Towards CERes in intuitionistic logic

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Abstract

Cut-elimination, introduced by Gentzen, plays an important role in automating the analysis of mathematical proofs. The removal of cuts corresponds to the elimination of intermediate statements (lemmas), resulting in an analytic proof. **CERes** is a method of cut-elimination by resolution that relies on global proof transformations, in contrast to reductive methods, which use local proof-rewriting transformations. By avoiding redundant operations, it obtains a speed-up over Gentzen's traditional method (and its variations). **CERes** has been successfully implemented and applied to mathematical proofs, and it is fully developed for classical logic (first and higher order), multi-valued logics and Gödel logic. But when it comes to mathematical proofs, intuitionistic logic also plays an important role due to its constructive characteristics and computational interpretation.

This paper presents current developments on adapting the **CERes** method to intuitionistic sequent calculus **LJ**. First of all, we briefly describe the **CERes** method for classical logic and the problems that arise when extending the method to intuitionistic logic. Then, we present the solutions found for the mentioned problems for the subclass $\mathbf{LJ}^-$ (the class of intuitionistic proofs of an end-sequent containing no strong quantifiers and no formula on the right). In addition, we explain, with an example, some ideas for improving the method and covering a bigger fragment of **LJ** proofs. Finally, we summarize the results and point the direction for future research.

1998 ACM Subject Classification F.4.1 Mathematical Logic

Keywords and phrases cut-elimination, resolution, **LJ**

Digital Object Identifier 10.4230/LIPIcs.CSL.2012.485

1 Introduction

Proof analysis is an essential part of mathematical activity, since it leads often to better proofs and occasionally to the discovery of important new mathematical concepts that allow the structuring of existing arguments [15]. Abstract notions such as *groups* and *probability*, for instance, are clear examples of concepts that are undoubtfully useful for organizing common patterns of mathematical reasoning.

The elimination of unnecessary lemmas from a proof is a prominent example of a technique for obtaining potentially simpler (or at least different) proofs. When a constructive mathematical proof is formalized in the sequent calculus **LJ**, lemmas correspond to cuts.

$$\frac{\Gamma \vdash A \quad \Gamma', A \vdash C}{\Gamma, \Gamma' \vdash C}\ cut$$

A proof without cuts has the subformula property: all formulas on the proof are (instances of) subformulas of end-sequent formulas. Consequently, cut-free proofs of a theorem will use only the theorem's theory itself. The main result on cut-elimination - the *Hauptsatz* - was proven by Gentzen [8, 9] in 1935. It states that the cut rule is admissible for the sequent

Computer Science Logic 2012 (CSL'12).
Editors: Patrick Cégielski, Arnaud Durand; pp. 485–499
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

calculi **LK** and **LJ**, for classical and intuitionistic logics respectively. Gentzen's proof of the *Hauptsatz* actually contains an algorithm for removing the cuts from a proof. Therefore, cut-elimination, seen as a method to remove lemmas from formalized proofs, is one of the most important techniques for automating proof analysis. Gentzen's method and some of its variants are often referred to as *reductive cut-elimination*, because they are based on local proof-rewriting rules that gradually reduce the grade and rank of cuts.

The method **CERes** [5] (cut elimination by resolution) is an alternative to reductive cut-elimination, and it is proven to display a non-elementary speed up over the latter. **CERes** was first developed for first order classical logic, and then extended to second and higher order logic [12, 11]. It has also been adapted to multi-valued logics [6] and Gödel logic [1]. The method has been implemented[1] and applied successfully to proofs of moderate size, such as the tape proof [2] and the lattice proof [13], in fully automatic mode. Also, Fürstenberg's proof of the infinitude of primes was successfully transformed, semi-automatically, into Euclid's argument of prime construction using **CERes** [3].

Intuitionistic logic, in contrast to classical logic, is based on a natural proof semantics [10] which is reflected in the rules of natural deduction. Consequently, from an intuitionistic proof of $A \vee B$, one can actually obtain a proof of one of the disjuncts, and from an intuitionistic proof of $\exists x.P(x)$, one can obtain a *witness* a such that $P(a)$ is true. This is not always the case in classical proofs. For this reason, intuitionistic logic is often referred to as a constructive logic. This is particularly useful in mathematics when one wants not only to guarantee the existence of a solution but to actually find it. This constructivism also makes intuitionistic logic more suitable for modeling computations, since constructive proofs can be directly related to algorithms.

The importance of intuitionistic logic for mathematics and computer science is the main motivation for extending the **CERes** method to **LJ**. This paper presents the results obtained so far, while pursuing this goal. More specifically, we present the **CERes** method for a subclass of **LJ** proofs, namely, proofs with end sequents having no strong quantifiers and no formula on the right side. This class represents proofs by contradiction in **LJ**. Observe that a proof of the end sequent $\Gamma \vdash F$ can be transformed into a proof by contradiction by applying the $\neg_l$ rule and obtaining $\Gamma, \neg F \vdash$ as an end-sequent.

The paper is organized as follows: Section 2 briefly describes the **CERes** method for classical logic and the problems that arise when extending the method to intuitionistic logic; Section 3 presents the solutions found for the mentioned problems and shows the new revised method applied to an example; Section 4 explains some ideas for improving the method and covering a bigger fragment of **LJ** proofs; and finally, Section 5 summarizes the results and points the direction for future research.

2 CERes in LK

The **CERes** method for classical logic is based on the computation of three structures from an **LK** proof φ: a characteristic clause set $CL(\varphi)$, a resolution refutation of this set and proof projections of φ w.r.t the elements in $CL(\varphi)$. By merging instances of the projections and the resolution refutation properly, one obtains a proof with only atomic cuts (ACNF - atomic cut normal form) of the same end sequent of φ. These three structures are informally explained in the subsections below. A more detailed and precise definition of **CERes** for **LK** is available in [7].

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[1] http://code.google.com/p/gapt/

The remaining atomic cuts on the final proof are inessential [16], and, since we use standard axioms ($A \vdash A$, where A is atomic), these can be eliminated using reductive cut-elimination.

