Modal Logics Definable by Universal Three-Variable Formulas

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Abstract

We consider the satisfiability problem for modal logic over classes of structures definable by universal first-order formulas with three variables. We exhibit a simple formula for which the problem is undecidable. This improves an earlier result in which nine variables were used. We also show that for classes defined by three-variable, universal Horn formulas the problem is decidable. This subsumes decidability results for many natural modal logics, including T, B, K4, S4, S5.

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1 Introduction

Modal logic for almost a hundred year has been an important topic in many academic disciplines, including philosophy, mathematics, linguistics, and computer science. Currently it seems to be most intensively investigated by computer scientists. Among numerous branches in which modal logic, sometimes in disguise, finds applications, are hardware and software verification, cryptography and knowledge representation.

Modal logic was introduced by philosophers to study modes of truth. The idea was to extend propositional logic by some new constructions, of which two most important were $\Diamond\varphi$ and $\Box\varphi$, originally read as φ *is possible* and φ *is necessary*, respectively. A typical question was, given a set of axioms $\mathcal{A}$, corresponding usually to some intuitively acceptable aspects of truth, what is the logic defined by $\mathcal{A}$, i.e. which formulas are provable from $\mathcal{A}$ in a Hilbert-like system.

One of the most important steps in the history of modal logic was inventing a formal semantics based on the notion of the so-called Kripke structures. Basically, a Kripke structure is a directed graph, called a *frame*, together with a valuation of propositional variables. Vertices of this graph are called *worlds*. For each world truth values of all propositional variables can be defined independently. In this semantics, $\Diamond\varphi$ means φ *is true in some world connected to the current world*; and $\Box\varphi$, equivalent to $\neg\Diamond\neg\varphi$, means φ *is true in all worlds connected to the current world.*

It appeared that there is a beautiful connection between syntactic and semantic approaches to modal logic [12]: logics defined by axioms can be equivalently defined by restricting classes of frames. E.g., the axiom $\Diamond\Diamond P \rightarrow \Diamond P$ (*if it is possible that P is possible, then P is possible*), is valid precisely in the class of transitive frames; the axiom $P \rightarrow \Diamond P$ (*if P is true, then P is possible*) – in the class of reflexive frames, $P \rightarrow \Box\Diamond P$ (*if P is true, then it is necessary*

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that P is possible) – in the class of symmetric frames, and the axiom $\Diamond P \rightarrow \Box\Diamond P$ (*if P is possible, then it is necessary that P is possible*) – in the class of Euclidean frames.

Thus we may think that every modal formula φ defines a class of frames, namely the class of those frames in which φ is valid. A formula φ is valid in a frame K if for any possible truth-assignment of propositional variables to the worlds of K, φ is true at every world. While this definition involves quantification over sets of worlds, many important classes of frames, in particular all the classes we mentioned above, can be defined by simple first-order formulas. For a given first-order sentence Φ over the signature consisting of a single binary symbol R we define $\mathcal{K}_\Phi$ to be the set of those frames which satisfy Φ.

In this paper we are interested in the satisfiability problem for modal logic over classes of frames definable by universal first-order formulas. The first result in this area was that there exists a universal first-order formula with equality Φ, such that the *global* satisfiability problem for modal logic over $\mathcal{K}_\Phi$ is undecidable [6]. By global satisfiability we mean the problem of determining if there exists a Kripke structure such that a given modal formula φ is true at every world of this structure. That result has been recently improved in [8] in two aspects: by removing equality and globalness. Namely, the authors exhibited a formula Φ' without equality, such that the standard, *local*, satisfiability problem for modal logic over $\mathcal{K}_{\Phi'}$ is undecidable.

The formula from [8] uses nine variables. A natural question arises, how many variables are necessary to obtain undecidability. Note that transitive, reflexive, symmetric, or equivalence frames are definable by formulas with just three variables. The satisfiability problem for modal logic over those classes is known to be decidable [9]. It appears however that there exists a universal first-order formula without equality with only three variables defining the class of frames over which satisfiability problem for modal logic is undecidable. Exhibiting such a formula is the first contribution of our paper.

▶ **Theorem 1.** *There exists a three-variable universal formula Γ', without equality, such that the local satisfiability problem for modal logic over $\mathcal{K}_{\Gamma'}$ is undecidable.*

Our formula, despite the fact that it uses much smaller number of variables, is also simpler than the formula from [8]. Actually, if we only want to show the undecidability of global satisfiability then we can use a formula Γ which is just a single, universally quantified clause consisting of six literals.

