $A^2(n_i) \ge i$ for $n_i = v(G_i)$. Note that $n_i = 3n_{i-1}+1$, where $n_0 = 1$. Therefore $n_i = 3^i + \frac{3^i-1}{2}$, which implies that $A^2(n_i) > \log_3 n_i - 1$.

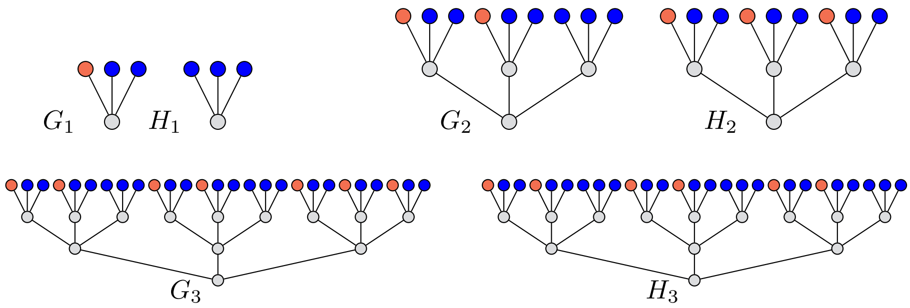


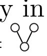
Figure 3 Proof of Theorem 3.1.

Consider now an arbitrary n and suppose that $n_i \le n < n_{i+1}$, i.e., $n_i \le n \le 3n_i$. We can increase the number of vertices in G_i and H_i to n by attaching $n - n_i$ new gray leaves at the root. Since this does not change the parameters $D^2_{\Sigma_i}(G_i, H_i)$ and $D^2_{\Pi_i}(G_i, H_i)$, we get $A^2(n) \ge A^2(n_i) > \log_3 n - 2$. ◀

Theorem 3.1 generalizes to any k-variable logic and, if $k > 2$, then no vertex coloring is needed any more.

▶ **Theorem 3.2.** *If $k \ge 3$, then $A^k(n) > \log_{k+1} n - 2$ over uncolored trees.*

Proof. Notice that the lifting construction of Lemma 2.2 generalizes to $k \ge 3$ variables by adding $k - 2$ extra copies of H_0 in G_1 and H_1 and $k - 2$ extra copies of G_i in G_{i+1} and H_{i+1}. Similarly to Theorem 3.1, this immediately gives us colored trees G_{i+1} and H_{i+1} such that $D^2_{\Sigma_i}(G_i, H_i) \le i$ and $D^k_{\Pi_i}(G_i, H_i) = \infty$ for all $i \ge 1$.

In order to remove colors from G_i and H_i, we construct these graphs recursively in the same way but now, instead of red and blue one-vertex graphs, we start with $G_0 =$  and $H_0 =$ ; see Fig. 4. Note that in the course of construction G_0 and H_0 will be handled as rooted trees (otherwise they are isomorphic).

We now claim that for the uncolored trees G_i and H_i it holds $D^3_{\Sigma_i}(G_i, H_i) \le i + 5$ and $D^k_{\Pi_i}(G_i, H_i) = \infty$. The latter claim is true exactly by the same reasons as in the colored case: since the number of Spoiler's jumps is bounded, Duplicator is always able to ensure playing on isomorphic branches. To prove the former bound, we will show that Spoiler can win similarly to the colored case playing with 3 pebbles.

Note that in the uncolored version of G_i and H_i, all formerly gray vertices have degree $k + 1$, red vertices have degree 3, and blue vertices have degree 2. A typical ending of the game on the colored trees was that Spoiler pebbles a red vertex while Duplicator is forced to pebble a blue one. Now this corresponds to pebbling a vertex u of degree 3 by Spoiler and a vertex v of degree 2 by Duplicator. Having 4 pebbles, Spoiler would win by pebbling the three neighbors of u. Having only 3 pebbles, Spoiler first pebbles two neighbors u_1 and

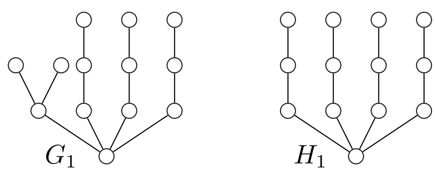


Figure 4 Proof of Theorem 3.2. The uncolored versions of G_1 and H_1 for 3-variable logic.

u_2 of u (in fact, one neighbor is already pebbled immediately before u). Duplicator must respond with the two neighbors v_1 and v_2 of v. In the next round Spoiler moves the pebble from u to its third neighbor u_3. Duplicator must remove the pebble from v and place it on some vertex v_3 non-adjacent to both v_1 and v_2. Note that, while the distance between any two vertices of u_1, u_2, and u_3 equals 2, there is a pair of indices s and t such that v_s and v_t are at the distance more than 2. Spoiler now wins by moving the pebble from u_q to u, where $\{q\} = \{1, 2, 3\} \setminus \{s, t\}$.

It remains to note that with 3 pebbles Spoiler is able to force climbing upwards in the trees and, hence, he can follow essentially the same winning strategy as in the colored case. Duplicator can deviate from this scenario only in the first round. Recall that in this round Spoiler pebbles a vertex u at the distance 1 from the root level, having degree at least 3. Suppose that Duplicator responds with pebbling a vertex v at the distance more than 1 from the root level. If $i = 1$, then v is of degree at most 2, and Spoiler wins as explained above. If $i \geq 2$, then v can have degree 3 or $k + 1$. In this case Spoiler forces climbing up and wins by pebbling a leaf above a formerly blue vertex because by this point Duplicator has already reached the highest possible level. Suppose now that in the first round Duplicator pebbles the root vertex. Then Spoiler puts a second pebble on the root of his graph, Duplicator is forced to pebble a vertex one level higher, and Spoiler again wins by forcing climbing up from the root to the highest leaf level.

Thus, we have shown that $A^k(n_i) \geq i$ for $n_i = v(G_i)$. Since $n_i = (k + 1)n_{i-1} + 1$ and $n_0 = 3$, we have $n_i = 3(k + 1)^i + \frac{(k+1)^i-1}{k}$, which implies that $A^2(n_i) > \log_{k+1} n_i - 1$. Like to the proof of Theorem 3.1, this bound extends to all n at the cost of decreasing it by 1. ◀

Theorems 3.1 and 3.2 are optimal in the sense that they cannot be extended to FO^2 over uncolored trees. The reason is that the quantifier alternation hierarchy of FO^2 over uncolored graphs collapses to the second level.

▶ **Theorem 3.3.** *If a class of uncolored graphs is definable by a first-order formula with two variables, then it is definable by a first-order formula with two variables and one quantifier alternation.*

We now show that the bound of Theorem 3.2 is tight up to a constant factor. The following theorem implies that, if $k \geq 3$, then $A^k(n) < (k + 3) \log_2 n$ over colored trees. The proof easily extends to the class of all binary structures whose Gaifman graph is a tree.

▶ **Theorem 3.4.** *Let $k \geq 3$. If $D^k(T, T') < \infty$ for colored trees T and T', then*

$$D^k(T, T') < (k + 3) \log_2 n \tag{2}$$

where n denotes the number of vertices in T.

Proof. Let $T - v$ denote the result of removal of a vertex v from the tree T. The component of $T - v$ containing a neighbor u of v will be denoted by T_{vu} and considered a rooted tree with the root at u. A similar notation will apply also to T'. The rooted trees T_{vu} will be called *branches of T at the vertex v*. Let $\tau(v)$ denote the maximum number of pairwise isomorphic branches at v. We define the *branching index* of T by $\tau(T) = \max_v \tau(v)$. In order to prove the theorem, we will show that the bound (2) is true for any non-isomorphic colored trees with branching index at most k and that $D^k(T, T') = D^k(T \bmod k, T' \bmod k)$ for $T \bmod k$ and $T' \bmod k$ being "truncated" versions of T and T' whose branching index is bounded by k. We first handle the latter task.

