Pattern Generation by Cellular Automata* (Invited Talk)

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Abstract

A one-dimensional cellular automaton is a discrete dynamical system where a sequence of symbols evolves synchronously according to a local update rule. We discuss simple update rules that make the automaton perform multiplications of numbers by a constant. If the constant and the number base are selected suitably the automaton becomes a universal pattern generator: all finite strings over its state alphabet appear from a finite seed. In particular we consider the automata that multiply by constants 3 and 3/2 in base 6. We discuss the connections of these automata to some difficult open questions in number theory, and we pose several further questions concerning pattern generation in cellular automata.

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1 Enumerating all patterns by a cellular automaton

Cellular Automata (CA) are parallel and synchronous rewrite systems with local dependencies. The system consists of a regular grid of *cells*, each storing a single symbol called the *state* of the cell. The cells change their states synchronously according to a *local update rule* that specifies the new state depending on the local pattern of states around the cell. As cellular automata obey fundamental principles of physics such as locality and uniformity in space and time, they have found applications in various modeling situations of natural systems [2].

Cellular automata were first introduced by John von Neumann, following a suggestion by Stanislaw Ulam, to demonstrate an abstract universal constructor in an artificial setting [9]. Since then, the complexity that can arise from simple local rules and simple seed patterns has been demonstrated several times. Most notably, the well-known *Game-of-Life* cellular automaton by John Conway supports universal computation [1], as does *Rule 110*, a one-dimensional cellular automaton with binary alphabet and radius-one local update rule [3].

In this talk we consider a question asked by Stanislaw Ulam about generating all patterns from a single finite seed [8, page 30]. The problem is to design a cellular automaton rule and an initial configuration with all but finitely many cells in null states such that in the evolution that follows all finite patterns over the state alphabet will appear. We use two simple facts to design such a rule [6]: (i) the powers of a number n written in base b contain all finite digit sequences if n is not a rational power of b, and (ii) the multiplication of numbers by n in base b is a cellular automaton if all prime factors of n also divide b. Smallest such example

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is the cellular automaton $F_{\times 3}$ that multiplies by $n = 3$ in base $b = 6$. The evolution from a single digit 1 looks as follows:

															1				
															3				
														1	3				
														4	3				
													2	1	3				
												1		4	3				
												3	2	1	3				
											1	4		4	3				
											5		2	1	3				
										2	3	1		4	3				
									1	1	3	3	2	1	3				
									3	4	4	4		4	3				
								1	5	2	2		2	1	3				
								5	4	1		1		4	3				
							2	5		3		3	2	1	3				
						1	2	3	1	3	1	4		4	3				
						4	1	3	4	3	5		2	1	3				
					2		4	5	1	5	3	1		4	3				
				1		2	2	3	5	4	3	3	2	1	3				

The automaton is time reversible, which means that another cellular automaton traverses the same configurations backwards in time. The existence of this simple solution raises other interesting questions to investigate:

- Do there exist analogous universal pattern generators also in two- and higher dimensional cellular spaces ?
- Does there exist a solution with fewer than six states ? In particular: is there a universal pattern generator over the binary alphabet ?
- Does there exist a solution that generates all patterns at all positions ? In terms of the usual product topology on the configuration space: Does there exist a cellular automaton with a dense orbit of finite configurations ?

2 Number theoretic questions

A candidate to solve the last problem is obtained by combining $F_{\times 3}$ with a suitable right shift. This suggests the automaton $F_{\times 3/2}$ that multiplies numbers in base 6 by constant 3/2. The problem arises to determine if all sequences of digits get generated next to the radix point, i.e., whether the fractional parts of the powers of 3/2 are dense in the interval $[0, 1]$. More precisely, one needs to find an integer m such that the fractional parts of $\xi(3/2)^i$ are dense for all $\xi = m/6^k$.

The automaton that multiplies by 3/2 relates also to other number theoretic problems in a natural way [5]. In [7], a Z-*number* was defined to be any real $\xi > 0$ with the property that the fractional part of $\xi(3/2)^i$ is less than one half for all $i = 0, 1, 2 \ldots$. The problem of whether any Z-numbers exist is still unsolved. If $\xi(3/2)^i$ is written in base 6, the requirement is simply that the first digit after the radix point is 0, 1 or 2 (with the minor exception that the fractional part may not be .2555 . . .). Hence the existence of Z-numbers can be rephrased as a question concerning time evolutions by $F_{\times 3/2}$ at a single site.

Finally, by adding a new state to represent a floating radix point, we modify $F_{\times 3}$ to simulate the Collatz-function

$$m \mapsto \begin{cases} m/2, & m \text{ even}, \\ 3m+1, & m \text{ odd}, \end{cases}$$

on base 6 representations of positive integers [5].

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