Lower Bounds on the Complexity of MSO_1 Model-Checking

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Abstract

One of the most important algorithmic meta-theorems is a famous result by Courcelle, which states that any graph problem definable in monadic second-order logic with edge-set quantifications (MSO_2) is decidable in linear time on any class of graphs of bounded tree-width. In the parlance of parameterized complexity, this means that MSO_2 model-checking is fixed-parameter tractable with respect to the tree-width as parameter. Recently, Kreutzer and Tazari [13] proved a corresponding complexity lower-bound—that MSO_2 model-checking is not even in XP wrt. the formula size as parameter for graph classes that are subgraph-closed and whose tree-width is poly-logarithmically unbounded. Of course, this is not an unconditional result but holds modulo a certain complexity-theoretic assumption, namely, the Exponential Time Hypothesis (ETH).

In this paper we present a closely related result. We show that even MSO_1 model-checking with a fixed set of vertex labels, but without edge-set quantifications, is not in XP wrt. the formula size as parameter for graph classes which are subgraph-closed and whose tree-width is poly-logarithmically unbounded unless the non-uniform ETH fails. In comparison to Kreutzer and Tazari, (1) we use a stronger prerequisite, namely non-uniform instead of uniform ETH, to avoid the effectiveness assumption and the construction of certain obstructions used in their proofs; and (2) we assume a different set of problems to be efficiently decidable, namely MSO_1-definable properties on vertex labeled graphs instead of MSO_2-definable properties on unlabeled graphs.

Our result has an interesting consequence in the realm of digraph width measures: Strengthening the recent result [8], we show that no subdigraph-monotone measure can be algorithmically useful, unless it is within a poly-logarithmic factor of (undirected) tree-width.

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1 Introduction

A famous result by Courcelle, proved in 1990, states that any graph property definable in monadic second-order logic with quantification over vertex- and edge-sets (MSO_2) can be decided in linear time on any class of graphs of bounded tree-width [2]. This result has a strong significance. As MSO_2 logic can express many interesting graph properties, we immediately

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get linear-time algorithms for important NP-hard problems, such as Hamiltonian Cycle, Vertex Cover, and 3-Colorability, on graphs of bounded tree-width. Such a result is called an *algorithmic meta-theorem*, and many other algorithmic meta-theorems have since appeared for other classes of graphs—see e.g. [9, 11] for a good survey.

As can be seen, Courcelle's theorem is a fast and relatively easy way of establishing that a problem can be solved efficiently on graphs of bounded tree-width. However, one may ask how far this result could be generalized. That is, is there a graph class of unbounded tree-width such that MSO_2 model-checking remains tractable on this class? Considering how important this question is for theoretical understanding of what makes some problems on certain graph classes hard, it is surprising that until recently there has not been much research in this direction.

The first result, by Kreutzer, providing a "lower bound" to Courcelle's theorem appeared in [12]. In that paper, Kreutzer used the following version of "unbounding" the tree-width of a graph class:

▶ **Definition 1** (Kreutzer and Tazari [12, 13]). The tree-width of a class $\mathcal{C}$ of graphs is strongly unbounded by a function $f\colon \mathbb{N} \to \mathbb{N}$ if there is an $\epsilon < 1$ and a polynomial $p(x)$ s.t. for all $n \in \mathbb{N}$ there is a graph $G_n \in \mathcal{C}$ with the following properties:

i) the tree-width of G_n is between n and $p(n)$ and is greater than $f(|G_n|)$, and
ii) given n, the graph G_n can be constructed in time 2^{n^ϵ}.

The degree of the polynomial p is called the *gap-degree* of $\mathcal{C}$ (with respect to f). The tree-width of $\mathcal{C}$ is *strongly unbounded poly-logarithmically* if it is strongly unbounded by $\log^c n$, for all $c \geq 1$.

In other words, the tree-width of $\mathcal{C}$ is *strongly unbounded* means that (i) there are no big gaps between the tree-width of witness graphs (those that certify that the tree-width of n-vertex graphs in $\mathcal{C}$ is greater than $f(n)$), and (ii) we can compute such witnesses effectively—in sub-exponential time wrt. n.

The main result of [12] is the following theorem (we postpone formal definitions to Sections 2 and 3): Let Γ be a fixed set of colors, and $\mathcal{C}$ be a class of graphs such that (1) the tree-width of $\mathcal{C}$ is strongly unbounded poly-logarithmically; (2) $\mathcal{C}$ is closed under Γ-colorings (i.e., if $G \in \mathcal{C}$ and G' is obtained from G by coloring some vertices or edges by colors from Γ, then $G' \in \mathcal{C}$); and, (3) $\mathcal{C}$ is constructible (i.e., given a witness graph in $\mathcal{C}$, a certain substructure can be computed in polynomial time). Then $\mathrm{MC}(\mathrm{MSO}_2\text{-}\Gamma, \mathcal{C})$, the MSO_2 model-checking problem on $\mathcal{C}$ with colors from Γ, is not in XP (and hence not in FPT—see Section 2 for a definition of these complexity classes), unless all problems in the polynomial-time hierarchy can be solved in sub-exponential time. This would, of course, mean that the Exponential-Time Hypothesis (ETH) fails. The results of [12] have been improved by Kreutzer and Tazari in [14], where the constructibility requirement (3) was dropped.