The following fact easily follows from the trivial observation that k pebbles can be placed on at most k isomorphic branches.

Claim A. Let T be a colored tree. Suppose that T has more than k isomorphic branches at a vertex v. Remove all but k of them from T and denote the resulting tree by $\hat{T}$. Then $D^k(T, G) = D^k(\hat{T}, G)$ for any colored graph G. ◁

The truncated tree $T \bmod k$ is obtained from T by a series of truncations as in Claim A. The truncations steps should be done from the top to the bottom in order to exclude appearance of new isomorphic branches in the course of the procedure. In order to define the "top and bottom" formally, recall that the *eccentricity* of a vertex v in a graph G is defined by $e(v) = \max_u \mathit{dist}(v, u)$, where $\mathit{dist}(v, u)$ denotes the distance between the two vertices. The *diameter* and the *radius* of G are defined by $d(G) = \max_v e(v)$ and $r(G) = \min_v e(v)$ respectively. A vertex v is *central* if $e(v) = r(G)$. For trees it is well known (e.g., [7, Chapter 4.2]) that if $d(T)$ is even, then T has a unique central vertex c. If $d(T)$ is odd, then T has exactly two central vertices c_1 and c_2, that are adjacent. Let us regard the central vertices as lying on the bottom level and the tree T as growing upwards. The height of a vertex is then its distance to the nearest central vertex. Starting from the highest level and going downwards, for each vertex v we cut off extra branches at v if their number exceeds k. Note that this operation can increase the number of isomorphic branches from vertices in lower levels but cannot do this for vertices in higher levels. Therefore, the resulting tree $T \bmod k$ has branching index at most k.

Applying repeatedly Claim A, we arrive at the equality $D^k(T, T') = D^k(T \bmod k, T' \bmod k)$. Note that $T \bmod k \not\cong T' \bmod k$ because it is assumed that $D^k(T, T') < \infty$. Thus, we have reduced proving the bound (2) to the case that T and T' are non-isomorphic and both have branching index at most k. Therefore, below we make this assumption.

We have to show that Spoiler is able to win the k-pebble game on such T and T' in less than $(k + 3) \log_2 n$ moves. Below we will actively exploit the following fact ensured by a standard halving strategy for Spoiler.

Claim B. Suppose that in the 3-pebble Ehrenfeucht-Fraïssé game on graphs G and H some two vertices $x, y \in V(G)$ at distance n are pebbled so that their counterparts $x', y' \in V(H)$ are at a strictly larger distance. Then Spoiler can win in at most $\lceil \log n \rceil$ extra moves. ◁

Every tree T has a single-vertex *separator*, that is, a vertex v such that no branch of T at v has more than $n/2$ vertices; see, e.g., [7, Chapter 4.2]. The idea of Spoiler's strategy is to pebble such a vertex and to force further play on some non-isomorphic branches of T and T', where the same strategy can be applied recursively. This scenario was realized in [8, Theorem 5.2] for first-order logic with counting quantifiers. Without counting, we have to use some additional tricks that are based on boundedness of the branching index. Below, by $N(v)$ we will denote the neighborhood of a vertex v.

Thus, in the first round Spoiler pebbles a separator v in T and Duplicator responds with a vertex v' somewhere in T'. Since $T \not\cong T'$, there is an isomorphism type $\mathcal{B}$ of a branch of T at v that appears with different multiplicity among the branches of T' at v'. Spoiler can use this fact to force pebbling vertices $u \in N(v)$ and $u' \in N(v')$ so that the rooted trees T_{vu} and $T'_{v'u'}$ are non-isomorphic (the pebbles on v and v' can be reused but, finally, v and v' have to remain pebbled as well). This is easy to do if the multiplicity of $\mathcal{B}$ in one of the trees is at most $k - 2$. If this multiplicity is $k - 1$ in one tree and k in the other, then Spoiler can do it still with k pebbles like as in the proof of Theorem 3.2. This phase of the game can take $k + 2$ rounds.

The next goal of Spoiler is to force pebbling adjacent vertices v_1 and u_1 in T_{vu} and adjacent vertices v'_1 and u'_1 in $T'_{v'u'}$ so that $T_{v_1u_1} \not\cong T'_{v'_1u'_1}$ and

$$v(T_{v_1u_1}) \leq v(T_{vu})/2 \text{ or } v(T'_{v'_1u'_1}) \leq v(T_{vu})/2. \tag{3}$$

Once this is done, the same will be repeated recursively (with the roles of T and T' swapped if only the second inequality in (3) is true).

To make the transition from T_{vu} to $T_{v_1u_1}$, Spoiler first pebbles a separator w of T_{vu}. Note that Duplicator is forced to respond with a vertex w' in $T'_{v'u'}$. Otherwise we would have $dist(w,u) = dist(w,v) - 1$ while $dist(w',u') = dist(w',v') + 1$. Therefore, some distances among the three pebbled vertices would be different in T and in T' and Spoiler could win in less than $\log v(T_{vu}) + 1$ moves by Claim B.

Let $T_{w\setminus u}$ denote the rooted tree obtained by removing from T the branch at w containing u and rooting the resulting tree at w. Note that $V(T_{w\setminus u}) \subset V(T_{vu})$. We consider a few cases.

Case 1: $T_{w\setminus u} \not\cong T'_{w'\setminus u'}$. In the trees $T_{w\setminus u}$ and $T'_{w'\setminus u'}$ we will consider branches at their roots w and w'.

Subcase 1-a: $T_{w\setminus u}$ contains a branch of isomorphism type $\mathcal{B}$ that has different multiplicity in $T'_{w'\setminus u'}$. As above, Spoiler can use k pebbles and $k+1$ moves to force pebbling vertices $x \in N(w)$ and $x' \in N(w')$ such that $T_{wx} \not\cong T'_{w'x'}$ and

$$T_{wx} \in \mathcal{B} \text{ or } T'_{w'x'} \in \mathcal{B}. \tag{4}$$

The pebbles occupying v, v' and u, u' can be released. The pebbles on w and w' can also be reused but, finally, w and w' have to remain pebbled. The branches T_{wx} and $T'_{w'x'}$ will now serve as $T_{v_1u_1}$ and $T'_{v'_1u'_1}$. Condition (3) follows from (4) because w is a separator of T_{vu}.

Subcase 1-b: $T_{w\setminus u}$ does not contain any branch as in Subcase 1-a. In this subcase there is a vertex $x' \in N(w')$ such that $T'_{w'x'}$ is a branch of $T'_{w'\setminus u'}$ and the isomorphism type of $T'_{w'x'}$ does not appear in $T_{w\setminus u}$. Spoiler moves the pebble from v' to x'. Suppose that Duplicator responds with $x \in N(w)$. If x lies on the path between u and w (while x' does not lie on the path between u' and w'), then equality of distances among the pebbled vertices cannot be preserved, and Spoiler wins by Claim B. If x does not lie between u and w, then T_{wx} is a branch of T_{vu} at the vertex w. The first equality in Condition (3) is then true because w is a separator of T_{vu}. In this case, T_{wx} and $T'_{w'x'}$ can serve as $T_{v_1u_1}$ and $T'_{v'_1u'_1}$.