2.1 Characteristic clause set

The characteristic clause set is computed by removing from φ all the rules that operate on end-sequent ancestors and the end-sequent ancestors themselves (including the end-sequent). After that, what is left is a derivation of the empty sequent from a set of axioms. These axioms contain only cut-ancestors and they compose the characteristic clause set. It is important to note that some branches of φ might be merged during this procedure, if they resulted from the application of a binary rule on an end-sequent ancestor. Consider the sub-derivation of a proof below, in which cut ancestors are marked with $\star$:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{I}
\UnaryInfC{$P(a)^\star \vdash P(a)$}
\AxiomC{}
\RightLabel{I}
\UnaryInfC{$Q(a) \vdash Q(a)^\star$}
\RightLabel{$\rightarrow_l$}
\BinaryInfC{$P(a)^\star, P(a) \rightarrow Q(a) \vdash Q(a)^\star$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$P(a) \rightarrow Q(a) \vdash (P(a) \rightarrow Q(a))^\star$}
\RightLabel{$\exists_r$}
\UnaryInfC{$P(a) \rightarrow Q(a) \vdash \exists x.(P(x) \rightarrow Q(x))^\star$}
\AxiomC{}
\RightLabel{I}
\UnaryInfC{$P(b)^\star \vdash P(b)$}
\AxiomC{}
\RightLabel{I}
\UnaryInfC{$Q(b) \vdash Q(b)^\star$}
\RightLabel{$\rightarrow_l$}
\BinaryInfC{$P(b)^\star, P(b) \rightarrow Q(b) \vdash Q(b)^\star$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$P(b) \rightarrow Q(b) \vdash (P(b) \rightarrow Q(b))^\star$}
\RightLabel{$\exists_r$}
\UnaryInfC{$P(b) \rightarrow Q(b) \vdash \exists x.(P(x) \rightarrow Q(x))^\star$}
\RightLabel{$\vee_l$}
\BinaryInfC{$(P(a) \rightarrow Q(a)) \vee (P(b) \rightarrow Q(b)) \vdash \exists x.(P(x) \rightarrow Q(x))^\star$}
\end{prooftree}
$$

By removing all inferences on end-sequent ancestors, we obtain the following derivation:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{I}
\UnaryInfC{$P(a)^\star, P(b)^\star \vdash Q(a)^\star, Q(b)^\star$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$P(a)^\star \vdash Q(a)^\star, (P(b) \rightarrow Q(b))^\star$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$\vdash (P(a) \rightarrow Q(a))^\star, (P(b) \rightarrow Q(b))^\star$}
\RightLabel{$\exists_r$}
\UnaryInfC{$\vdash (P(a) \rightarrow Q(a))^\star, \exists x.(P(x) \rightarrow Q(x))^\star$}
\RightLabel{$\exists_r$}
\UnaryInfC{$\vdash \exists x.(P(x) \rightarrow Q(x))^\star, \exists x.(P(x) \rightarrow Q(x))^\star$}
\end{prooftree}
$$

In this case, the sequent $P(a), P(b) \vdash Q(a), Q(b)$ would be in the characteristic clause set.

▶ Remark. Observe that, in classical logic, this is not a problem, but in intuitionistic logic (**LJ** - Figure 1) this is not a well-formed sequent since it has more than one formula on the right side. Note also that the original derivation could easily be part of an **LJ** proof, but the transformed derivation contains non-intuitionistic sequents. Thus, sequents of the characteristic clause set might be classical, even if we start with an intuitionistic proof. As they will be part of the final proof, it is not desirable that they are classical, because we expect to obtain a cut-free proof in **LJ**.

2.2 Resolution refutation

By the transformation exemplified above, there exists an **LK** derivation of the empty sequent from the clauses of $CL(\varphi)$. Since **LK** is sound, the set $CL(\varphi)$ is unsatisfiable. And since the resolution calculus is complete, there exists a resolution refutation of $CL(\varphi)$.

The resolution refutation, which can be obtained with a resolution theorem prover, is used as a basis for the final proof, and can be seen as its skeleton. The resolution steps will correspond to the atomic cuts.

2.3 Projections

The projections are derivations of a sequent from $CL(\varphi)$ merged with the end-sequent. Dually to what was done for the characteristic clause set, they are constructed by removing the rules

$$\frac{}{A \vdash A}\,[\mathrm{I}] \qquad \frac{\Gamma_1 \vdash P \quad \Gamma_2, P \vdash C}{\Gamma_1, \Gamma_2 \vdash C}\,[\mathrm{Cut}]$$

$$\frac{\Gamma \vdash P}{\Gamma, \neg P \vdash}\,[\neg_l] \qquad \frac{\Gamma, P \vdash}{\Gamma \vdash \neg P}\,[\neg_r]$$

$$\frac{P_i, \Gamma \vdash C}{P_1 \wedge P_2, \Gamma \vdash C}\,[\wedge_{li}] \qquad \frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}\,[\wedge_r]$$

$$\frac{P, \Gamma \vdash C \quad Q, \Gamma \vdash C}{P \vee Q, \Gamma \vdash C}\,[\vee_l] \qquad \frac{\Gamma \vdash P_i}{\Gamma \vdash P_1 \vee P_2}\,[\vee_{ri}]$$

$$\frac{\Gamma_1 \vdash P \quad Q, \Gamma_2 \vdash C}{P \rightarrow Q, \Gamma_1, \Gamma_2 \vdash C}\,[\rightarrow_l] \qquad \frac{\Gamma, P \vdash Q}{\Gamma \vdash P \rightarrow Q}\,[\rightarrow_r]$$

$$\frac{P\{x \leftarrow \alpha\}, \Gamma \vdash C}{\exists x.P, \Gamma \vdash C}\,[\exists_l] \qquad \frac{\Gamma \vdash P\{x \leftarrow t\}}{\Gamma \vdash \exists x.P}\,[\exists_r]$$

$$\frac{P\{x \leftarrow t\}, \Gamma \vdash C}{\forall x.P, \Gamma \vdash C}\,[\forall_l] \qquad \frac{\Gamma \vdash P\{x \leftarrow \alpha\}}{\Gamma \vdash \forall x.P}\,[\forall_r]$$

$$\frac{P, P, \Gamma \vdash C}{P, \Gamma \vdash C}\,[C_l] \qquad \frac{\Gamma \vdash C}{P, \Gamma \vdash C}\,[W_l] \qquad \frac{\Gamma \vdash}{\Gamma \vdash P}\,[W_r]$$

Figure 1 **LJ**: Sequent calculus for intuitionistic logic. It is assumed that α is a variable not contained in P, Γ or C and t does not contain variables bound in P.

applied to cut-ancestors. Each sequent (clause) in $CL(\varphi)$ will generate a projection, possibly with variables that can later be instantiated to form the final proof. Using the same example as before, the projection corresponding to the clause $P(a), P(b) \vdash Q(a), Q(b)$ is the following:

$$\dfrac{\dfrac{\dfrac{}{P(a)^\star \vdash P(a)}\,I \quad \dfrac{}{Q(a) \vdash Q(a)^\star}\,I}{P(a)^\star, P(a) \rightarrow Q(a) \vdash Q(a)^\star}\,\rightarrow_l \quad \dfrac{\dfrac{}{P(b)^\star \vdash P(b)}\,I \quad \dfrac{}{Q(b) \vdash Q(b)^\star}\,I}{P(b)^\star, P(b) \rightarrow Q(b) \vdash Q(b)^\star}\,\rightarrow_l}{P(a)^\star, P(b)^\star, (P(a) \rightarrow Q(a)) \vee (P(b) \rightarrow Q(b)) \vdash Q(a)^\star, Q(b)^\star}\,\vee_l$$

▶ Remark. Note that, once again, the resulting derivation is classical. Since these will be directly used on the final proof, it is also a problem that should be solved if we expect the output of **CERes** to be intuitionistic when applied to **LJ** proofs.