A further improvement by the same authors appeared in [13]. The main result in [13] can be stated as follows: Let $\mathcal{C}$ be a class of graphs such that (1) the tree-width of $\mathcal{C}$ is strongly unbounded poly-logarithmically; and (2′) $\mathcal{C}$ is closed under taking subgraphs, i.e. $G \in \mathcal{C}$ and $H \subseteq G$ implies $H \in \mathcal{C}$. Then $\mathrm{MC}(\mathrm{MSO}_2, \mathcal{C})$, the (ordinary) MSO_2 model-checking problem on $\mathcal{C}$, is not in XP unless all problems in the polynomial-time hierarchy can be solved in sub-exponential time. Note that (2′), to be closed under subgraphs, is a strictly weaker condition than (2), to be closed under Γ-colorings (of edges, too).

Our results

In this paper we prove a result closely related to Kreutzer–Tazari's [12, 14, 13] but for MSO_1 logic with a fixed set of vertex labels. The role of *vertex labels* in our paper is similar to that of colors in [12, 14], but weaker in the sense that the labels are not assigned to edges.[1] In contrast to the work by Kreutzer and Tazari, we assume a different set of problems—those expressible by MSO_1-L on graphs with vertex labels from a fixed finite set L—to be efficiently solvable on a graph class in order to derive an analogous conclusion.

Before stating our main result, we mention one more fact. There exist classes $\mathcal{C}$ of L-labeled graphs of unbounded tree-width on which $\mathrm{MC}(\mathrm{MSO}_1\text{-}L, \mathcal{C})$, the MSO_1 model-checking problem on $\mathcal{C}$, is polynomial time solvable, e.g. classes of bounded clique-width or rank-width. But it is important to realize that these classes are *not* closed under taking subgraphs. Our main result then reads—cf. Section 4:

▶ **Theorem 2** (reformulated as Theorem 12). *Assume a (suitable but fixed) finite label set L, and a graph class $\mathcal{G}$ satisfying the following two properties:*

a) $\mathcal{G}$ is closed under taking subgraphs,
b) the tree-width of $\mathcal{G}$ is densely unbounded poly-logarithmically (see Def. 8).

Then $\mathrm{MC}(\mathrm{MSO}_1\text{-}L, \mathcal{G}^L)$*, the* MSO_1-L *model-checking problem on all L-vertex-labeled graphs from $\mathcal{G}^L$, is* not *in* XP *unless the non-uniform Exponential-Time Hypothesis fails.*

Our general approach follows that by Kreutzer and Tazari in [12, 14, 13] but differs from theirs in three main ways:

I) Kreutzer and Tazari require witnesses as in (ii) of Definition 1 of [13] to be computable effectively in their proofs. It is unclear how this can be done and hence they simply add this as a natural requirement. Furthermore, the construction of certain obstructions (grid-like minors) used in their proof requires an involved machinery [14]. We adopt a different position and avoid (note our "densely unbounded" vs. "strongly unbounded") both aspects by using a stronger complexity-theoretic assumption, namely the non-uniform ETH instead of the ordinary ETH. In this way, we can get the obstructions as *advice* "for free." This makes our proof shorter and exhibits its structure more clearly.

II) Our result applies to MSO_1-L model-checking on *L-vertex-labeled graphs*, while the result of [13] applies to MSO_2 over unlabeled graphs. There are problems that can be expressed in MSO_1-L and not in MSO_2 and vice versa (take RED-BLUE DOMINATING SET vs. HAMILTONIAN CYCLE, for instance). If, however, the set of labels L is fixed for both, MSO_1-L has much weaker expressive power than MSO_2-L due to missing edge-set quantifications (see Section 2). In particular, note that many of the existing algorithmic meta-theorems (e.g. [2, 4]) that deal with MSO-definable properties handle unlabeled as well as (vertex-)labeled inputs with equal ease. However, extending e.g. the results of [4] from MSO_1-L to MSO_2 is not possible unless EXP = NEXP.

III) Finally, because of the free advice, our proof does not need technically involved machinery such as the simulation of a run of a Turing machine encoded in graphs [13].

Theorem 2 raises the open question whether poly-logarithmically unbounded tree-width along with closure under subgraphs is a strong enough condition for even the *bare* MSO_1 *model-checking to be intractable* (modulo appropriate complexity-theoretic assumptions).

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[1] The reason we use the term labels and not colors is to be able to clearly distinguish between vertex-labeled graphs and the colored graphs used in [12, 14], where colors are assigned to edges and vertices.

If we assume that the label set L is "unbounded" we obtain an even stronger result: MSO_1-L model-checking with vertex labels L is not tractable for a graph class satisfying (a) and (b) of Theorem 2 unless *every* problem in the polynomial-time hierarchy is in $\mathrm{DTIME}(2^{o(n)})/\textsc{SubEXP}$ (cf. Theorem 13).

Finally, as a corollary, we obtain an interesting consequence in the area of directed graph (digraph) width measures, improving upon [8]. Informally, digraph width measures that are subdigraph-monotone and algorithmically "powerful" is at most a poly-logarithmic factor of the tree-width of the underlying undirected graph—cf. Section 5. In this context, we let $U(D)$ denote the underlying undirected graph of a digraph D. Given a digraph width measure δ, we let $U_\delta(d) := \{U(D) \mid \delta(D) \leq d\}$ to be the set of underlying undirected graphs of digraphs of δ-width at most d.

▶ **Theorem 3** (reformulated as Theorem 15). *Assume a (suitable but fixed) finite label set L, and a digraph width measure δ such that*

a) δ is monotone under taking subdigraphs, and
b) $\mathrm{MC}(\mathrm{MSO}_1\text{-}L, \mathcal{D}^L)$, the MSO_1-L model-checking problem on all L-vertex-labeled digraphs $\mathcal{D}^L$ is in XP wrt. $\delta(D)$ and the input formula $\varphi \in \mathrm{MSO}_1$-$L$ as parameters.