Case 2: $T_{w\setminus u} \cong T'_{w'\setminus u'}$. We assume that $dist(u,w) = dist(u',w')$ because otherwise Spoiler wins by Claim B. For a vertex y on the path between u and w, let $T_{y\setminus u,w}$ denote the rooted tree obtained by removing from T the branches at y containing u and w and rooting the resulting tree at y. The rooted tree $T_{u\setminus v,w}$ is defined similarly. Note that $T_{u\setminus v,w}$ and each $T_{y\setminus u,w}$ are parts of a branch of T_{vu} at the vertex w and, therefore, have at most $v(T_{vu})/2$ vertices. Given y between u and w, by y' we will denote the vertex lying between u' and w' at the same distance to these vertices as y to u and w. Since $T_{vu} \not\cong T'_{v'u'}$, we must have

$$T_{y\setminus u,w} \not\cong T'_{y'\setminus u',w'} \quad \text{for some } y \text{ or} \tag{5}$$

$$T_{u\setminus v,w} \not\cong T'_{u'\setminus v',w'}. \tag{6}$$

Assume that Condition (5) is true and fix such y.

Subcase 2-a: $T_{y\setminus u,w}$ contains a branch of isomorphism type $\mathcal{B}$ that has different multiplicity in $T'_{y'\setminus u',w'}$. Spoiler moves the pebble from v to y. Duplicator is forced to move the pebble from v' to y'. The pebbles occupying u, u' and w, w' can now be released. Spoiler proceeds similarly to Subcase 1-a and forces pebbling vertices $z \in N(y)$ and $z' \in N(y')$ such that $T_{yz} \not\cong T'_{y'z'}$ and one of these trees has isomorphism type $\mathcal{B}$ and, hence, is as small as desired.

Subcase 2-b: $T_{y\setminus u,w}$ does not contain any branch as in Subcase 2-a. In this subcase there is a vertex $z' \in N(y')$ such that $T'_{y'z'}$ is a branch of $T'_{y'\setminus u',w'}$ whose isomorphism type does not appear in $T_{y\setminus u,w}$. Similarly to Subcase 1-b, Spoiler aims to pebble y' and z' while forcing Duplicator to respond with y and $z \in N(y)$ such that T_{yz} is a part of $T_{y\setminus u,w}$. This will ensure that $T_{yz} \not\cong T'_{y'z'}$ and that T_{yz} is small enough. Now Spoiler's task is more complicated because he has to prevent Duplicator from pebbling z on the path between u and w. Since this requires keeping the pebbles on u, u' and w, w', Spoiler cannot pebble both y' and z' if there are only $k = 3$ pebbles. In this case he first pebbles the vertex z' by the pebble released from v. Let z be Duplicator's response. If z is in $N(y)$ and does not lie between u and w, Spoiler succeeds by moving the pebble from u' to y'. Duplicator is forced to move the pebble from u to y because w' remains pebbled and, therefore, the position of y is determined by the distances to z and w. If z is not in $N(y)$ or lies between u and w, then Spoiler wins because $dist(z, u) \neq dist(z', u')$ or $dist(z, w) \neq dist(z', w')$

An analysis of the case (6) is quite similar. The role of the triple (u, y, w) is now played by the triple (v, u, w).

Note that the transition from T_{vu} to $T_{v_1u_1}$ takes at most $k + 3$ rounds. Also, 2 rounds suffice to win the game once the current subtree T_{vu} has at most 2 vertices. The number of transitions from the initial branch of order at most $n/2$ to one with at most 2 vertices is bounded by $\log_2 n - 1$ because $v(T_{vu})$ becomes twice smaller each time. It follows that Spoiler wins the game on T and T' in less than $k + 2 + (\log_2 n - 1)(k + 3) + 2 \leq (k + 3)\log_2 n + 1$ moves. The additive term of 1 can be dropped because if pebbling the initial branch takes no less than $k + 2$ moves, then the size of this branch will actually not exceed n/k. ◀

4 Alternation function for FO^2 over colored graphs

Theorem 3.1 gives us a logarithmic lower bound on the alternation function $A^2(n)$, which is true even for trees. Over all colored graphs, we now prove a linear lower bound. Along with the general upper bound $A^2(n) \leq n + 1$, it shows that $A^2(n)$ has a linear growth.

▶ **Theorem 4.1.** *$A^2(n) > n/8 - 2$.*

Proof. For each integer $m \geq 2$, we will construct colored graphs G and H, both with $n = 8m - 4$ vertices, that can be distinguished in FO^2 with $m - 2$, but no less than that, alternations. The graph $G = 2G_m$ is the union of two disjoint copies of the same graph G_m and, similarly, $H = 2H_m$ where G_m and H_m are defined as follows. Each of G_m and H_m is obtained by merging two building blocks A_m and B_m shown in Fig. 5. The colored graph A_m is a "ladder" with m horizontal rungs, each having 2 vertices. The vertices on the bottom rung are colored in green, the vertices on the top rung are colored one in red and the other in blue, the remaining $2m - 4$ vertices are white (uncolored). The graph B_m is obtained from A_m by recoloring red in apricot and blue in cyan. A_m and B_m are glued together at the green vertices. There are two ways to do this, and the resulting graphs G_m and H_m are non-isomorphic. Let α^+ (resp. α^-) denote the partial isomorphism from G_m to H_m identifying the A_m-parts (resp. the B_m-parts) of these graphs.

We will design a strategy allowing Spoiler to win the $(m - 2)$-alternation (i.e., Σ_{m-1} or Π_{m-1}) 2-pebble Ehrenfeucht-Fraïssé game on G and H and a strategy allowing Duplicator to win the $(m - 3)$-alternation game. Before playing on G and H, we analyse the 2-pebble game on G_m and H_m. Spoiler can win this game as follows. In the first round he pebbles the

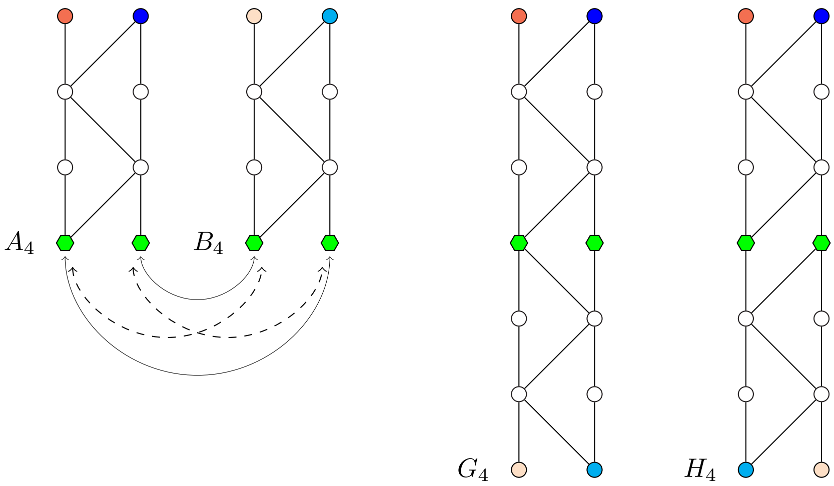


Figure 5 Proof of Theorem 4.1.

left green vertex in G_m; see Fig. 5. Not to lose immediately, Duplicator responds either with the left or with the right green vertex in H_m. The corresponding partial isomorphism can be extended to α^+ in the former case and to α^- in the latter case (but not to both α^+ and α^-). These two cases are similar, and we consider the latter of them, where there is no extension to α^+ and hence Spoiler has a chance to win playing in the A_m-parts of G_m and H_m.

In the second round Spoiler pebbles the upright neighbor of the left green vertex in G_m. His goal in subsequent rounds is to force pebbling, one by one, edges along the upright paths to the red vertex in G_m and to the blue vertex in H_m. If Duplicator makes a step down, Spoiler wins by reaching the top rung sooner than Duplicator. If Duplicator moves all the time upward, starting from the third round of the game she has a possibility to slant. Spoiler prevents this by changing the graph. Note that in one of the graphs there is only one way upstairs, and Spoiler always leaves this graph for Duplicator. In this way Spoiler wins by making m moves and alternating between the graphs $m-2$ times.