Even if the resulting projections were intuitionistic, they are merged with the resolution refutation of $CL(\varphi)$, and if two sequents with one formula on the right side are merged, the resulting sequent will have two formulas on the right side and will then be classical.

3 CERes in LJ

The problems described in Section 2 were addressed and solved for a subclass of **LJ**, namely **LJ**$^-$ (Definition 2). The resulting **iCERes** method is presented in this section.

▶ **Definition 1** (Strong and weak quantifiers)**.** Let F be a formula. If $\forall x$ occurs positively (negatively) in F, then $\forall x$ is called a *strong (weak) quantifier*. If $\exists x$ occurs positively (negatively) in F, then $\exists x$ is called a *weak (strong) quantifier*. Let $A_1, ..., A_n \vdash B_1, ..., B_m$ be a sequent. A quantifier is called strong (weak) on this sequent if it is strong (weak) on the corresponding formula $A_1 \wedge ... \wedge A_n \to B_1 \vee ... \vee B_m$.

▶ **Definition 2** (**LJ**$^-$)**.** **LJ**$^-$ is the set of **LJ** proofs of end-sequents with no formula on the right side and no strong quantifiers.

Note that, in principle, the condition of absence of formulas on the right side of the end-sequent, can be satisfied by simply applying the $\neg_l$ inference rule at the bottom of the proof, in order to negate and shift to the left the formula that occurred in the right side. The other requirement, the absence of strong quantifiers, can be achieved by using methods of skolemization of **LJ** proofs [4]. A more detailed discussion of the implications of using these transformations is postponed to Section 4.

This class contains proofs by contradiction in **LJ**, which are exactly those proofs of $\Gamma \vdash F$ transformed into a proof of $\Gamma, \neg F \vdash$ with the application of a $\neg_l$ rule. It is also worth mentioning that **LJ**$^-$ is a "nontrivial" class of proofs, in the sense that there exist sequents of the form $\Gamma \vdash$ which are provable classically but not intuitionistically (e.g. $\neg \exists x. \forall y. (P(x) \to P(y)) \vdash$).

Although the problem was solved only for a subclass of **LJ** proofs, all definitions and proofs on this section are valid for full **LJ**, with the exception of Theorem 16.

▶ **Definition 3** (Intuitionistic Clause)**.** An *intuitionistic clause* is a sequent composed only of atoms or negated atoms and with the right hand side containing at most one formula.

▶ **Definition 4** (Intuitionistic Clause Set with Negations)**.** The *intuitionistic characteristic clause set* is built analogously to the usual characteristic clause set, except that all the formulas on the right hand side are negated and added to the left hand side:

- If ν is an axiom, then $CL^I(\nu)$ is the set containing the sub-sequent composed only of the formulas that are cut-ancestors, such that all the formulas that would appear on the right-hand side are negated and added to the left-hand side.
- If ν is the result of the application of a unary rule on μ, then $CL^I(\nu) = CL^I(\mu)$
- If ν is the result of the application of a binary rule on μ_1 and μ_2, we have to distinguish two cases:
  - If the rule is applied to ancestors of a cut formula, then $CL^I(\nu) = CL^I(\mu_1) \cup CL^I(\mu_2)$.
  - If the rule is applied to ancestors of the end-sequent, then $CL^I(\nu) = CL^I(\mu_1) \times CL^I(\mu_2)$.

Where[2]:

$$CL^I(\mu_1) \times CL^I(\mu_2) = \{C \circ D | C \in CL^I(\mu_1), D \in CL^I(\mu_2)\}$$

Note that since the formulas on the right hand side are moved to the left hand side already on the axioms, the clauses always have the right side empty. This guarantees that we always have intuitionistic sequents and no conflicts arise while merging.

▶ **Theorem 5** (Refutability of the Intuitionistic Clause Set)**.** *The intuitionistic clause set is* ***LJ****-refutable.*

Proof. Let φ be an **LJ** proof and $CL^I(\varphi)$ be its intuitionistic clause set built according to Definition 4 and $CL(\varphi)$ be its classical clause set obtained by the classical version of **CERes**. For each clause $C_i = A^i_1, ..., A^i_{n_i} \vdash B^i_1, ..., B^i_{m_i}$ of the classical clause set, we build the closed formula $F_i = \forall \overline{x}. \neg (A^i_1 \wedge ... \wedge A^i_{n_i} \wedge \neg B^i_1 \wedge ... \wedge \neg B^i_{m_i})$.

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[2] The operation $\circ$ represents the merging of sequents, i.e., $(\Gamma \vdash \Delta) \circ (\Gamma' \vdash \Delta') = \Gamma, \Gamma' \vdash \Delta, \Delta'$.

By previous results, summarized in Section 2, we know that there is an **LK** refutation ψ of $CL(\varphi)$:

$$\dfrac{\dfrac{C_1}{\vdots} \quad \cdots \quad \dfrac{C_k}{\vdots}}{\vdash}$$

By merging each formula F_i to its corresponding clause C_i and propagating it down the refutation, we obtain an LK proof ψ_1 from the formulas $F_i, A^i_1, ..., A^i_{n_i} \vdash B^i_1, ..., B^i_{m_i}$ of the end-sequent $F_1, ..., F_k \vdash$:

$$\dfrac{\dfrac{\overset{\varphi_1}{F_1 \circ C_1}}{\vdots} \quad \cdots \quad \dfrac{\overset{\varphi_k}{F_k \circ C_k}}{\vdots}}{F_1, ..., F_k \vdash}$$

where each φ_i is a derivation of $F_i \circ C_i$ from tautological axioms. We can transform the proof ψ_1 into a proof ψ_2 of $\vdash \neg(F_1 \wedge ... \wedge F_k)$:

$$\dfrac{\dfrac{\dfrac{\psi_1}{F_1, ..., F_k \vdash}}{F_1 \wedge ... \wedge F_k \vdash}\ \wedge_l \times (k-1)}{\vdash \neg(F_1 \wedge ... \wedge F_k)}\ \neg_r$$

Since the axioms of this proof are tautological, we can transform this into an **LJ** proof ψ_3 via the following negative translation [14]:

$$\begin{array}{rl} A & \rightarrow \neg\neg A^* \\ A^* & \rightarrow A \text{ (if A is an atom)} \\ (\neg A)^* & \rightarrow \neg A^* \text{ (if A is an atom)} \\ (A \square B)^* & \rightarrow (A^* \square B^*), \square \in \{\wedge, \vee, \Rightarrow\} \\ (\exists x.A)^* & \rightarrow \exists x.A^* \\ (\forall x.A)^* & \rightarrow \forall x.\neg\neg A^* \end{array}$$

The end-sequent of ψ_3 is $\vdash \neg(\tilde{F}_1 \wedge ... \wedge \tilde{F}_k)$, where each $\tilde{F}_i$ is the negative translation of F_i. Note that $\vdash \neg\neg\neg A$ is **LJ**-equivalent to $\vdash \neg A$, so there is still only one negation on this end-sequent.