We emphasize that our result is optimal with respect to the number of variables. Indeed, if Φ is an arbitrary (not necessarily universal) first-order sentence with two variables, then the satisfiability problem for modal logic over $\mathcal{K}_\Phi$ can be reduced to the satisfiability problem for the two-variable fragment of first-order logic, FO^2, using the standard translation of modal logic into FO^2. The latter problem is known to be decidable [10, 4]. For details about the standard translation see e.g. [2].

Decidable classes of frames we mentioned earlier can be defined by three-variable first-order sentences even if we further restrict the language to universal Horn formulas, UHF. Universal Horn formulas were considered in [7], where a dichotomy result was proved, that the satisfiability problem for modal logic over the class of structures defined by an UHF formula (with an arbitrary number of variables) is either in NP or PSpace-hard. In the same paper decidability is shown for a rich subclass of UHF, including in particular all formulas which imply reflexivity. However, the problem remained open for formulas involving variants of transitivity. The authors of [7] conjecture that the problem is decidable, and in PSpace for all universal Horn formulas. Our second contribution is confirming this conjecture for the case of formulas with at most three-variables, UHF^3.

▶ **Theorem 2.** *Let Φ be a* UHF^3 *sentence. Then the local and the global satisfiability problems for modal logic over $\mathcal{K}_\Phi$ are decidable.*

This theorem extends the decidability results for the classes we mentioned earlier in this introduction, in particular for modal logics T, B, K4, S4, S5. It also works for some interesting classes of frames, for which, up to our knowledge, decidability has not been established so far. An example is the class defined by $\forall xyz(xRy \wedge yRz \rightarrow zRx)$.

We provide a full classification of UHF^3 sentences, with respect to the complexity of satisfiability of modal logic over the classes of frames they define. It appears, that except for the trivial case of inconsistent formulas for which the problem is in P, local satisfiability is either NP-complete or PSPACE-complete, and global satisfiability is NP-complete, PSPACE-complete, or EXPTIME-complete.

2 Preliminaries

As we work with both first-order logic and modal logic we help the reader by distinguishing them in our notation: we denote first-order formulas with Greek capital letters, and modal formulas with Greek small letters. We assume that the reader is familiar with first-order and propositional logic.

Modal logic extends propositional logic with the operator $\Diamond$ and its dual $\Box$. Formulas of modal logic are interpreted in Kripke structures, which are triples of the form $\langle W, R, \pi\rangle$, where W is a set of worlds, $\langle W, R\rangle$ is a directed graph called a *frame*, and π is a function that assigns to each world a set of propositional variables which are true at this world. We say that a structure $\langle W, R, \pi\rangle$ is *based* on the frame $\langle W, R\rangle$.

The semantics of modal logic if defined recursively. A modal formula φ is (locally) *satisfied* in a world w of a model $\mathfrak{M} = \langle W, R, \pi\rangle$, denoted as $\mathfrak{M}, w \models \varphi$ if (i) φ is a variable and $\varphi \in \pi(w)$, (ii) $\varphi = \varphi_1 \vee \varphi_2$ and $\mathfrak{M}, w \models \varphi_1$ or $\mathfrak{M}, w \models \varphi_2$, (iii) $\varphi = \neg\varphi'$ and $\mathfrak{M}, w \not\models \varphi'$, or (iv) $\varphi = \Diamond\varphi'$ and there exists a world $v \in W$ such that $(w, v) \in R$ and $\mathfrak{M}, v \models \varphi'$. We abbreviate $\neg\Diamond\neg\varphi$ by $\Box\varphi$. By $|\varphi|$ we denote the length of φ measured as the total number of occurrences of propositional variables. We say that a formula φ is *globally* satisfied in $\mathfrak{M}$, denoted as $\mathfrak{M} \models \varphi$, if for all worlds w of $\mathfrak{M}$, we have $\mathfrak{M}, w \models \varphi$.

For a given class of frames $\mathcal{K}$, we say that a formula φ is *locally* (resp. *globally*) $\mathcal{K}$-*satisfiable* if there exists a frame $K \in \mathcal{K}$, a structure $\mathfrak{M}$ based on K, and a world $w \in W$ such that $\mathfrak{M}, w \models \varphi$ (resp. $\mathfrak{M} \models \varphi$). We define the *local* (resp. *global*) *satisfiability problem* $\mathcal{K}$-SAT (resp. global $\mathcal{K}$-SAT) as follows. For a given modal formula, is this formula locally (resp. globally) $\mathcal{K}$-satisfiable?