Then, unless the non-uniform ETH fails, for all $d \in \mathbb{N}$ the tree-width of the class $U_\delta(d)$ is not densely unbounded poly-logarithmically.

Proof outline and organization

We are going to show via a suitable (multi-step) reduction that the potential tractability of MSO_1-L model-checking on our graph class implies sub-exponential time algorithms for problems which are not believed to have one (cf. ETH). The success of the reduction, of course, rests on the assumptions of $\mathcal{G}$ being subgraph-closed and of unbounded tree-width. So, at a high level, our proof technique is similar to that of Kreutzer and Tazari.

However, there are some crucial differences. While [13] uses the effectiveness assumption in Definition 1. ii and some further technically involved algorithms to construct a "skeleton" in the class $\mathcal{C}$ suitable for their reduction, in our reduction we obtain a corresponding labeled skeleton in the class $\mathcal{G}^L$ "for free" from an oracle advice function which comes with the non-uniform (fixed-sized circuits) computing model. That is why our complete proof is also significantly shorter than that in [13]. Additionally, our arguments employ a result on strong edge colorings of graphs in order to "simulate" certain edge sets within the MSO_1-L language, thus avoiding the need for a more expressive logic such as MSO_2.

The rest of the paper is organized as follows: In Section 2 we overview some standard terminology and notation. Section 3 then includes the core technical concepts: unbounding tree-width (Definition 8), the grid-like graphs of Reed and Wood [16] (Proposition 2), and a new way of interpreting arbitrary graphs in labeled grid-like graphs of sufficiently high order (Lemma 10). These then lead to the proof of our main result, equivalently formulated as Theorem 12, in Section 4. In this section, we also show the stronger collapse result in Theorem 13, that of $\mathrm{PH} \subseteq \mathrm{DTIME}(2^{o(n)})/\textsc{SubEXP}$. The consequences for directed width measures are then discussed in Section 5, followed by concluding remarks in Section 6.

2 Preliminaries

The graphs we consider in this paper are *simple*, i.e. they do not contain loops and parallel edges. Given a graph G, we let $V(G)$ denote its vertex set and $E(G)$ its edge set. A *path P of length $r > 0$* in G is a sequence of vertices $P = (x_0, \dots, x_r)$ such that all x_i are pairwise distinct and $(x_i, x_{i+1}) \in E(G)$ for every $0 \leq i < r$. Let $\mathcal{S}$ be a family of sets S_i for $i = 1, 2, \ldots$. Then the *intersection graph on* $\mathcal{S}$ is the graph $I(\mathcal{S})$ where $V(I(\mathcal{S})) = \mathcal{S}$ and $S_iS_j \in E(I(\mathcal{S}))$ iff $S_i \cap S_j \neq \emptyset$.

Let $L = \{L_1, \ldots, L_k\}$ be a set of labels. A *L-vertex-labeled graph*, or *L-graph* for short, is a graph G together with a function $\lambda\colon V(G) \to 2^L$, assigning each vertex a set of labels, and we write (G, λ) to denote this graph. For a graph class $\mathcal{G}$, we shortly write $\mathcal{G}^L$ for the class of all L-graphs over $\mathcal{G}$, i.e. $\mathcal{G}^L$ contains all (G, λ) where $G \in \mathcal{G}$ and λ is an arbitrary L-vertex-labelling of G. Note that, unlike in e.g. [12], we do not allow labels for edges, which is in accordance with our focus on MSO_1 logic of graphs (defined next).

Monadic second-order logic (MSO) is an extension of first-order logic by quantification over sets. On the one-sorted adjacency model of graphs it reads as follows:

▶ **Definition 4.** The language of MSO_1, *monadic second-order logic of graphs*, contains the expressions built from the following elements:

i) variables $x, y, \ldots$ for vertices, and $X, Y, \ldots$ for sets of vertices,
ii) the predicates $x \in X$ and $adj(x, y)$ with the standard meaning,
iii) equality for variables, the connectives $\wedge, \vee, \neg, \rightarrow$ and the quantifiers $\forall, \exists$.

Note that we do not allow quantification over sets of edges (as edges are not elements). If we considered the two-sorted incidence graph model (in which the edges formed another sort of elements), we would obtain aforementioned MSO_2, *monadic second-order logic of graphs with edge-set quantification*, which is strictly more powerful than MSO_1, cf. [6]. Yet even MSO_1 has strong enough expressive power to describe many common problems.

▶ **Example 5.** The 3-Coloring problem can be expressed in MSO_1 as follows: $\exists V_1, V_2, V_3$ $\big[\forall v\, (v \in V_1 \vee v \in V_2 \vee v \in V_3) \wedge \bigwedge_{i=1,2,3} \forall v, w\, (v \notin V_i \vee w \notin V_i \vee \neg\, adj(v, w))\,\big]$.

The MSO_1 logic can naturally be extended to L-graphs. The *monadic second-order logic* on L-vertex-labeled graphs, denoted by MSO_1-L, is the natural extension of MSO_1 with unary predicates $L_i(x)$ for each label $L_i \in L$, such that $L_i(x)$ holds iff $L_i \in \lambda(x)$.

Parameterized complexity and MSO_1 model-checking

Throughout the paper we are interested in the problem of checking whether a given input graph satisfies a property specified by a fixed formula. This problem can be thought of as an instance of a parameterized problem, studied in the field of *parameterized complexity* (see e.g. [7] for a background on parameterized complexity).