The strategy we just described is inoptimal with respect to the alternation number. In fact, Spoiler can win the game on G_m and H_m with no alternation at all by pebbling in the first round the right green vertex in G_m. If Duplicator responds with the left green vertex in H_m, Spoiler puts the second pebble on the non-adjacent vertex in the next upper rung. Duplicator is forced to play in a different rung of H_m because otherwise she would violate the non-adjacency relation. If in the first round Duplicator responds with the right green vertex, Spoiler plays similarly, but in the lower rung of G_m. In any case, the second pebble is closer to the red or to the apricot vertex in G_m than in H_m, which makes Spoiler's win easy.

Nevertheless, the former, $(m-2)$-alternation strategy has an advantage: Spoiler ensures that the two pebbled vertices are always adjacent. By this reason, the same strategy can be used by Spoiler to win also the game on $G = 2G_m$ and $H = 2H_m$. Once Duplicator steps aside to another copy of G_m or H_m, she immediately loses.

The partial isomorphism α^+ from G_m to H_m determines two partial isomorphisms α_0^+ and α_1^+ from $G = 2G_m$ to $H = 2H_m$ identifying the two A_m-parts of G with the two A_m-parts of H. Similarly, α^- gives rise to two partial isomorphisms α_0^- and α_1^-.

We now show that the number of alternations $m-2$ is optimal for the game on G and H. Fix an integer a such that Spoiler has a winning strategy in the a-alternation 2-pebble game on G and H. For this game, let us fix an arbitrary winning strategy for Spoiler and a strategy for Duplicator satisfying the following conditions.

- Duplicator always respects vertex rungs.
- Additionally, Duplicator respects adjacency.
- Duplicator respects also non-adjacency. Moreover, whenever Spoiler violates adjacency of the vertices pebbled in one graph, Duplicator responds so that the vertices pebbled in the other graph are not only non-adjacent but even lie in different G_m- or H_m-components.
- If Spoiler pebbles a vertex above the green rung and the three preceding rules still do not determine Duplicator's response uniquely, then she responds according to α_0^+ or α_1^+; in a similar situation below the green rung, she plays according to α_0^- or α_1^-.

Note that these rules uniquely determine Duplicator's moves on non-green vertices provided one pebble is already on the board. In particular, the choice of α_0^+ or α_1^+ in the last rule depends on the component where this pebble is placed.

Let $u_i \in V(G)$ and $v_i \in V(H)$ denote the vertices pebbled in the i-th round of the game. We now highlight a crucial property of Duplicator's strategy. Suppose that u_i, v_i and u_{i+1}, v_{i+1} are in the A_m-parts of G and H and that u_{i+1} and v_{i+1} are non-green. Then the following conditions are met.

- If u_i and u_{i+1} are non-adjacent, then $\alpha_s^+(u_{i+1}) = v_{i+1}$ for $s = 0$ or $s = 1$.
- If u_i and u_{i+1} (as well as v_i and v_{i+1}) are adjacent and $\alpha_s^+(u_i) = v_i$ for $s = 0$ or $s = 1$, then $\alpha_s^+(u_{i+1}) = v_{i+1}$ for the same s.

The similar property holds if the pebbles are in the B_m parts.

Suppose that Spoiler wins in the r-th round. Note that Duplicator's strategy allows Spoiler to win only when u_r and v_r are on the top or on the bottom rungs and have different colors. Since the two cases are similar, assume that Spoiler wins on the top.

Let p be the smallest index such that all vertices in the sequence $u_p, v_p, \ldots, u_r, v_r$ are above the green level. By assumption, $\alpha_s^+(u_r) \neq v_r$ for both $s = 0, 1$. The aforementioned property of Duplicator's strategy implies that, furthermore,

$$\alpha_0^+(u_i) \neq v_i \text{ and } \alpha_1^+(u_i) \neq v_i \text{ for all } i \geq p. \tag{7}$$

Therefore, u_{i+1} and u_i as well as v_{i+1} and v_i are adjacent for all $i \geq p$ (for else Duplicator plays so that $\alpha_s^+(u_{i+1}) = v_{i+1}$ for $s = 0$ or $s = 1$). By the same reason, $p > 1$ and u_{p-1} and u_p are also adjacent. It follows that u_{p-1} and v_{p-1} are green and $\alpha_s^+(u_{p-1}) \neq v_{p-1}$ for both $s = 0, 1$.

Another consequence of (7) is that both vertex sequences $u_{p-1}, u_p, \ldots, u_r$ and $v_{p-1}, v_p, \ldots, v_r$ lie on upright paths. This follows from the fact that either from u_i or from v_i there is only one edge emanating upstairs (also downstairs), and it is upright.

It remains to notice that after each transition to the adjoining rung (i.e., from u_i, v_i to u_{i+1}, v_{i+1} for $i \geq p-1$) Spoiler has to jump to the other graph because otherwise Duplicator will choose the neighbor that ensures $\alpha_s^+(u_{i+2}) = v_{i+2}$ for some value of $s = 0, 1$. This observation readily implies that the number of alternations a cannot be smaller than $m - 2$.

We have shown that $A^2(n) \geq m - 1$ if $n = 8m - 4$. Adding up to seven isolated vertices to both G and H, we get the same bound also for $n = 8m - 3, \ldots, 8m + 3$. Therefore, $A^2(n) \geq (n - 11)/8$ for all n. ◀

5 Succinctness results

Since $D^k_{\Sigma_i}(G, H) = D^k_{\Pi_i}(H, G)$, the following result holds true as well for $\Pi_i \cap \mathrm{FO}^k$.

▶ **Theorem 5.1.** *Let G and H be structures over the same vocabulary. If G is distinguishable from H in $\Sigma_i \cap \mathrm{FO}^k$, then $D^k_{\Sigma_i}(G, H) \leq (v(G)v(H))^{k-1} + 1$.*

In particular, if binary structures G and H have n elements each and G is distinguishable from H in existential two-variable logic, then $D^2_\exists(G,H) \le n^2+1$. We now show that this bound is tight up to a constant factor. For the existential-positive fragment of FO^2, a quadratic lower bound can be obtained from the benchmark instances for the arc consistency problem going back to [4, 9]; see [1] where also an alternative approach is suggested. We here elaborate on the construction presented in [1]. To implement this idea for existential two-variable logic, we need to undertake a more delicate analysis as the existential-positive fragment is more restricted and simpler.

▶ **Theorem 5.2.** *There are infinitely many colored graphs G and H, both on n vertices, such that G is distinguishable from H in existential two-variable logic and $D^2_\exists(G,H) > n^2/11$.*

Proof. Our construction will depend on an integer parameter $m \ge 2$. We construct a pair of colored graphs G_m and H_m such that G_m is distinguishable from H_m in the existential two-variable logic, both $v(G_m) = O(m)$ and $v(H_m) = O(m)$, and $D^2_\exists(G_m,H_m) = \Omega(m^2)$. Though $v(G_m) < v(H_m)$, later we will be able to increase the number of vertices in G_m to $v(H_m)$.

The graphs have vertices of 4 colors, namely apricot, blue, cyan, and dandelion. G_m contains a cycle of length $3(2m-1)$ where apricot, blue, and cyan alternate in this order; see Fig. 6. H_m contains a similar cycle of length $3 \cdot 2m$. Successive apricot, blue, and cyan vertices will be denoted by a_i, b_i, and c_i in G_m, where $0 \le i < 2m-1$, and by a'_i, b'_i, and c'_i in H_m, where $0 \le i \le 2m-1$. Furthermore, the vertex a_0 is adjacent to a dandelion vertex d_0, and every a'_i except for $i = m$ is adjacent to a dandelion vertex d'_i. This completes the description of the graphs.