From the proof ψ_3, we can construct the proof ψ_4:

$$\dfrac{\dfrac{\psi_3}{\vdash \neg(\tilde{F}_1 \wedge ... \wedge \tilde{F}_k)} \qquad \dfrac{\dfrac{\overset{\Xi_1}{\vdash \tilde{F}_1} \quad ... \quad \overset{\Xi_n}{\vdash \tilde{F}_k}}{\vdash \tilde{F}_1 \wedge ... \wedge \tilde{F}_k}\ \wedge_r \times k}{\neg(\tilde{F}_1 \wedge ... \wedge \tilde{F}_k) \vdash}\ \neg_l}{\vdash}\ cut$$

Note that the end-sequent of each derivation Ξ_i is of the form:

$$\vdash \neg\neg\forall x_1....\neg\neg\forall x_r.\neg\neg\neg(A^i_1 \wedge ... \wedge A^i_{n_i} \wedge \neg B^i_1 \wedge ... \wedge \neg B^i_{m_i})$$

And each Ξ_i is:

$$\begin{array}{c l}
A_1^i, \ldots, A_{n_i}^i, \neg B_1^i, \ldots, \neg B_{m_i}^i \vdash & \\ \hline\hline
A_1^i \wedge \ldots \wedge A_{n_i}^i \wedge \neg B_1^i \wedge \ldots \wedge \neg B_{m_i}^i \vdash & \wedge_l \times (n_i + m_i - 1) \\ \hline\hline
\vdash \neg\neg\neg(A_1^i \wedge \ldots \wedge A_{n_i}^i \wedge \neg B_1^i \wedge \ldots \wedge \neg B_{m_i}^i) & \neg_r, \neg_l, \neg_r \\ \hline
\vdots & \\ \hline\hline
\vdash \neg\neg\forall x_2 . \ldots \neg\neg\forall x_r . \neg\neg\neg(A_1^i \wedge \ldots \wedge A_{n_i}^i \wedge \neg B_1^i \wedge \ldots \wedge \neg B_{m_i}^i) & \forall_r, \neg_l, \neg_r \\ \hline\hline
\vdash \neg\neg\forall x_1 . \ldots \neg\neg\forall x_r . \neg\neg\neg(A_1^i \wedge \ldots \wedge A_{n_i}^i \wedge \neg B_1^i \wedge \ldots \wedge \neg B_{m_i}^i) & \forall_r, \neg_l, \neg_r
\end{array}$$

So, we obtain an **LJ**-refutation of the clauses $A_1^i, \ldots, A_{n_i}^i, \neg B_1^i, \ldots, \neg B_{m_i}^i \vdash$ for every i, which are exactly the elements of $CL^I(\varphi)$. ◀

▶ **Definition 6** ($\mathbf{R}^{\neg}$)**.** The $\mathbf{R}^{\neg}$ calculus is a resolution calculus with the following rules:

$$\dfrac{\Gamma \vdash A \quad \Gamma', A' \vdash \Delta}{\Gamma\sigma, \Gamma'\sigma \vdash \Delta\sigma}\ R \qquad \dfrac{\Gamma \vdash \neg A \quad \Gamma', \neg A' \vdash \Delta}{\Gamma\sigma, \Gamma'\sigma \vdash \Delta\sigma}\ R\neg$$

$$\dfrac{A, A' \vdash \Delta}{A\sigma \vdash \Delta\sigma}\ C \qquad \dfrac{\neg A, \neg A' \vdash \Delta}{\neg A\sigma \vdash \Delta\sigma}\ C\neg$$

$$\dfrac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ \neg_r \qquad \dfrac{\Gamma \vdash A}{\Gamma, \neg A \vdash}\ \neg_l$$

Where Δ is a multi-set with at most one formula[3] and σ is the most general unifier of A and A'.

The choice of a modified resolution calculus is justified by the fact that a proof in this calculus will be used as a part of the final **LJ** proof. In fact, any calculus for intuitionistic logic could be used for the proof search itself, but then we would need a translation of the corresponding proof object into an **LJ**-proof to use in this method.

▶ **Lemma 7.** *If φ is an **LJ**-refutation of a set of intuitionistic clauses (Definition 3) $\mathcal{S}$ and φ' is a normal form of φ with respect to reductive cut-elimination, then any cut-formula in φ' is either an atom or a negated atom.*

Proof. Assume, for the sake of contradiction, that φ' contains a cut whose cut-formula F is neither an atom nor a negated atom. Since the axioms of φ' contain only atoms or negated atoms, it must be the case that the left and right occurrences of F in this cut are introduced, respectively, by inferences ρ_l and ρ_r occurring somewhere in φ'. Two cases can be distinguished:

1. Both ρ_l and ρ_r occur immediately above the cut: in this case, either a grade reduction rule can be applied, if both ρ_l and ρ_r are logical inferences, or a reduction over weakening, if at least one of them is a weakening.
2. At least one of ρ_l and ρ_r does not occur immediately above the cut: in this case, a rank reduction rule can be applied.

In both cases, the assumption contradicts the fact that φ' is in normal form. Therefore, it must be the case that all cut-formulas in φ' are either atoms or negated atoms. ◀

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[3] Throughout the paper, Δ stands as a multi-set with at most one formula.

▶ **Lemma 8.** *If φ is an **LJ**-refutation of a set of intuitionistic clauses $\mathcal{S}$ and φ' is a normal form of φ with respect to reductive cut-elimination, then the only inference rules used in φ' are $\neg_l$, $\neg_r$, cut and left contraction.*

Proof. Assume, for the sake of contradiction, that there is an inference ρ in φ' that is neither a $\neg_l$, nor a $\neg_r$, nor a *cut* inference, nor a left contraction, and let F be its main formula. Since φ' is an **LJ**-refutation, its end-sequent is empty. Hence, F must be the ancestor of a cut-formula, and since F is neither an atom nor a negated atom, its descendant cut-formula is also neither an atom nor a negated atom. However, this contradicts Lemma 7, according to which any cut-formula in φ' must be either an atom or a negated atom. Therefore, inferences that are neither $\neg_l$, nor $\neg_r$, nor *cut*, nor left contraction cannot occur in φ'. ◀

▶ Remark. All logical inferences that are neither $\neg_l$ nor $\neg_r$ disappear when φ is rewritten to φ' due to grade reduction rules. This is exemplified below for the conjunction case:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash A} \quad \overset{\varphi_2}{\Gamma_2 \vdash B}}{\Gamma_1, \Gamma_2 \vdash A \wedge B}\wedge_r \quad \dfrac{\overset{\varphi_3}{\Gamma_3, A, B \vdash \Delta}}{\Gamma_3, A \wedge B \vdash \Delta}\wedge_l}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\mathit{cut} \quad \Rightarrow \quad \dfrac{\overset{\varphi_1}{\Gamma_1 \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma_2 \vdash B} \quad \overset{\varphi_3}{\Gamma_3, A, B \vdash \Delta}}{\Gamma_2, \Gamma_3, A \vdash \Delta}\mathit{cut}}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\mathit{cut}$$