A parameterized problem Q is a subset of $\Sigma \times \mathbb{N}_0$, where Σ is a finite alphabet and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A parameterized problem Q is said to be *fixed-parameter tractable* if there is an algorithm that given $(x, k) \in \Sigma \times \mathbb{N}_0$ decides whether (x, k) is a yes-instance of Q in time $f(k) \cdot p(|x|)$ where f is some computable function of k alone, p is a polynomial and $|x|$ is the size measure of the input. The class of such problems is denoted by FPT. The class XP is the class of parameterized problems that admit algorithms with a run-time of $O(|x|^{f(k)})$ for some computable f, i.e. polynomial-time for every fixed value of k.

We are dealing with a parameterized model-checking problem $MC(MSO_1, \mathcal{C})$ where $\mathcal{C}$ is a class of graphs; the task is to decide, given a graph $G \in \mathcal{C}$ and a formula $\phi \in MSO_1$, whether $G \models \phi$. The parameter is $k = |\phi|$, the size of the formula ϕ. We actually consider the labeled variant $MC(MSO_1\text{-}L, \mathcal{C})$ for $\mathcal{C}$ being a class of L-graphs.

Interpretability of logic theories

One of our main tools is the classical interpretability of logic theories [15] (which in this setting is analogical to transductions as used e.g. by Courcelle [3]). To describe the simplified setting, assume that two classes of *relational structures* $\mathscr{K}$ and $\mathscr{L}$ are given. The basic idea of an *interpretation* I of the theory $\mathrm{Th}_{\mathrm{MSO}}(\mathscr{K})$ into $\mathrm{Th}_{\mathrm{MSO}}(\mathscr{L})$ is to transform MSO formulas ϕ over $\mathscr{K}$ into MSO formulas ϕ^I over $\mathscr{L}$ in such a way that "truth is preserved":

- First, one chooses a formula $\alpha(x)$ intended to define in each structure $G \in \mathscr{L}$ a set of individuals (new domain) $G[\alpha] := \{a : a \in dom(G) \text{ and } G \models \alpha(a)\}$, where $dom(G)$ denotes the set of individuals (domain) of G.
- Then, one chooses for each s-ary relational symbol R from $\mathscr{K}$ a formula $\beta^R(x_1, \ldots, x_s)$, with the intention to define a corresponding relation $G[\beta^R] := \{(a_1, \ldots, a_s) \colon a_1, \ldots, a_s \in dom(G) \text{ and } G \models \beta^R(a_1, \ldots, a_s)\}$. With these formulas one defines for each $G \in \mathscr{L}$ the relational structure $G^I := \big(G[\alpha], G[\beta^R], \ldots\big)$ intended to correspond with structures in $\mathscr{K}$.
- Finally, there is a natural way to translate each formula ϕ (over $\mathscr{K}$) into a formula ϕ^I (over $\mathscr{L}$), by induction on the structure of formulas. The atomic ones are substituted by corresponding chosen formulas (such as β^R) with the corresponding variables. Then one proceeds via induction simply as follows:

$$(\neg\phi)^I \;\mapsto\; \neg(\phi^I) \quad , \qquad (\phi_1 \wedge \phi_2)^I \;\mapsto\; (\phi_1)^I \wedge (\phi_2)^I,$$
$$\big(\exists x\, \phi(x)\big)^I \;\mapsto\; \exists y\, \big(\alpha(y) \wedge \phi^I(y)\big) \quad , \qquad \big(\exists X\, \phi(X)\big)^I \;\mapsto\; \exists Y\, \phi^I(Y).$$

The whole concept is shortly illustrated in by the following scheme

$$\begin{array}{ccc} \phi \in \text{MSO over } \mathscr{K} & \xrightarrow{\;I\;} & \phi^I \in \text{MSO over } \mathscr{L} \\ H \in \mathscr{K} & & G \in \mathscr{L} \\ G^I \cong H & \xleftarrow{\;I\;} & G \end{array}$$

▶ **Definition 6** (Interpretation between theories). Let $\mathscr{K}$ and $\mathscr{L}$ be classes of relational structures. Theory $\mathrm{Th}_{\mathrm{MSO}}(\mathscr{K})$ is *interpretable* in theory $\mathrm{Th}_{\mathrm{MSO}}(\mathscr{L})$ if there exists an interpretation I as above such that the following two conditions are satisfied:

i) For every structure $H \in \mathscr{K}$, there is $G \in \mathscr{L}$ such that $G^I \cong H$, and
ii) for every $G \in \mathscr{L}$, the structure G^I is isomorphic to some structure of $\mathscr{K}$.

Furthermore, $\mathrm{Th}_{\mathrm{MSO}}(\mathscr{K})$ is *efficiently interpretable* in $\mathrm{Th}_{\mathrm{MSO}}(\mathscr{L})$ if the translation of each ϕ into ϕ^I is computable in polynomial time and the structure $G \in \mathscr{L}$, where $G^I \cong H$, can be computed from any $H \in \mathscr{K}$ in polynomial time.

Exponential-Time Hypothesis

The *Exponential-Time Hypothesis (ETH)*, formulated in [10], states that there exists no algorithm that can solve n-variable 3-SAT in time $2^{o(n)}$. It was shown in [10] that the hypothesis can be formulated using one of the many equivalent problems (e.g. k-Colorability or Vertex Cover)—i.e. sub-exponential complexity for one of these problems would imply the same for all the others.

ETH can be formulated in the *non-uniform* version: There is no family of algorithms (one for each input length) which can solve n-variable 3-SAT in time $2^{o(n)}$. In theory of computation literature, "non-uniform algorithms" are often referred to as "fixed-sized input circuits" where for each length of the input a different circuit is used. Yet another way of thinking about non-uniform algorithms is as having an algorithm that is allowed to receive an oracle advice, which depends only on the length of the input. As mentioned in [1], the results of [10] hold also for the non-uniform ETH.