By Lemma 2.1, we have to show that Spoiler is able to win the 2-pebble Σ_1 game on G_m and H_m and that Duplicator is able to prevent losing the game for $\Omega(m^2)$ rounds.

Note that, once the pair (a_0, a'_m) is pebbled, Spoiler wins in the next move by pebbling d_0. He is able to force pebbling (a_0, a'_m) as follows. In the first round he pebbles a_0. Suppose that Duplicator responds with a'_s, where $0 \le s < m$. In a series of subsequent moves, Spoiler goes around the whole circle in G_m, visiting $c_{2m-2}, b_{2m-2}, a_{2m-2}, c_{2m-2}, \ldots$ and using the two pebbles alternately (if $m < s < 2m$, he does the same but in the other direction). As Spoiler comes back to a_0, Duplicator is forced to arrive at a'_{s+1}. The next Spoiler's tour around the circle brings Duplicator to a'_{s+2} and so forth. Thus, the most successful moves for Duplicator in the first round is a'_0. Then Spoiler needs to play $1 + m \cdot 3(2m-1) + 1 = 6m^2 - 3m + 2$ rounds in order to win.

Our next task is to design a strategy for Duplicator allowing her to survive $\Omega(m^2)$ rounds, no matter how Spoiler plays. We will show that Duplicator is able to force Spoiler to pass

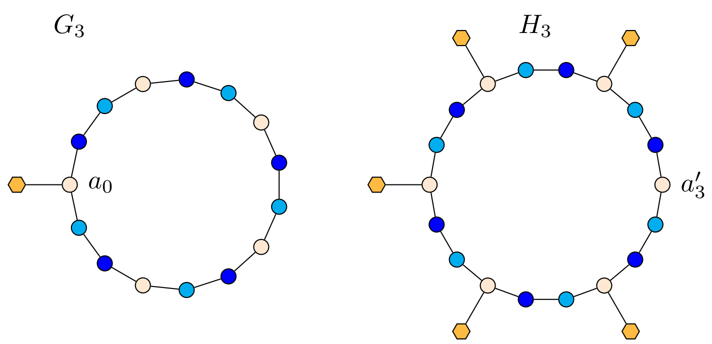


Figure 6 Proof of Theorem 5.2.

around the cycle in G_m many times. A crucial observation is that (a_0, a'_m) is the only pair whose pebbling allows Spoiler to win in one extra move.

Let us regard the additive group $\mathbb{Z}_{2m}$ as a cycle graph with i and j adjacent iff $i - j = \pm 1$. Denote the distance between vertices in this graph by Δ. The same letter will denote the following partial function $\Delta : V(G_m) \times V(H_m) \to \mathbb{Z}$. For two vertices of the same color, say, for a_i and a'_j, we set $\Delta(a_i, a'_j) = \Delta(i, j)$. Note that $\Delta(a_0, a'_m) = m$, which is the largest possible value. Duplicator' strategy will be to keep the value of the Δ-function on the pebbled pair as small as possible.

Specifically, in the first round Duplicator responds to Spoiler's move x with pebbling a vertex x' such that $\Delta(x, x') = 0$ (that is, if $x = a_i, b_i, c_i, d_0$, then $x' = a'_i, b'_i, c'_i, d'_0$ respectively). Suppose that a pair (y, y') is pebbled in the preceding round and Duplicator is still alive. If Spoiler pebbles x in the current round, Duplicator chooses her response x' by the following criteria. Below, $\sim$ denotes the adjacency relation.

- x' should have the same color as x and, moreover, $x' \sim y'$ iff $x \sim y$ (this is always possible unless $(y, y') = (a_0, a'_m)$ and $x = d_0$);
- if there is still more than one choice, x' should minimize the parameter $\Delta(x, x')$.

We do not consider the cases when $x = y$ or when x is pebbled by the pebble removed from y because, in our analysis, we can assume that Spoiler uses an *optimal* strategy, allowing him to win the 2-pebble Σ_1 game on G_m and H_m from the initial position (y, y') in the smallest possible number of rounds (if he does not play optimally, Duplicator survives even longer).

Claim C. If $x \not\sim y$ and $x \neq y$, then $\Delta(x, x') \leq 1$.

Proof of Claim C. Assume first that $x \neq d_0$ and $y \neq d_0$. W.l.o.g., suppose that y and y' are apricot and, specifically, $y' = a'_j$ (the blue and the cyan cases are symmetric to the apricot case). Not to lose immediately, Duplicator cannot pebble x' in $\{c'_{j-1}, a'_j, b'_j\}$, where $j - 1$ is supposed to be an element of $\mathbb{Z}_{2m}$. This can obstruct attaining $\Delta(x, x') = 0$ (if $x \in \{c_{j-1}, a_j, b_j\}$), but then there is a choice of x' with $\Delta(x, x') = 1$.

Assume now that $x = d_0$. Then $x' = d'_0$ if $y' \neq a'_0$ and $x' = d'_1$ otherwise. In both cases $\Delta(x, x') \leq 1$. Finally, let $y = d_0$ and $y' = d'_j$. Then the value $x' = a'_j$ is forbidden and, if this prevents $\Delta(x, x') = 0$, then we have $\Delta(x, x') = 1$. ◁

Consider now the dynamical behaviour of $\Delta(x, x')$, assuming that Duplicator uses the above strategy and Spoiler follows an optimal winning strategy. We have $\Delta(x, x') = 0$ at the beginning of the game and $\Delta(x, x') = m$ at the end (that is, in the round immediately before Spoiler wins). Consider the last round of the game where $\Delta(x, x') \leq 1$. By Claim C, starting from the next round Spoiler always moves along an edge in G_m. Note that, from now on, visiting d_0 earlier than in the very last round would be inoptimal. Therefore, Spoiler walks along the circle. Another consequence of optimality is that he moves always in the same direction.

W.l.o.g., we can suppose that Spoiler moves in the ascending order of indices. Note that $\Delta(x, x')$ increases by 1 only under the transition from $x = a_{2m-2}$ to $x = a_0$ (at this point, the index of x makes a jump in $\mathbb{Z}_{2m}$, while the index of x' moves along $\mathbb{Z}_{2m}$ always continuously). In order to increase $\Delta(x, x')$ from 1 to m, the edge $a_{2m-2}a_0$ must be passed $m - 1$ times. It follows that, before Spoiler wins, the game lasts at least $2 + (m-2) \cdot 3(2m-1) = 6m^2 - 15m + 8$ rounds.

Note that $v(G_m) = 6m - 2$ and $v(H_m) = 8m - 1$. In order to make the number of vertices in both graphs $n = 8m - 1$, let m be multiple of 3 and add two new connected components to G_m, namely the cycle of length $2m$ with alternating colors apricot, blue, and cyan and one isolated vertex of any color. Spoiler can still win by playing in the old component. Since playing in the new components does not help him, the game on the modified G_m and the same H_m lasts at least $6m^2 - 15m + 8 = \frac{3}{32}\,n^2 - O(n)$ rounds. ◀

Lifting it higher

Since $D^2_{\Sigma_i}(G, H) = D^2_{\Pi_i}(H, G)$, the following results hold true as well for $\Pi_i \cap \mathrm{FO}^2$.