For a given formula φ, a Kripke structure $\mathfrak{M}$, and a world $w \in W$ we define the *type* of w (with respect to φ) in $\mathfrak{M}$ as $tp^{\varphi}_{\mathfrak{M}}(w) = \{\psi : \mathfrak{M}, w \models \psi$ and ψ is subformula of $\varphi\}$. We write $tp_{\mathfrak{M}}(w)$ if the formula is clear from the context. Note that $|tp^{\varphi}_{\mathfrak{M}}(w)| < |\varphi|$.

The set of *universal Horn formulas* with three variables without equality, UHF^3, is defined as the set of those Φ which are of the form $\forall xyz.\Phi_1 \wedge \Phi_2 \wedge \ldots \wedge \Phi_i$, where each Φ_i is a Horn clause. A Horn clause is a disjunction of literals of which at most one is positive. We usually present Horn clauses as implications. For example, the formula $\forall xyz.(xRy \wedge yRz \Rightarrow xRz) \wedge (xRx \Rightarrow \bot)$ defines the set of transitive and irreflexive frames. We often skip the quantifiers and represent such formulas as sets of clauses, e.g.: $\{xRy \wedge yRz \Rightarrow xRz, xRx \Rightarrow \bot\}$. We assume without loss of generality that each Horn clause is of the form $\Psi \Rightarrow \bot$, $\Psi \Rightarrow xRx$, or $\Psi \Rightarrow xRy$. We define $\Psi(v_1, v_2, v_3)$ as the instantiation of Ψ with $x = v_1$, $y = v_2$, and $z = v_3$, e.g. $(xRy \wedge yRz)(a, b, c) = aRb \wedge bRc$. We denote by Φ^p the set of the clauses of Φ containing a positive literal, i.e. all clauses of Φ except those of the form $\Psi \Rightarrow \bot$.

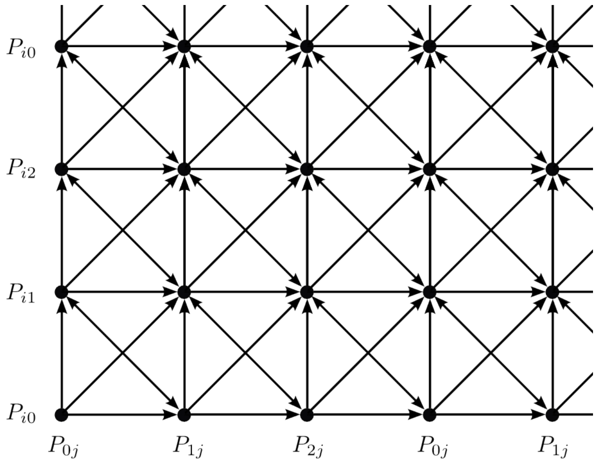

Figure 1 The structure $\mathfrak{G}_{\mathbb{N}}$. Its universe is $\mathbb{N} \times \mathbb{N}$. Reflexive arrows are omitted for clarity.

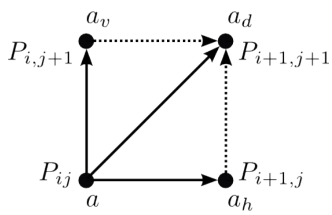

Figure 2 Completing the grid. Dotted arrows are enforced by Γ and τ.

3 Undecidability

In this section we work with signatures consisting of a single binary symbol R, and a number of unary symbols, including P_{ij}, for $0 \leq i, j \leq 2$. Structures over such signatures can be naturally viewed as Kripke structures in which R is the accessibility relation, and unary relations describe valuations of propositional variables. To simplify our notation we assume that subscripts in P_{ij} are always taken modulo 3, e.g. if $i = 2, j = 2$, then $P_{i+1,j+1}$ denotes P_{00}.

Let

$$\Gamma = \forall xyz.\neg xRy \vee yRx \vee \neg xRz \vee zRx \vee yRz \vee zRy.$$

First, we prove that global $\mathcal{K}_\Gamma$-SAT is undecidable. Then we use the trick from [8] and show that also local $\mathcal{K}_{\Gamma'}$-SAT is undecidable, for Γ' being a modification of Γ, using still only three variables.

3.1 General idea

Note that Γ can be rewritten as $\forall xyz.(xRy \wedge \neg yRx \wedge xRz \wedge \neg zRx) \rightarrow (yRz \vee zRy)$, i.e. it says, that if there are one-way connections from a world x to worlds y, z, then there is also a connection (not necessarily one-way) between y and z. The structure $\mathfrak{G}_{\mathbb{N}}$ illustrated in Fig. 1 (we assume that this structure is reflexive) is a model of Γ. Note that it is important that some connections are two-way. In $\mathfrak{G}_{\mathbb{N}}$ we can define the horizontal adjacency relation by the following formula with free variables x, y: $\bigvee_{ij}(P_{ij}x \wedge P_{i+1,j}y \wedge xRy)$. Analogously, we define the vertical adjacency: $\bigvee_{ij}(P_{ij}x \wedge P_{i,j+1}y \wedge xRy)$. $\mathfrak{G}_{\mathbb{N}}$ can be now viewed as an expansion of the standard grid on $\mathbb{N} \times \mathbb{N}$.