The same cannot be done with negation inferences. Observe that, as usual, the grade reduction for negation requires the cut-formulas to be introduced by $\neg_l$ and $\neg_r$:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1, A \vdash}}{\Gamma_1 \vdash \neg A}\neg_r \quad \dfrac{\overset{\varphi_2}{\Gamma_2 \vdash A}}{\Gamma_2, \neg A \vdash}\neg_l}{\Gamma_1, \Gamma_2 \vdash}\mathit{cut} \quad \Rightarrow \quad \dfrac{\overset{\varphi_1}{\Gamma_1, A \vdash} \quad \overset{\varphi_2}{\Gamma_2 \vdash A}}{\Gamma_1, \Gamma_2 \vdash}\mathit{cut}$$

However, since the intuitionistic clause can have negated atoms, it may be the case that, (at least) one of the cut-formulas was directly introduced by an axiom, as shown in the example proof below:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1, A \vdash}}{\Gamma_1 \vdash \neg A}\neg_r \quad \dfrac{}{\Gamma_2, \neg A \vdash}}{\Gamma_1, \Gamma_2 \vdash}\mathit{cut}$$

In such cases, the grade reduction rule for negation cannot be applied, and hence the negation inference and the cut with a negated atomic formula remain.

▶ **Lemma 9.** *If φ is an **LJ**-refutation of an unsatisfiable set of intuitionistic clauses $\mathcal{S}$ and φ' is a normal form of φ with respect to reductive cut-elimination, then the axioms of φ' are instances of the clauses of $\mathcal{S}$.*

Proof. On applying the rewriting rules for cut-elimination, the initial sequents are not altered, except for the quantifier grade reduction rules, shown below:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash F(\alpha)}}{\Gamma_1 \vdash \forall x.F(x)}\forall_r \quad \dfrac{\overset{\varphi_2}{\Gamma_2, F(t) \vdash \Delta}}{\Gamma_2, \forall x.F(x) \vdash \Delta}\forall_l}{\Gamma_1, \Gamma_2 \vdash \Delta}\mathit{cut} \quad \Rightarrow \quad \dfrac{\overset{\varphi_1\{\alpha \leftarrow t\}}{\Gamma_1 \vdash F(t)} \quad \overset{\varphi_2}{\Gamma_2, F(t) \vdash \Delta}}{\Gamma_1, \Gamma_2 \vdash \Delta}\mathit{cut}$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash F(t)}}{\Gamma_1 \vdash \exists x.F(x)}\,\exists_r \qquad \dfrac{\overset{\varphi_2}{\Gamma_2, F(\alpha) \vdash \Delta}}{\Gamma_2, \exists x.F(x) \vdash \Delta}\,\exists_l}{\Gamma_1, \Gamma_2 \vdash \Delta}\,cut \quad \Rightarrow \quad \dfrac{\overset{\varphi_1}{\Gamma_1 \vdash F(t)} \qquad \overset{\varphi_2\{\alpha \leftarrow t\}}{\Gamma_2, F(t) \vdash \Delta}}{\Gamma_1, \Gamma_2 \vdash \Delta}\,cut$$

In order to eliminate the quantifier of the cut formula, the instantiated version of the formulas must be used. But this imposes no problem, since we can apply the substitution $\sigma = \{\alpha \leftarrow t\}$ on the proof.

If X is an axiom clause in φ_2, $X\{\alpha \leftarrow t\}$ will be an axiom clause in $\varphi_2\{\alpha \leftarrow t\}$. Finally, φ' will have axioms that are, in fact, instances of the clauses in $\mathcal{S}$.

◀

Next, we prove the completeness of the $R^{\neg}$ resolution calculus. In order to do that, we need the lifting lemma for this calculus. Intuitively, this lemma guarantees that if there is a resolution of instantiated terms, it is possible to transform ("lift") this into a resolution of the same terms with variables and a substitution.

▶ **Definition 10.** Let X and Y be clauses, then $X \leq_s Y$ iff there exists a substitution Θ with $X\Theta = Y$.

▶ **Lemma 11** (Lifting). *Let C and D be clauses with $C \leq_s C'$ and $D \leq_s D'$. Assume that C' and D' have a $R^{\neg}$-resolvent E'. Then, there exists a $R^{\neg}$-resolvent E of C and D such that $E \leq_s E'$.*

The proof of Lemma 11 is analogous to the one for the ordinary resolution calculus and will not be described here.

▶ **Theorem 12** (Completeness of $\mathbf{R}^{\neg}$). *Let $\mathcal{S}$ be an **LJ**-refutable set of intuitionistic clauses. Then $\mathcal{S}$ is $R^{\neg}$-refutable.*

Proof. Let φ be an **LJ**-refutation of $\mathcal{S}$. By applying Gentzen's proof-rewriting rules for cut-elimination exhaustively, φ is rewritten to a normal form φ', whose existence is guaranteed by the fact that Gentzen's proof-rewriting system is terminating (see Gentzen's Hauptsatz [8, 9]). By Lemmas 7 and 8, φ' has only $\neg_l$, $\neg_r$, *cut* and left contraction inferences. As these inference rules correspond, respectively, to the rules $\neg_l$, $\neg_r$, $\{R, R\neg\}$ and $\{C, C\neg\}$ (without unification) of the $\mathbf{R}^{\neg}$ calculus, φ' can be immediately converted to a ground $\mathbf{R}^{\neg}$-refutation δ. By Lemma 9, the axioms of φ' and hence also of δ are instances of the clauses in $\mathcal{S}$. Therefore, by the lifting lemma (Lemma 11), δ can be lifted into an $\mathbf{R}^{\neg}$-refutation δ^* of $\mathcal{S}$. ◀

Due to the way the intuitionistic clause set is constructed, all the clauses have no formula on the right hand side. This means that the rule $\neg_l$ can be dropped from $\mathbf{R}^{\neg}$ and the clause sets used in our scenario will still be refutable. Also, the resolution rule on non-negated atoms could also be eliminated in our case, since we could always replace any (non-negated) resolution by negation inferences and negated resolution.

▶ **Definition 13** (Intuitionistic Projection). An *intuitionistic projection* is built analogously to a usual projection, except that all the formulas on the right side are negated and added to the left side.

Let φ be a proof in **LJ** and $C \in CL^I(\varphi)$. Then the **LJ**-proof $\varphi(C)$ is called an *intuitionistic projection* and it is build inductively on the number of inferences of φ. Let ν be a node in φ and $\varphi_\nu(C)$ the projection for clause C until node ν:

1. ν is a leaf: then $\varphi_\nu(C)$ is the derivation consisting of applying a negation rule ($\neg_l$) to the atoms which are cut-ancestors in order to shift them from the right to the left side (if there is a cut-ancestor on the right).
2. ν is the result of a unary rule ξ applied to μ:
   - 2.a. ξ operates on a cut ancestor: $\varphi_\nu(C) = \varphi_\mu(C)$
   - 2.b. ξ operates on an end sequent ancestor: $\varphi_\nu(C)$ is $\varphi_\mu(C)$ plus the application of ξ to its end-sequent
3. ν is the result of a binary rule ξ applied to μ_1 and μ_2:
   - 3.a. ξ operates on a cut ancestor: $\varphi_\nu(C)$ is $\varphi_{\mu_i}(C)$ (i depends on which branch C is coming from) plus some weakenings to obtain formulas that were used in the other branch.
   - 3.b. ξ operates on an end sequent ancestor: $\varphi_\nu(C)$ is the result of applying ξ to the end-sequents of $\varphi_{\mu_1}(C)$ and $\varphi_{\mu_2}(C)$.