3 Key Technical Concepts

Unbounding Tree-width

Following Definition 1, we aim to formally describe what it means to say that the tree-width of a graph class is not bounded by a function g. Recall (see also [12, 13]) that it is not enough just to assume $tw(G) > g(|V(G)|)$ for some sporadic values of tw with huge gaps between them, but a reasonable density of the surpassing tree-width values is also required. Hence we suggest the following alternative definition:

▶ **Definition 7** (Densely unbounded tree-width)**.** For a graph class $\mathcal{G}$, we say that the tree-width of $\mathcal{G}$ is *densely unbounded by a function g* if there is a constant $\gamma > 1$ such that, for every $m \in \mathbb{N}$, there exists a graph $G \in \mathcal{G}$ whose tree-width is $tw(G) \geq m$ and $|V(G)| < \mathcal{O}\big(g^{-1}(m^{\gamma})\big)$. The constant γ is called the *gap-degree* of this property.

▶ Remark. Comparing to Definition 1 one can easily check that if the tree-width of a class $\mathcal{G}$ is strongly unbounded by a function g, then the tree-width is densely unbounded by g with the same gap-degree, and the witnessing graphs G of Definition 7 can be computed for all m efficiently—in sub-exponential time wrt. m. Hence our definition is weaker in this respect.

For simplicity we are interested in graph classes whose tree-width is densely unbounded by every poly-logarithmic function of the graph size. That is expressed by the following simpler definition:

▶ **Definition 8** (Densely unbounded tree-width II)**.** For a graph class $\mathcal{G}$, we say that the tree-width of $\mathcal{G}$ is *densely unbounded poly-logarithmically* if it is densely unbounded by $\log^c m$ for every $c \in \mathbb{N}$. That is, for every $c > 0$ the following holds: for all $m \in \mathbb{N}$ there exists a graph $G \in \mathcal{G}$ whose tree-width is $tw(G) \geq m$ and with size $|V(G)| < \mathcal{O}\big(2^{m^{1/c}}\big)$. (The gap-degree becomes irrelevant in this setting.)

Grid-like graphs

The notion of a grid-like minor was introduced by Reed and Wood in [16], and extensively used by Kreutzer and Tazari [14, 13]. In what follows, we avoid use of the word "minor" in our definition of the same concept, since "H-minors" where H is grid-like are always found as subgraphs of the target graph, which might cause some confusion.

▶ **Definition 9** (Grid-like [16])**.** A graph G together with a collection $\mathcal{P}$ of paths, formally the pair $(G, \mathcal{P})$, is called *grid-like* if the following is true:

i) G is the union of all the paths in $\mathcal{P}$,
ii) each path in $\mathcal{P}$ has at least two vertices, and
iii) the *intersection graph* $I(\mathcal{P})$ of the path collection is bipartite.

The *order* of such grid-like graph $(G, \mathcal{P})$ is the maximum integer ℓ such that the intersection graph $I(\mathcal{P})$ contains a K_ℓ-minor. When convenient, we refer to a grid-like graph simply as to G.

Note that the condition (ii) is not explicitly stated in [16], but its validity implicitly follows from the point to get a K_ℓ-minor in $I(\mathcal{P})$, cf. Theorem 2. One can easily observe the following:

▶ Proposition 1. Let $(G, \mathcal{P})$ be a grid-like graph. Then the collection $\mathcal{P}$ can be split into $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ such that each $\mathcal{P}_i$, $i = 1, 2$, consists of pairwise disjoint paths. Consequently, the maximum degree in G is $\Delta(G) \leq 4$.

The next result is crucial for our paper:

▶ Proposition 2 (Reed and Wood [16]). Every graph with tree-width at least $c\ell^4\sqrt{\log \ell}$ contains a subgraph which is grid-like of order ℓ, for some constant c.

MSO_1 interpretation on grid-like graphs

Now we prove the core new technical tool of our paper. We show how the subgraphs of $I(\mathcal{P})$ of any grid-like graph $(G, \mathcal{P})$ can be efficiently MSO_1-interpreted in G itself with a suitable vertex labelling. First, we state a useful result about strong edge colorings of graphs—a *strong edge-coloring* is an assignment of colors to the edges of a graph such that no path of length three contains the same color twice.

▶ Proposition 3 (Cranston [5]). Every graph of maximum degree 4 has a strong edge-coloring using at most 22 colors. This coloring can be found with a polynomial-time algorithm.

For a class of grid-like graphs $\mathcal{G}$, let $I^{\subseteq}(\mathcal{G}) = \{H : H \subseteq I(\mathcal{P}),\ (G, \mathcal{P}) \in \mathcal{G}\}$ denote the class of all subgraphs of their intersection graphs. Our core tool is the following lemma.

▶ **Lemma 10.** *Let $\mathcal{G}$ be any class of grid-like graphs. There exists a fixed finite set L of labels, with $|L| \geq 47$, and a graph class $\mathcal{I} \supseteq I^{\subseteq}(\mathcal{G})$, such that the following holds. The* MSO_1 *theory of $\mathcal{I}$ has an efficient interpretation in the* MSO_1 *theory of $\mathcal{G}^L$—the class of all L-vertex-labeled graphs over $\mathcal{G}$. Stated differently, any $H \subseteq I(\mathcal{P})$ where $(G, \mathcal{P}) \in \mathcal{G}$ is interpreted in some L-graph of G.*

Proof. Note that the use of a class $\mathcal{I}$ in the statement of the lemma is only a technicality related to (ii) of Definition 6. We are actually interested only in interpreting the graphs from $I^{\subseteq}(\mathcal{G})$, and $\mathcal{I}$ then simply contains all the graphs that (also accidentally) result from the presented interpretation.