▶ **Theorem 5.3.** *Let $i \geq 1$. There are infinitely many colored graphs G and H, both on n vertices, such that G is distinguishable from H in $\Sigma_i \cap \mathrm{FO}^2$ and $D^2_{\Sigma_i}(G; H) > \frac{1}{11 \cdot 9^i}\,n^2 - \frac{1}{11 \cdot 3^i}n$.*

Proof. For infinitely many values of an integer parameter n_0, Theorem 5.2 provides us with colored graphs G_0 and H_0 on n_0 vertices each such that Spoiler has a continuous winning strategy in the 2-pebble Σ_1 game on G_0 and H_0, and $D^2_{\exists}(G_0, H_0) > \frac{1}{11}\,n_0^2$. Let G_i and H_i be now the graphs obtained from G_0 and H_0 by the lifting construction described in Section 2. Note that $v(G_i) = 3\,v(G_{i-1}) + 1$, where $G_0 = G$. It follows that $n = v(G_i) = 3^i n_0 + \frac{3^i - 1}{2}$. The graph G_i is distinguishable from H_i in $\Sigma_i \cap \mathrm{FO}^2$ by part 1 of Lemma 2.2. By part 2 of this lemma, we have $D^2_{\Sigma_i}(G_i, H_i) > \frac{1}{11}\,n_0^2$, which implies the bound stated in terms of n. ◀

Using a similar sequence of graphs, we can also show that $\Sigma_i \cap \mathrm{FO}^2$ is more succinct than $\Sigma_{i-1} \cap \mathrm{FO}^2$. Given i, let us construct G_i and H_i starting from the same G_0 as in the proof of Theorem 5.3 and a slightly modified H_0. Specifically, we make all dandelion vertices in H_0 adjacent; see Fig. 6 for G_0 and H_0, where H_0 is still unmodified. This makes part 4 of Lemma 2.2 applicable, which along with part 2 gives us the following result.

▶ **Theorem 5.4.** *For each $i \geq 2$ there are infinitely many colored graphs G and H, both on n vertices, such that $D^2_{\Sigma_i}(G, H) = O(1)$ while $D^2_{\Sigma_{i-1}}(G, H) < \infty$ and $D^2_{\Pi_i}(G, H) = \Omega(n^2)$.*

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A The proof of Theorem 3.3

The *complement* of a graph G is the graph on the same vertex set $V(G)$ with any two vertices adjacent if and only if they are not adjacent in G. We call a graph *normal* if it has neither isolated nor universal vertex. Note that a graph is normal iff its complement is normal. For every graph G with at least 2 vertices we inductively define its *rank* $\operatorname{rk} G$.

- Graphs of rank 1 are exactly the empty, the complete, and the normal graphs.
- Graphs of rank 2 are exactly the graphs obtained by adding universal vertices to empty graphs, or isolated vertices to complete graphs, or either universal or isolated vertices to normal graphs.
- If $i \geq 2$, disconnected graphs of rank $i+1$ are obtained from connected graphs of rank i by adding a number of isolated vertices.
- For every i, connected graphs of rank i are exactly complements of disconnected graphs of rank i.

A simple inductive argument on the number of vertices shows that all graphs with at least two vertices get ranked. Indeed, if a graph G is normal, complete, or empty, it receives rank 1. This includes the case that G has two vertices. If G does not belong to any of these three classes, it has either isolated or universal vertices. Since graphs with universal vertices are connected and are the complements of graphs with isolated vertices, it suffices to consider the case that G has isolated vertices. Remove all of them from G and denote the result by G'. Note that G' has less vertices than G but still more than one vertex. By the induction assumption, G' is ranked. If $\operatorname{rk} G' = 1$, then $\operatorname{rk} G = 2$ by definition. If $\operatorname{rk} G' > 1$, then G' must be connected (for else it would be normal). Therefore, $\operatorname{rk} G = \operatorname{rk} G' + 1$ by definition.

We now introduce a ranking of vertices in a graph G. If $\operatorname{rk} G = 1$, then all vertices of G get *rank* 1. Suppose that $\operatorname{rk} G > 1$. If G is disconnected, it has at least one isolated vertex; if G is connected, there is at least one universal vertex. Denote the set of such vertices by ∂G. Every vertex in ∂G is assigned *rank* 1. If $u \notin \partial G$, then it is assigned *rank* one greater than the rank of u in the graph $G - \partial G$. The rank of a vertex u in G will be denoted by $\operatorname{rk} u$. It ranges from the lowest value 1 to the highest value $\operatorname{rk} G$. Note that a vertex u with $\operatorname{rk} u < \operatorname{rk} G$ has the same adjacency to all other vertices of equal or higher rank.

Given an integer $m \geq 1$ and a graph G with $\operatorname{rk} G > m$, we define the *m-tail type* of G to be the sequence $(t_0, t_1, \ldots, t_m)$ where $t_0 \in \{conn, disc\}$ depending on whether G is connected or disconnected and, for $i \geq 1$, $t_i \in \{thin, thick\}$ depending on whether G has one or more vertices of rank i.

Furthermore, we define the *kernel* of a graph G to be its subgraph induced on the vertices of rank $\operatorname{rk} G$. Note that the kernel of any G is a graph of rank 1. We define the *head type* of G to be *empty*, *compl*, or *norma* depending on the kernel. We say that graphs G and H are of the same *type* if $\operatorname{rk} G = \operatorname{rk} H$, G and H have the same head type, and if $\operatorname{rk} G > 1$, then they also have the same m-tail type for $m = \operatorname{rk} G - 1$. The single-vertex graph has its own type.

▶ **Lemma 1.1.**

1. *If G and H are of the same type, then $D^2(G, H) = \infty$.*
2. *If G and H have the same m-tail type, then $D^2(G, H) \geq m$.*

▶ **Lemma 1.2.**

1. *For each m-tail type, the class of graphs of this type is definable by a first-order formula with two variables and one quantifier alternation.*
2. *For each G, the class of graphs of the same type as G is definable by a first-order formula with two variables and one quantifier alternation.*

Let C be a class of graphs definable by a formula with two variables of quantifier depth less than m. By Lemma 1.1, C is the union of finitely many classes of graphs of the same type (each of rank at most m) and finitely many classes of graphs of the same m-tail type. By Lemma 1.2, C is therefore definable by a first-order formula with two variables and one quantifier alternation. To complete the proof of Theorem 3.3, it remains to prove the lemmas.

Proof of Lemma 1.1. 1. Let $\operatorname{rk} G = \operatorname{rk} H = m + 1$. Let $V(G) = U_1 \cup \ldots U_{m+1}$ and $V(H) = V_1 \cup \ldots V_{m+1}$ be the partitions of the vertex sets of G and H according to the ranking of vertices. We will describe a winning strategy for Duplicator in the two-pebble game on G and H. We will call a pair of pebbled vertices $(u, v) \in V(G) \times V(H)$ *straight* if $u \in U_i$ and $v \in V_i$ for the same i. Note that both the kernels U_{m+1} and V_{m+1} contain at least 2 vertices and, since G and H are of the same type, $|U_i| = 1$ iff $|V_i| = 1$. This allows Duplicator to play so that the vertices pebbled in each round form a straight pair and the equality relation is never violated. If the head type of G and H is *empty* or *compl*, this strategy is winning because the adjacency of vertices $u \in U_i$ and $u' \in U_j$ depends only on the indices i and j and is the same as the adjacency of any vertices $v \in V_i$ and $v' \in V_j$. It remains to notice that Duplicator can resist also when the game is played inside the normal kernels U_{m+1} and V_{m+1}. In this case she never loses because, for every vertex in a normal graph, she can find another adjacent or non-adjacent vertex, as she desires.