▶ **Definition 14** (NACNF)**.** A proof is said to be in negated atomic cut normal form (NACNF) when all the cuts are on atoms or negated atoms.

▶ **Definition 15** (**iCERes**)**.** Let φ be a proof in **LJ** of a sequent S, $CL^I(\varphi)$ its intuitionistic clause set (Definition 4) and $\pi_1, ..., \pi_n$ the intuitionistic projections (Definition 13) of the clauses of $CL^I(\varphi)$. By Theorems 5 and 12, there exists a grounded refutation φ^* of $CL^I(\varphi)$. We define **iCERes** as the procedure of computing the elements $CL^I(\varphi)$, $\pi_1, ..., \pi_n$, and φ^* from φ and then merging (instances of) $\pi_1, ..., \pi_n$ with φ^* in the following way:

Let C_i be the clause of a leaf in φ^*. Then, C_i is replaced by the projection π_i (with the proper substitution of variables), which is actually a derivation of $C_i \circ S$. Moreover, the formulas of S are propagated down the refutation.

▶ **Theorem 16.** *Let φ be an proof in **LJ**$^-$ (Definition 2). Then **iCERes**, applied to φ, produces an intuitionistic negated atomic cut normal form.*

Proof. From Definition 15, we can observe that the result of applying **iCERes** to an **LJ**-proof consists of the resolution refutation in $R^\neg$ merged with the projections. These last elements have no cuts and are derivations in **LJ** by definition. The refutation has resolution rules on atoms and negated atoms, which will be the cuts on the final proof. Since the projections have no formula on the right side of their end sequents, and the resolution sequents have no more than one formula on the right side of each sequent, the final proof is an **LJ**-proof of an end-sequent equal to the one of φ up to some contractions on the left. ◀

3.1 Example

In order to illustrate the **iCERes** method, we will apply it to the following **LJ**$^-$ proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P\alpha^\star \vdash P\alpha}\,I \quad \dfrac{\dfrac{}{Pf\alpha \vdash Pf\alpha}\,I \quad \dfrac{}{Pf^2\alpha \vdash Pf^2\alpha^\star}\,I}{Pf\alpha, Pf\alpha \to Pf^2\alpha \vdash Pf^2\alpha^\star}\,\to_l}{P\alpha^\star, P\alpha \to Pf\alpha, Pf\alpha \to Pf^2\alpha \vdash Pf^2\alpha^\star}\,\to_l}{P\alpha^\star, \forall x.(Px \to Pfx), \forall x.(Px \to Pfx) \vdash Pf^2\alpha^\star}\,\forall_l}{\forall x.(Px \to Pfx), \forall x.(Px \to Pfx) \vdash (P\alpha \to Pf^2\alpha)^\star}\,\to_r}{\forall x.(Px \to Pfx) \vdash (P\alpha \to Pf^2\alpha)^\star}\,c_l}{\forall x.(Px \to Pfx) \vdash \forall x.(Px \to Pf^2x)^\star}\,\forall_r \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Pa \vdash Pa^\star}\,I \quad \dfrac{\dfrac{}{Pf^2a^\star \vdash Pf^2a^\star}\,I \quad \dfrac{}{Pf^4a^\star \vdash Pf^4a}\,I}{Pf^2a^\star, (Pf^2a \to Pf^4a)^\star \vdash Pf^4a}\,\to_l}{Pa, (Pa \to Pf^2a)^\star, (Pf^2a \to Pf^4a)^\star \vdash Pf^4a}\,\to_l}{Pa, \forall x.(Px \to Pf^2x)^\star, \forall x.(Px \to Pf^2x)^\star \vdash Pf^4a}\,\forall_l}{Pa, \forall x.(Px \to Pf^2x)^\star \vdash Pf^4a}\,c_l}{Pa, \forall x.(Px \to Pf^2x)^\star \vdash \exists z.Pf^4z}\,\exists_r}{\dfrac{Pa, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z}{Pa, \forall x.(Px \to Pfx), \neg\exists z.Pf^4z \vdash}\,\neg_l}\,cut$$

Note that the cut formulas and cut ancestors are superscribed with $\star$. By removing the rules applied on end-sequent ancestors and merging the branches as was described on Definition 4, the intuitionistic clause set obtained is:

$$CL^I(\varphi) = \{P\alpha, \neg Pf^2\alpha \vdash \;;\; \neg Pa \vdash \;;\; Pf^4a \vdash\}$$

As was proved previously, there is a resolution refutation of this clause set on $R^\neg$:

$$\dfrac{\dfrac{\dfrac{P\alpha, \neg Pf^2\alpha \vdash}{\neg Pf^2\alpha \vdash \neg P\alpha}\,\neg_r \qquad \neg Pa \vdash}{\neg Pf^2a \vdash}\,R^\neg\{\alpha \leftarrow a\} \qquad \dfrac{\dfrac{\dfrac{Pf^4a \vdash}{\vdash \neg Pf^4a}\,\neg_r \qquad P\alpha, \neg Pf^2\alpha \vdash}{Pf^2a \vdash}\,R^\neg\{\alpha \leftarrow f^2a\}}{\vdash \neg Pf^2a}\,\neg_r}{\vdash}\,R^\neg$$

The projections of the three clauses of CL^I are:

$\pi_1[\alpha]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P\alpha \vdash P\alpha}\,I \qquad \dfrac{\dfrac{}{Pf\alpha \vdash Pf\alpha}\,I \qquad \dfrac{\dfrac{}{Pf^2\alpha \vdash Pf^2\alpha}\,I}{\neg Pf^2\alpha, Pf^2\alpha \vdash}\,\neg_l}{Pf\alpha, \neg Pf^2\alpha, Pf\alpha \to Pf^2\alpha \vdash}\,\to_l}{P\alpha, \neg Pf^2\alpha, P\alpha \to Pf\alpha, Pf\alpha \to Pf^2\alpha \vdash}\,\to_l}{P\alpha, \neg Pf^2\alpha, \forall x.(Px \to Pfx), \forall x.(Px \to Pfx) \vdash}\,\forall_l \times 2}{P\alpha, \neg Pf^2\alpha, \forall x.(Px \to Pfx) \vdash}\,c_l}{\dfrac{P\alpha, \neg Pf^2\alpha, Pa, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z}{P\alpha, \neg Pf^2\alpha, Pa, \forall x.(Px \to Pfx), \neg\exists z.Pf^4z \vdash}\,\neg_l}\,w \times 2$$