Hence we choose an arbitrary $(G, \mathcal{P}) \in \mathcal{G}$ and $H \subseteq I(\mathcal{P})$. The task is to find a vertex labeling $\lambda_H : V(G) \to 2^L$ such that H has an efficient MSO_1 interpretation in the labeled graph $(G, \lambda_H) \in \mathcal{G}^L$. By Proposition 3 (cf. also Proposition 1), let $\gamma : E(G) \to \{1, \ldots, 22\}$ be a strong edge-coloring of the chosen graph G. Let $\mathcal{P} = \mathcal{P}_w \cup \mathcal{P}_b$ be the bipartition of the paths forming G corresponding to the partite sets of $I(\mathcal{P})$. We call the paths of $\mathcal{P}_w \cap V(H)$ "white" and those of $\mathcal{P}_b \cap V(H)$ "black". The remaining paths not in the vertex set of H are irrelevant. The edges of white/black paths are also called white/black, respectively, with the understanding that some edges of G may be both white and black. For $x \in V(G)$, we let $w(x) = \{\gamma(f) : f \text{ is a white edge incident to } x\}$ and $b(x) = \{\gamma(f) : f \text{ is a black edge incident to } x\}$. According to Proposition 1, $|w(x)| \leq 2$, $|b(x)| \leq 2$.

The key observation, derived directly from the definition of a strong edge-coloring, is that any edge $f = xy \in E(G)$ is a white edge iff $w(x) \cap w(y) \neq \emptyset$, and analogously for black edges. This allows us to speak separately about the white and black edges in G using only the language of MSO_1. Another easy observation is that the vertex sets of the paths in $\mathcal{P}$ have a system of distinct representatives by Hall's theorem. For if $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{P}'$ contains p white paths and q black paths, then $|V(\mathcal{P}')| \geq 2 \cdot \max\{p, q\} \geq p + q$, proving Hall's criterion. We assign a marker $r(x) \in \{\emptyset, w, b\}$ to each $x \in V(G)$ such that $r^{-1}(w)$ is the set of the representatives of white paths and $r^{-1}(b)$ is that of black paths (i.e., $r^{-1}(\emptyset)$ are not representatives). Finally, we assign another vertex marker $m(x) \in \{0, 1\}$ to each vertex $x \in V(G)$ such that $m(x) = 1$ iff $x \in V(P_1) \cap V(P_2)$ where $P_1, P_2 \in V(H) \subseteq \mathcal{P}$ and $\{P_1, P_2\} \in E(H)$.

Hence the label set L consists of 22 "light" colors coming from γ values on white paths, another 22 "dark" colors from black paths, and the three singletons w, b, m described above (altogether 47 binary labels). Note that the actual size of the needed label space over L is even much smaller; at most $\left[\binom{22}{2} + 22 + 1\right]^2 \cdot 3 \cdot 2 < 2^{19}$. The label $\lambda_H(x)$ of a vertex $x \in V(G)$ then contains the disjoint union $w(x)\dot\cup b(x)$, the label $r(x)$ if $\neq \emptyset$, and finally m if $m(x) = 1$.

Now, the interpretation of H in (G, λ_H) is simply as follows: The domain, i.e. the vertex set of H, is identified within $V(G)$ by a predicate $\alpha(x)$ expressing that "$r(x) = w \vee r(x) = b$" in MSO$_1$-L. In formal logic language (cf. Section 2), it is $L_w(x) \vee L_b(x)$. The relational symbol *adj* of H is then replaced, for $x, y \in V(G)$ s.t. $\alpha(x) \wedge \alpha(y)$, with $\beta^{adj}(x, y) \equiv \exists z \left[\text{“}m(z) = 1\text{”} \wedge \varrho(x, z) \wedge \varrho(y, z)\right]$, where $\varrho(t, z) \equiv \left[\text{“}r(t) = w\text{”} \rightarrow con_w(t, z)\right] \wedge \left[\text{“}r(t) = b\text{”} \rightarrow con_b(t, z)\right]$ and where con_w (con_b) routinely expresses in MSO$_1$-L the fact that t, z belong to the same component induced by white (black) edges in G. Precisely, $con_w(t, z) \equiv \forall Z\big[\, z \in Z \wedge t \notin Z \rightarrow \exists\, u, v\big(v \in Z \wedge u \notin Z \wedge adj(u, v) \wedge \text{“}w(u) \cap w(v) \neq \emptyset\text{”}\big)\big]$. Clearly, in this interpretation $(G, \lambda_H)^I \simeq H$ thanks to our choice of λ_H. ◀

Lemma 10 will be coupled with the next technical tool of similar flavor used in our previous [8]. We remark that its original formulation was even stronger, making the target graph class planar, but we are content with the following weaker formulation here. We call a graph G $\{1, 3\}$*-regular* if all the vertices of G have degree either one or three.

▶ **Lemma 11** ([8, in Theorem 5.5]). *The* MSO$_1$ *theory of all simple graphs has an efficient interpretation in the* MSO$_1$ *theory of all simple* $\{1, 3\}$*-regular graphs. Furthermore, this efficient interpretation* I *can be chosen such that, for every* MSO$_1$ *formula* ψ*, the resulting property* ψ^I *is invariant under subdivisions of edges; i.e. for every* $\{1, 3\}$*-regular graph* G *and any subdivision* G_1 *of* G *it holds* $G \models \psi^I$ *iff* $G_1 \models \psi^I$*.*