2. We have to show that Duplicator can survive in at least $m - 1$ rounds. Note that both $\operatorname{rk} G \geq m+1$ and $\operatorname{rk} H \geq m+1$. Similarly to part 1, consider partitions $V(G) = U_1 \cup \ldots U_{m+1}$ and $V(H) = V_1 \cup \ldots V_{m+1}$, where U_{m+1} and V_{m+1} now consist of the vertices whose rank is higher than m. In the first round Duplicator plays so that the pebbled vertices form a straight pair. However, starting from the second round it can be for her no more possible to keep the pebbled pairs straight. Call a pair of pebbled vertices $(u, v) \in V(G) \times V(H)$ *skew* if $u \in U_i$ and $v \in V_j$ for different i and j. Assume that Spoiler uses his two pebbles alternatingly (playing with the same pebble in two successive rounds gives him no advantage). Let (u_r, v_r) denote the pair of vertices pebbled the r-th round. If (u_r, v_r) is skew, let S_r denote the minimum s such that $u_r \in U_s$ or $v_r \in V_s$. If (u_r, v_r) is straight, we set $S_r = m+1$. Our goal is to show that, if $S_r = m + 1$, then Duplicator has a non-losing move in the next round such that $S_{r+1} \geq m - 1$ and that, as long as $1 < S_r \leq m$, she has a non-losing move such that $S_{r+1} \geq S_r - 1$. This readily implies that Duplicator does not lose the first $m - 1$ rounds.

To avoid multiple treatment of symmetric cases, we use the following notation. Let $\{G_1, G_2\} = \{G, H\}$. Let $y_1 \in G_1$ and $y_2 \in G_2$ denote the vertices being pebbled in the round $r + 1$, and let $x_1 \in G_1$ and $x_2 \in G_2$ be the vertices pebbled in the round r (in the previous notation, $\{x_1, x_2\} = \{u_r, v_r\}$ and $\{y_1, y_2\} = \{u_{r+1}, v_{r+1}\}$).

Suppose first that $\{x_1, x_2\}$ is a straight pair contained in the slice $U_i \cup V_i$. If $i \leq m$, it makes no problem for Duplicator to move so that the pair $\{y_1, y_2\}$ is also straight. This holds true also if $i = m + 1$ and Spoiler pebbles $y_a \in U_j \cup V_j$ with $j \leq m$. Thus, in these cases $S_{r+1} = S_r = m + 1$. However, if $i = j = m + 1$, moving straight can be always Duplicator's loss. In this case she survives by pebbling a vertex y_{3-a} of rank m or $m - 1$, depending on the adjacency relation between x_a and y_a. In this case $S_{r+1} \geq m - 1$.

Let us accentuate the property of the vertex ranking that is beneficial to Duplicator in the last case. Recall that, if a vertex u is not in the graph kernel, it has the same adjacency to all other vertices of equal or higher rank. If u is adjacent to all such vertices, we say that u is of *universal type*; otherwise we say that it is of *isolated type*. Duplicator uses the fact that the type of a vertex gets flipped when its rank increases by one.

Suppose now that $\{x_1, x_2\}$ is a skew pair. Let $x_1 \in U_i \cup V_i$ and $x_2 \in U_j \cup V_j$ and, w.l.o.g., assume that $i > j$. Since $j = S_r$, it is supposed that $j > 1$. We consider three cases depending on Spoiler's move y_a. In the most favorable for Duplicator case, $\operatorname{rk} y_a < j$. Then Duplicator responds with a vertex y_{3-a} of the same rank, resetting S_{r+1} back to the initial value $m + 1$. If Spoiler pebbles a vertex y_2 of $\operatorname{rk} y_2 \geq j$, then Duplicator responds with a vertex y_1 of $\operatorname{rk} y_1 = j$, keeping $S_{r+1} \geq j = S_r$ (unchanged or reset to $m + 1$). Finally, consider the case when Spoiler pebbles a vertex y_1 of $\operatorname{rk} y_1 \geq j$. Assume that x_2 is of universal type (the other case is symmetric). If y_1 and x_1 are adjacent, then Duplicator responds with a vertex y_2 of $\operatorname{rk} y_2 = i$, keeping $S_{r+1} \geq S_r$. If y_1 and x_1 are not adjacent, then Duplicator responds with y_2 of $\operatorname{rk} y_2 = j - 1$, which is of isolated type. This is the only case when $S_{r+1} = S_r - 1$ decreases. ◀

Proof of Lemma 1.2. 1. Consider an m-tail type $(t_0, t_1, \ldots, t_m)$. Assume that $t_0 = \mathit{conn}$ (the case of $t_0 = \mathit{disc}$ is similar). Let $\sim$ denote the adjacency relation. We inductively define a sequence of formulas $\Phi_s(x)$ with occurrences of two variables x and y and with one free variable:

$$
\begin{aligned}
\Phi_1(x) &\stackrel{\text{def}}{=} \forall y\,(y \sim x \vee y = x),\\
\Phi_{2k}(x) &\stackrel{\text{def}}{=} \forall y\,(\Phi_{2k-1}(y) \vee y \not\sim x),\\
\Phi_{2k+1}(x) &\stackrel{\text{def}}{=} \forall y\,(\Phi_{2k}(y) \vee y \sim x \vee y = x).
\end{aligned}
$$

Here $\Phi_{2k-1}(y)$ is obtained from $\Phi_{2k-1}(x)$ by swapping x and y. A simple inductive argument shows that, if G is a connected graph and $\operatorname{rk} G$ is greater than an odd (resp. even) integer s, then $G, v \models \Phi_s(x)$ exactly when the vertex v is of universal (resp. isolated) type and $\operatorname{rk} v \leq s$.

Furthermore, we define a sequence of closed formulas Ψ_s with alternation number 1:

$$
\begin{aligned}
\Psi_1 &\stackrel{\text{def}}{=} \exists x \Phi_1(x) \wedge \exists x \neg \Phi_1(x),\\
\Psi_2 &\stackrel{\text{def}}{=} \exists x \Phi_1(x) \wedge \exists x \Phi_2(x) \wedge \exists x (\neg \Phi_1(x) \wedge \neg \Phi_2(x)),\\
\Psi_s &\stackrel{\text{def}}{=} \exists x \Phi_1(x) \wedge \exists x \Phi_2(x) \wedge \bigwedge_{i=3}^{s} (\Phi_i(x) \wedge \neg \Phi_{i-2}(x)) \wedge \exists x (\neg \Phi_{s-1}(x) \wedge \neg \Phi_s(x)), \quad s \geq 3.
\end{aligned}
$$

Note that a graph G satisfies Ψ_s if and only if G is connected and $\operatorname{rk} G > s$.

We are now able to define the class of graphs of m-tail type $(t_0, t_1, \ldots, t_m)$ by the conjunction

$$
\Psi_m \wedge \bigwedge_{i=1}^{m} T_i,
$$

where

$$
T_i \stackrel{\text{def}}{=} \exists x \exists y\,(x \neq y \wedge \Phi_i(x) \wedge \neg \Phi_{i-2}(x) \wedge \Phi_i(y) \wedge \neg \Phi_{i-2}(y))
$$

if $t_i = \mathit{thick}$ and

$$
T_i \stackrel{\text{def}}{=} \forall x \forall y\,(\neg \Phi_i(x) \vee \neg \Phi_i(y) \vee \Phi_{i-2}(x) \vee \Phi_{i-2}(y) \vee x = y)
$$

if $t_i = \mathit{thin}$ (if $i \leq 2$, the subformulas with non-positive indices should be ignored).