π_2 :

$$\dfrac{\dfrac{\dfrac{\dfrac{}{Pa \vdash Pa}\,I}{\neg Pa, Pa \vdash}\,\neg_l}{\neg Pa, Pa, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z}\,w \times 2}{\neg Pa, Pa, \forall x.(Px \to Pfx), \neg\exists z.Pf^4z \vdash}\,\neg_l$$

π_3 :

$$\dfrac{\dfrac{\dfrac{\dfrac{}{Pf^4a \vdash Pf^4a}\,I}{Pf^4a \vdash \exists z.Pf^4z}\,\exists_r}{Pf^4a, Pa, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z}\,w \times 2}{Pf^4a, Pa, \forall x.(Px \to Pfx), \neg\exists z.Pf^4z \vdash}\,\neg_l$$

By merging the appropriate instances of the projections on the resolution refutation, we obtain the final proof, depicted in Figure 3. The projections are colored accordingly. The projection π_1 used on the left side had α replaced with a and the one used on the right side had α replaced with f^2a. Note that this proof is in NACNF, containing only cuts on atoms or negated atoms, and it is still a proof in **LJ**.

4 On the possibility of extending iCERes to a larger class of proofs

On the example of Section 3.1, the last application of the rule $\neg_l$ was used in order to make the end-sequent fulfill the condition of not having formulas on the right. This is a simple operation, but, as we mentioned before, it is not trivial how to transform the final proof into a proof of the sequent where the shifted formula is on the right side. In this section we analyse a possible solution to deal with end-sequents without this restriction.

Applying **iCERes** to a proof in **LJ** where the end-sequent has a formula on the right will yield intuitionistic projections and a refutation in the $R^{\neg}$ calculus. But when these elements are put together, it might be the case that some classical sequents appear (in the sense of having more than one formula on the right). Nevertheless, it seems to be often the case that applying reductive cut-elimination on this final **LK** proof and removing the cuts on atoms and negated atoms will again result in an **LJ** proof. We present here an example to illustrate this.

The proof used is the same as the one on Section 3.1, except for the last application of $\neg_l$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P\alpha^\star \vdash P\alpha}\,{\scriptstyle I} \quad \dfrac{\dfrac{}{Pf\alpha \vdash Pf\alpha}\,{\scriptstyle I} \quad \dfrac{}{Pf^2\alpha \vdash Pf^2\alpha^\star}\,{\scriptstyle I}}{Pf\alpha, Pf\alpha \to Pf^2\alpha \vdash Pf^2\alpha^\star}\,{\scriptstyle \to_l}}{P\alpha^\star, P\alpha \to Pf\alpha, Pf\alpha \to Pf^2\alpha \vdash Pf^2\alpha^\star}\,{\scriptstyle \to_l}}{P\alpha^\star, \forall x.(Px \to Pfx), \forall x.(Px \to Pfx) \vdash Pf^2\alpha^\star}\,{\scriptstyle \forall_l}}{\forall x.(Px \to Pfx), \forall x.(Px \to Pfx) \vdash (P\alpha \to Pf^2\alpha)^\star}\,{\scriptstyle \to_r}}{\dfrac{\forall x.(Px \to Pfx) \vdash (P\alpha \to Pf^2\alpha)^\star}{\forall x.(Px \to Pfx) \vdash \forall x.(Px \to Pf^2x)^\star}\,{\scriptstyle \forall_r}}\,{\scriptstyle c_l} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Pa \vdash Pa^\star}\,{\scriptstyle I} \quad \dfrac{\dfrac{}{Pf^2a^\star \vdash Pf^2a^\star}\,{\scriptstyle I} \quad \dfrac{}{Pf^4a^\star \vdash Pf^4a}\,{\scriptstyle I}}{Pf^2a^\star, (Pf^2a \to Pf^4a)^\star \vdash Pf^4a}\,{\scriptstyle \to_l}}{Pa, (Pa \to Pf^2a)^\star, (Pf^2a \to Pf^4a)^\star \vdash Pf^4a}\,{\scriptstyle \to_l}}{Pa, \forall x.(Px \to Pf^2x)^\star, \forall x.(Px \to Pf^2x)^\star \vdash Pf^4a}\,{\scriptstyle \forall_l}}{\dfrac{Pa, \forall x.(Px \to Pf^2x)^\star \vdash Pf^4a}{Pa, \forall x.(Px \to Pf^2x)^\star \vdash \exists z.Pf^4z}\,{\scriptstyle \exists_r}}\,{\scriptstyle c_l}}{Pa, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z}\,{\scriptstyle cut}$$

The intuitionistic characteristic clause set is (the same as before):

$$CL^I = \{P\alpha, \neg Pf^2\alpha \vdash \;\; ; \;\; \neg Pa \vdash \;\; ; \;\; Pf^4a \vdash\}$$

The projections for each element of CL^I are:

$$\pi_1[\alpha] :$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P\alpha \vdash P\alpha}\,{\scriptstyle I} \quad \dfrac{\dfrac{}{Pf\alpha \vdash Pf\alpha}\,{\scriptstyle I} \quad \dfrac{\dfrac{}{Pf^2\alpha \vdash Pf^2\alpha}\,{\scriptstyle I}}{\neg Pf^2\alpha, Pf^2\alpha \vdash}\,{\scriptstyle \neg_l}}{Pf\alpha, \neg Pf^2\alpha, Pf\alpha \to Pf^2\alpha \vdash}\,{\scriptstyle \to_l}}{P\alpha, \neg Pf^2\alpha, P\alpha \to Pf\alpha, Pf\alpha \to Pf^2\alpha \vdash}\,{\scriptstyle \to_l}}{P\alpha, \neg Pf^2\alpha, \forall x.(Px \to Pfx), \forall x.(Px \to Pfx) \vdash}\,{\scriptstyle \forall_l \times 2}}{\dfrac{P\alpha, \neg Pf^2\alpha, \forall x.(Px \to Pfx) \vdash}{P\alpha, \neg Pf^2\alpha, Pa, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z}\,{\scriptstyle w \times 2}}\,{\scriptstyle c_l}$$

$$\pi_2 :$$

$$\dfrac{\dfrac{\dfrac{}{Pa \vdash Pa}\,{\scriptstyle I}}{\neg Pa, Pa \vdash}\,{\scriptstyle \neg_l}}{\neg Pa, Pa, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z}\,{\scriptstyle w \times 2}$$

$$\pi_3 :$$

$$\dfrac{\dfrac{\dfrac{}{Pf^4a \vdash Pf^4a}\,{\scriptstyle I}}{Pf^4a \vdash \exists z.Pf^4z}\,{\scriptstyle \exists_r}}{Pf^4a, Pa, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z}\,{\scriptstyle w \times 2}$$