4 The Main Theorem

▶ **Theorem 12** (cf. Theorem 2). *Let* L *be a finite set of labels,* $|L| \geq 47$*. Unless the nonuniform Exponential-Time Hypothesis fails, there exists no graph class* $\mathcal{G}$ *satisfying all the three properties*

a) $\mathcal{G}$ *is closed under taking subgraphs,*
b) the tree-width of $\mathcal{G}$ *is densely unbounded poly-logarithmically,*
c) the MC(MSO$_1$-$L, \mathcal{G}^L$) *model-checking problem is in* XP*, i.e. testing whether* $G \models \varphi$ *is solvable in time* $\mathcal{O}\big(|V(G)|^{f(|\varphi|)}\big)$ *for some computable function* f*.*

Proof. We will show that if there exists a graph class $\mathcal{G}$ satisfying all three properties stated above, then we contradict the non-uniform ETH. Fix $b \in \mathbb{N}$ (to be determined later) and any sufficiently large $c \in \mathbb{N}$ such that $c > 5b$. By (b) and Definition 8, we have that for all $m \in \mathbb{N}$ there is $G'_m \in \mathcal{G}$ such that $tw(G'_m) \geq m^{5b}$ and $|V(G'_m)| < \mathcal{O}\big(2^{m^{5b/c}}\big)$. By Proposition 2, the graph G'_m contains a subgraph $G_m \subseteq G'_m$ which is grid-like as $(G_m, \mathcal{P}_m)$ of order m^b, for all sufficiently large m. Also $G_m \in \mathcal{G}$ by (a). We fix (one of) the K_{m^b}-minor in $I(\mathcal{P}_m)$, and denote by $\mathcal{V}_m$ the partition of the vertex set of $I(\mathcal{P}_m)$ into connected subgraphs that define this minor. Furthermore, by Proposition 3, there exists a strong edge coloring $\gamma_m \colon E(G_m) \rightarrow \{1, \ldots, 22\}$ of G_m. Define an advice function A that acquires the values $A(m) := \langle G_m, \mathcal{P}_m, \mathcal{V}_m, \gamma_m \rangle$ (whenever m is large enough for G_m to be defined as above). Since $c > 5b$ and $|V(G_m)| < \mathcal{O}\big(2^{m^{5b/c}}\big)$, our advice function A is sub-exponentially bounded.

Now we get to the core of the proof. Assume that we get an arbitrary graph F and any MSO$_1$ formula φ as input. We show that the model-checking instance $F \models \varphi$ can be solved in sub-exponential time wrt. $m = |V(F)|$ with help of our advice function A. For starters we query the oracle advice value $A(m) = \langle G_m, \mathcal{P}_m, \mathcal{V}_m, \gamma_m \rangle$. Then, by Lemma 11, there is an interpretation I_1 such that there exists a $\{1,3\}$-regular graph H and $H^{I_1} \simeq F$. Moreover, since I_1 is efficient, we can compute H efficiently and $|V(H)| \leq m^b$ for a suitable fixed b and sufficiently large m. Since our advice $(G_m, \mathcal{P}_m)$ is a grid-like graph of order m^b—i.e., its intersection graph $I(\mathcal{P}_m)$ has a K_{m^b}-minor— $I(\mathcal{P}_m)$ has a minor isomorphic to H, too. But H is $\{1,3\}$-regular and, in particular, has maximum degree three. Hence there exists a subgraph $H_1 \subseteq I(\mathcal{P}_m)$ that is isomorphic to a subdivision of H (in other words, H is a topological minor of $I(\mathcal{P}_m)$). This subgraph H_1 can be straightforwardly computed from the advice $\mathcal{V}_m$ over $(G_m, \mathcal{P}_m)$ in polynomial time.

By Lemma 10 there is another efficient interpretation I_2 assigning to H_1 a labeling λ_1 such that $(G_m, \lambda_1)^{I_2} \simeq H_1$. This λ_1 can actually be computed very easily with help of the advice γ_m from $A(m)$ along the lines of the proof of Lemma 10, not even using the algorithmic part of Proposition 3. Finally, we compute in polynomial time the formula $\psi \equiv (\varphi^{I_1})^{I_2}$. According to Lemma 11, ψ is invariant under subdivisions of edges, and so $H \models \varphi^{I_1} \iff H_1 \models \varphi^{I_1}$. Then, by the interpretation principle, $F \models \varphi \iff H \models \varphi^{I_1} \iff H_1 \models \varphi^{I_1} \iff (G_m, \lambda_1) \models \psi$. The final task is to run the algorithm of (c) on the instance $(G_m, \lambda_1) \models \psi$. The run-time is $|V(G_m)|^p$ for some p depending only on ψ, i.e. only on φ. Hence we get a solution to the model-checking instance $F \models \varphi$ in time $\mathcal{O}\big(|V(G_m)|^{f(|\varphi|)}\big) < \mathcal{O}\big(2^{f(|\varphi|)\cdot m^{5b/c}}\big) \in 2^{\mathcal{O}(m^{1-\varepsilon})}$ for any fixed φ, with a sub-exponentially bounded oracle advice function A.

In particular, if φ expresses the fact that a graph is 3-colorable (Example 5), then this shows that 3-Colorability $\in$ DTIME$(2^{o(m)})$/SubEXP, contradicting non-uniform ETH. ◀

▶ Proposition 4. Theorem 12 remains valid even if (b) is replaced with "the tree-width of $\mathcal{G}$ is densely unbounded by $\log^{q\cdot\gamma}$ with gap degree γ" for any $q > 8$.

Proof sketch. This follows from Definition 7 and since Lemma 11 works letting $b = 2$ (cf., [8]). Combining with Proposition 2, we see that any exponent $q > 2 \cdot 4$ suffices for our arguments to work, modulo the gap degree. ◀

We can strengthen Theorem 12 by showing that every problem in the Polynomial-Time Hierarchy (PH) is in DTIME$(2^{o(n)})$/SubEXP. But this stronger result comes at the price of a stricter assumption on the graph class $\mathcal{G}$: we assume that the MC(MSO$_1$-$L, \mathcal{G}^L$) model-checking problem is in XP (wrt. the formula size $|\varphi|$ as parameter) for *every* finite set of labels L such that $|L| = O(|\varphi|)$. Note that in Theorem 12, L was a fixed finite set of labels.