2. The single-vertex graph is defined by a formula $\forall x \forall y\,(x = y)$. The three classes of graphs of rank 1 are defined by the following three formulas:

$$
\begin{aligned}
&\exists x \exists y\,(x \neq y) \wedge \forall x \forall y\,(x \not\sim y),\\
&\exists x \exists y\,(x \neq y) \wedge \forall x \forall y\,(x = y \vee x \sim y),\\
&\forall x \exists y\,(x \sim y) \wedge \forall x \exists y\,(x \neq y \vee x \not\sim y).
\end{aligned}
$$

Suppose that $\text{rk}\, G = m + 1$ and $m \geq 1$. Let $(t_0, t_1, \ldots, t_m)$ be the m-tail type of G. Assume that G is connected, that is, $t_0 = conn$ (the disconnected case is similar). We use the formulas $\Phi_s(x)$, Ψ_s, and T_i constructed in the first part. If the head type of G is *compl* or *empty* (the former is possible if m is even and the latter if m is odd), then the type of G is defined by

$$\Psi_m \wedge \bigwedge_{i=1}^{m} T_i \wedge \forall x \Phi_{m+1}(x).$$

If the head type of G is *norma*, then the type of G is defined by

$$\Psi_m \wedge \bigwedge_{i=1}^{m} T_i \wedge \neg\exists x \Phi_{m+1}(x).$$

Indeed, $\Psi_m \wedge \bigwedge_{i=1}^{m} T_i$ is true on a graph H if and only if H has the m-tail type $(t_0, t_1, \ldots, t_m)$ and $\text{rk}\, H \geq m + 1$. Let $Q \subset V(H)$ denote the set of vertices not in the tail part. Then Q is a homogeneous set exactly when H satisfies $\forall x \Phi_{m+1}(x)$, and Q spans a normal subgraph exactly when H satisfies $\neg\exists x \Phi_{m+1}(x)$. ◀

B Proof of Theorem 5.1

By Lemma 2.1, we have to prove that, if Spoiler has a winning strategy in the r-round k-pebble Σ_i game on G and H for some r, then he has a winning strategy in the game with $v(G)v(H) + 1$ rounds.

The proof is based on a general game-theoretic argument. Consider a two-person game, where the players follow some fixed strategies and one of them wins. Then the length of the game cannot exceed the total number of all possible positions because once a position occurs twice, the play falls into an endless loop. Here it is assumed that the players' strategies are *positional*, that is, that a strategy of a player maps a current *position* (rather than the sequence of all previous positions) to one of the moves available for the player.

Implementing this scenario for the Σ_i game, we have to overcome two complications. First, we have to "reduce" the space $V(G)^k \times V(H)^k$ of all possible positions in the game, which has size $(v(G)v(H))^k$. Second, we have take care of the fact that, if $i > 1$, then Spoiler's play can hardly be absolutely memoryless in the sense that he apparently has to remember the number of jumps left to him or, at least, the graph in which he moved in the preceding round.

We begin with some notation. Let $\bar{u}$ and $\bar{v}$ be tuples of vertices in G and H, respectively, having the same length no more than k. Given $\Xi \in \{\Sigma, \Pi\}$ and $a \geq 1$, let $R(\Xi, a, \bar{u}, \bar{v})$ be the minimum r such that Spoiler has a winning strategy in the Ξ_a game on G and H starting from the initial position $(\bar{u}, \bar{v})$. Given a k-tuple $\bar{w}$ and $j \leq k$, let $\sigma_j \bar{w}$ denote the $(k-1)$-tuple obtained from $\bar{w}$ by removal of the j-th coordinate. Note that, if $\bar{u} \in V(G)^k$ and $\bar{v} \in V(H)^k$, then

$$R(\Xi, a, \bar{u}, \bar{v}) = \min_{1 \leq j \leq k} R(\Xi, a, \sigma_j \bar{u}, \sigma_j \bar{v}). \quad (8)$$

In order to estimate the length of the k-pebble Σ_i game on G and H, we fix a strategy for Duplicator arbitrarily and consider the strategy for Spoiler as described below. For $i \geq 1$, we will say that $\bar{C}_s = (\Xi_s, a_s, \bar{u}_s, \bar{v}_s)$ is the *position after the s-th round* if

- $\Xi_s = \Sigma$ if in the s-th round Spoiler moved in G and $\Xi_s = \Pi$ if he moved in H;
- during the first s rounds Spoiler jumped from one graph to another $i - a_s$ times;

- after the s-th round the pebbles $p_1, \ldots, p_k$ are placed on the vertices $\bar{u} \in V(G)^k$ and $\bar{v} \in V(H)^k$ (we suppose that in the first round Spoiler puts all k pebbles on one vertex).

Furthermore, we will say that $\tilde{C}_s = (\Xi_s, a_s, \tilde{u}_s, \tilde{v}_s)$ is the *position before the* $(s+1)$*-th move* if in the $(s+1)$-th round Spoiler moves the pebble p_j and $\tilde{u}_s = \sigma_j \bar{u}_s$ and $\tilde{v}_s = \sigma_j \bar{v}_s$.

Let us describe Spoiler's strategy. He makes the first move according to an arbitrarily prescribed strategy that is winning for him in the $D^k_{\Sigma_i}(G, H)$-round k-pebble Σ_i game on G and H. If this move is in G, let $\Xi_1 = \Sigma$ and $a_1 = i$; otherwise $\Xi_1 = \Pi$ and $a_1 = i - 1$. After Duplicator responses, the position $\bar{C}_1$ is specified. Note that $R(\bar{C}_1) < D^k_{\Sigma_i}(G, H)$.

Suppose that the s-th round has been played and after this we have the position $\bar{C}_s = (\Xi_s, a_s, \bar{u}_s, \bar{v}_s)$. In the next round Spoiler plays with the pebble p_j for the smallest value of j such that

$$R(\tilde{C}_s) = R(\bar{C}_s). \tag{9}$$

Such index j exists by (8). Spoiler makes his move according to a prescribed strategy that is winning for him in the $R(\bar{C}_s)$-round k-pebble $(\Xi_s)_{a_s}$ game on G and H with the initial position $(\tilde{u}_s, \tilde{v}_s)$. If he moves in the same graph as in the s-th round, then $\Xi_{s+1} = \Xi_s$ and $a_{s+1} = a_s$; otherwise Ξ_{s+1} gets flipped and $a_{s+1} = a_s - 1$.

Note that $a_{s+1} \le a_s$ and, if $\Xi_{s+1} \ne \Xi_s$, then $a_{s+1} < a_s$. Since Spoiler in each round uses a strategy optimal for the rest of the game,

$$R(\bar{C}_{s+1}) < R(\bar{C}_s). \tag{10}$$

It follows that the described strategy allows Spoiler to win the Σ_i game on G and H in at most $D^k_{\Sigma_i}(G, H)$ moves.

We now estimate the length of the game from above. Suppose that after the t-th round Duplicator is still alive. Due to (9) and (10),

$$R(\tilde{C}_1) > R(\tilde{C}_2) > \ldots > R(\tilde{C}_t).$$

It follows that the elements of the sequence $\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_t$ are pairwise distinct. We conclude from here that the elements of the sequence $(\tilde{u}_1, \tilde{v}_1), (\tilde{u}_2, \tilde{v}_2), \ldots, (\tilde{u}_t, \tilde{v}_t)$ are pairwise distinct too. Indeed, let $s' > s$. If $a_{s'} = a_s$, then $\Xi_s = \Xi_{s'}$. Since $\tilde{C}_s \ne \tilde{C}_{s'}$, we have $(\tilde{u}_s, \tilde{v}_s) \ne (\tilde{u}_{s'}, \tilde{v}_{s'})$. If $a_{s'} < a_s$, the same inequality follows from the fact that $R(\Xi, a, \tilde{u}, \tilde{v}) \le R(\Xi', a', \tilde{u}, \tilde{v})$ whenever $a' < a$.

Since $(\tilde{u}_s, \tilde{v}_s)$ ranges over $V(G)^{k-1} \times V(H)^{k-1}$, we conclude that $t \le (v(G)v(H))^{k-1}$ and, therefore, Spoiler wins in the round $(v(G)v(H))^{k-1} + 1$ at latest.