The resolution refutation is (the same as before):

$$\dfrac{\dfrac{\dfrac{P\alpha, \neg Pf^2\alpha \vdash}{\neg Pf^2\alpha \vdash \neg P\alpha}\,{\scriptstyle \neg_r} \quad \neg Pa \vdash}{\neg Pf^2a \vdash}\,{\scriptstyle R^{\neg}\{\alpha \leftarrow a\}} \quad \dfrac{\dfrac{\dfrac{Pf^4a \vdash}{\vdash \neg Pf^4a}\,{\scriptstyle \neg_r} \quad P\alpha, \neg Pf^2\alpha \vdash}{Pf^2a \vdash}\,{\scriptstyle R^{\neg}\{\alpha \leftarrow f^2a\}}}{\vdash \neg Pf^2a}\,{\scriptstyle \neg_r}}{\vdash}\,{\scriptstyle R^{\neg}}$$

$$
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\dfrac{}{Pfa \vdash Pfa}\, I
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\dfrac{}{Pf^4a \vdash Pf^4a}\, I
}{Pf^4a \vdash \exists z.Pf^4z}\, \exists_r
}{Pa, Pf^3a, Pf^3a \to Pf^4a \vdash \exists z.Pf^4z}\, \to_l
}{Pf^2a, Pa, Pf^2a \to Pf^3a, Pf^3a \to Pf^4a \vdash \exists z.Pf^4z}\, \to_l
}{Pf^2a, Pa, \forall x.(Px \to Pfx), \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z}\, \forall_l \times 2
}{Pf^2a, Pa, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z}\, c_l
}{Pa, Pfa, Pfa \to Pf^2a, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z}\, \to_l
}{Pa, Pa, Pa \to Pfa, Pfa \to Pf^2a, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z}\, \to_l
}{Pa, Pa, \forall x.(Px \to Pfx), \forall x.(Px \to Pfx), \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z}\, \forall_l \times 2
}{Pa, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z}\, c_l \times 3
$$

Figure 2 Proof obtained after eliminating the atomic cuts.

Given the projections and grounded resolution, the final proof is depicted in Figure 4. Note that, since the end-sequent had a formula on the right side, the final proof is not in **LJ**, but in **LK**. Even though the refutation and projections were intuitionistic, when they are put together some sequents end up having more than one formula on the right.

But if reductive cut-elimination is applied to the proof in Figure 4 and all useless weakenings and contractions are removed, the result is again a proof in **LJ**, depicted in Figure 2.

The second condition, the absence of strong quantifiers in the end-sequent, can be satisfied by removing strong quantifiers with methods of skolemization of **LJ**-proofs [4]. It is important to note, however, that the end-sequent of a final proof φ obtained by applying **iCERes** to an **LJ**-proof that has been skolemized will also contain skolem terms. Depending on the application, it might be desirable to transform φ into a proof of the original non-skolemized end-sequent. Unfortunately, it is not yet clear how to perform such a deskolemization.

5 Conclusions and future work

This paper presents a method of cut-elimination by resolution for a fragment of **LJ**. In order to develop this method, it was necessary to define intuitionistic clause sets, intuitionistic projections and a new resolution calculus. It was proved that this calculus is complete with respect to **LJ** on clause logic. Applying the **CERes** method (previously developed for classical logic) with these new definitions on an **LJ** proof, such that its end-sequent does not contain a formula on the right side, will yield an intuitionistic proof with only atomic or negated atomic cuts.

It is important to observe that, for any **LJ**-proof, the projections and refutation of the clause set, as defined in this paper, are both intuitionistic. But when these elements are assembled together to compose the final proof, a classical sequent might occur. This is why the current method has to be restricted to a fragment of **LJ** for which this undesirable effect cannot occur. Interestingly, this fragment corresponds to the class of intuitionistic proofs by contradiction.

Immediate future work will consist of extending **iCERes** to larger classes of (and hopefully all) **LJ**-proofs. It seems likely that this will require more sophisticated ways of assembling projections and refutations.

In addition to extending **iCERes** to larger classes of proofs, we also intend to eventually apply it to a real mathematical proof, as we have done with the classical **CERes** for Fürstenberg's proof of the infinitude of primes.

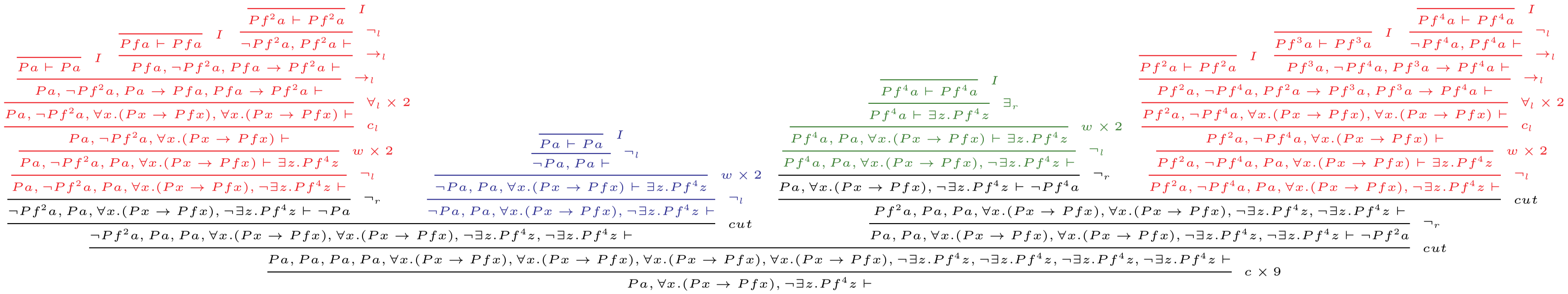

Figure 3 Final proof after applying the method to an LJ-proof of an end-sequent with no formula on the right.

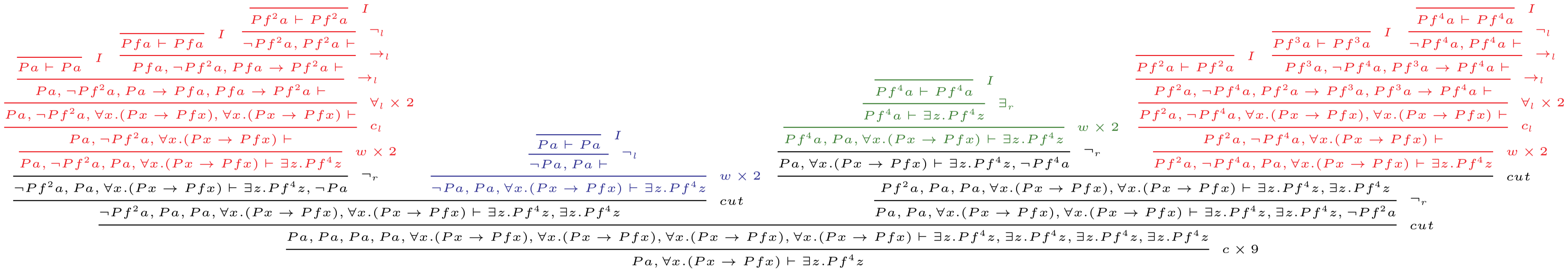

Figure 4 Final proof after applying the method to an LJ-proof of an end-sequent with a formula on the right.

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