▶ **Theorem 13.** *Unless* PH $\subseteq$ DTIME$(2^{o(n)})$/SubEXP, *there exists no graph class $\mathcal{G}$ satisfying all three properties*

a) $\mathcal{G}$ is closed under taking subgraphs,
b) the tree-width of $\mathcal{G}$ is densely unbounded poly-logarithmically,
c) the MC(MSO$_1$-$L, \mathcal{G}^L$) *model-checking problem is in* XP*, i.e. testing whether $G \models \varphi$, where G is a vertex-labeled graph with $O(|\varphi|)$ labels, is solvable in time $\mathcal{O}\big(|V(G)|^{f(|\varphi|)}\big)$ for some computable function f.*

5 Implications for Directed Width Measures

In this section, we briefly foray into the area of digraph width measures and discuss the implications of the results in the previous section. This part follows on our earlier [8]. An important goal in the design of a "good" width measure is for it to satisfy two seemingly contradictory requirements: (1) a large class of problems must be efficiently solvable on the graphs of bounded width; and, (2) the class of the graphs of bounded width should have a nice, reasonably rich and natural structure. In contrast to the undirected graph case, where e.g. tree-width has become a true success story, this effort has largely failed for digraph width measures. A partial answer for the reasons of this failure was provided in [8] where it was shown that any digraph width measure that is different from the undirected tree-width and monotone under directed topological minors is not algorithmically powerful. The phrase "different from tree-width" is defined by the property that there exists a constant $c \in \mathbb{N}$ such that the class of the underlying undirected graphs of digraphs of width at most c has unbounded tree-width. Algorithmic "powerfulness" has been defined as the property of admitting XP algorithms (wrt. the width as parameter) for all problems in MSO_1.

We improve upon this result by showing that even if the underlying undirected graphs corresponding to digraphs of bounded width have poly-logarithmically unbounded tree-width, and the digraph width measure is monotone just under subdigraphs, then the width measure is not algorithmically powerful. First note that we relax *unbounded* tree-width by *poly-logarithmically unbounded* tree-width. This is a somehow stronger assumption, and the strengthening is unavoidable due to a negative example shown in [8]. Secondly, we require the directed width measure to be closed under *subdigraphs* and not *directed topological minors* as in [8]; which is, on the other hand, a much weaker requirement. Thirdly, our interpretation of algorithmic powerfulness is that all problems in MSO_1-L can be solved on *L-vertex-labeled graphs* in XP-time wrt. the width and formula size as parameters. This again is a dilution of the notion of algorithmic power as defined in [8], where only plain MSO_1 over unlabeled digraphs has been exploited.

We start by defining what it means for a digraph width measure to have poly-logarithmically unbounded tree-width. We shortly denote by $U(D)$ the underlying undirected graph of a digraph D.

▶ **Definition 14.** A directed width measure δ *largely surpasses tree-width* if there exists $d \in \mathbb{N}$ such that the tree-width of the undirected graph class $\{U(D) : \delta(D) \leq d\}$ is densely unbounded poly-logarithmically.

Then the main result of this section reads:

▶ **Theorem 15.** *Let L be a finite set of labels, $|L| \geq 47$. Unless the non-uniform Exponential-Time Hypothesis fails, there exists no directed width measure δ satisfying all three properties:*

a) δ is monotone under taking subdigraphs;
b) δ largely surpasses the tree-width of underlying undirected graphs; and
c) for all L-vertex-labeled digraphs D and all formulas $\varphi \in MSO_1$-L, the problem of deciding whether $D \models \varphi$ is solvable in time $O(|D|^{f(\delta(D),|\varphi|)})$ for some computable f.

6 Concluding Remarks

Our paper contributes to Kreutzer and Tazari's impressive results in this area. Our proof is shorter and holds for MSO_1-L logic instead of MSO_2 at the price of a stronger assumption in computational complexity. The expressive power of MSO_2 over graphs with labels from a set L and MSO_1 with the same label set is huge—for instance, the latter is not able to express some natural graph problems like Hamiltonian cycle. However, one cannot directly compare the expressive power of bare MSO_2 without labels and MSO_1-L over graphs with vertex labels from L, as there are problems which can be expressed in MSO_1-L but not in

MSO_2 and vice versa. We have proved that it is not possible to efficiently process latter MSO_1-L on graph classes with "very" unbounded tree-width which are subgraph-closed.

Besides the implications discussed in Section 5, there is also an implication for another width measure—clique-width. Clique-width [4] (as well as rank-width) is a graph parameter which allows efficient (FPT time) model-checking of (labeled) MSO_1-L formulas, however it has received some criticism for not having nice structural properties such as being monotone under taking subgraphs. Our results indicate that it is unlikely any parameter exists with the desirable properties of clique-width which is monotone under taking subgraphs.

Finally, let us briefly mention the possibility of extending Theorem 12 to unlabeled graphs, i.e., using plain MSO_1 over $\mathcal{G}$ in Theorem 12 (c). It is not known whether there exists any natural and nontrivial graph class where unlabeled MSO_1 is efficiently solvable and yet MSO_1-L model-checking is hard. Such a graph class would necessarily contain graphs of unbounded clique-width (since otherwise MSO_1-L could be efficiently model-checked) and yet with sufficient structure to allow efficient model-checking of bare MSO_1.

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