To get the undecidability we construct a modal formula τ, capturing some properties of $\mathfrak{G}_{\mathbb{N}}$, such that any model $\mathfrak{M} \models \tau$ from $\mathcal{K}_\Gamma$ locally looks like a grid. Namely, τ says that every element satisfying P_{ij} has three R-successors: one in $P_{i+1,j}$, one in $P_{i,j+1}$, and one in $P_{i+1,j+1}$, and forbids connections from $P_{i+1,j+1}$ to $P_{i,j+1}$, $P_{i+1,j}$, and P_{ij}. If we consider now any element a in a model, we see that τ enforces the existence of its horizontal successor a_h, its vertical successor a_v and its upper-right diagonal successor a_d (see Fig. 2). By τ, the connections to these successors are one-way, so we need, by Γ, connections between a_h and

a_d, and a_v and a_d. Again, by τ, these connections has to go from a_h to a_d, and from a_v to a_d, so a_d is indeed a horizontal successor of a_v, and a vertical successor of a_h.

Below we present a more detailed proof covering also the case of finite satisfiability, i.e. satisfiability in the class of finite models. The technique we employ is quite standard. It is similar e.g. to the technique used in [11].

3.2 Domino systems

In the proof we use some well known results on domino systems.

▶ **Definition 3.** A *domino system* is a tuple $\mathcal{D} = (D, D_H, D_V)$, where D is a set of domino pieces and $D_H, D_V \subseteq D \times D$ are binary relations specifying admissible horizontal and vertical adjacencies. We say that $\mathcal{D}$ *tiles* $\mathbb{N} \times \mathbb{N}$ if there exists a function $t : \mathbb{N} \times \mathbb{N} \to \mathcal{D}$ such that $\forall i, j \in \mathbb{N}$ we have $(t(i, j), t(i + 1, j)) \in D_H$ and $(t(i, j), t(i, j + 1)) \in D_V$. Similarly, $\mathcal{D}$ tiles $\mathbb{Z}_k \times \mathbb{Z}_l$, for $k, l \in \mathbb{N}$, if there exists $t : \mathbb{Z}_k \times \mathbb{Z}_l \to \mathcal{D}$ such that $(t(i, j), t(i+1 \mod k, j)) \in D_H$ and $(t(i, j), t(i, j + 1 \mod l)) \in D_V$.

The following lemma comes from [1, 5].

▶ **Lemma 4.** *The following problems are undecidable:*

(i) For a given domino system $\mathcal{D}$ determine if $\mathcal{D}$ tiles $\mathbb{N} \times \mathbb{N}$.
(ii) For a given domino system $\mathcal{D}$ determine if there exists $k \in \mathbb{N}$ such that $\mathcal{D}$ tiles $\mathbb{Z}_k \times \mathbb{Z}_k$.

3.3 Grid definition

We capture some properties of $\mathfrak{G}_{\mathbb{N}}$ by a modal formula τ.

$$\tau = \tau_0 \wedge \bigwedge_{0 \le i,j \le 2} (\tau^{\Diamond}_{ij} \wedge \tau^{\Box}_{ij}),$$

where τ_0 says that each element satisfies one of P_{ij}, $\tau^{\Diamond}_{ij}$ ensure that all elements have appropriate horizontal, vertical and upper-right diagonal successors, and $\tau^{\Box}_{ij}$ forbid reversing the horizontal, vertical and upper-right diagonal arrows.

$$\tau^{\Diamond}_{ij} = P_{ij} \to (\Diamond P_{i+1,j} \wedge \Diamond P_{i,j+1} \wedge \Diamond P_{i+1,j+1}),$$

$$\tau^{\Box}_{ij} = P_{ij} \to \Box(\neg P_{i-1,j} \wedge \neg P_{i,j-1} \wedge \neg P_{i-1,j-1}).$$

Note that $\tau^{\Box}_{ij}$ allow for reflexive edges.

3.4 Domino encoding

We encode an instance of the domino problem by a modal formula in a standard way. For a given domino system $\mathcal{D} = (D, D_H, D_V)$ we define

$$\lambda^{\mathcal{D}} = \lambda_0 \wedge \bigwedge_{0 \le i,j \le 2} (\lambda^H_{ij} \wedge \lambda^V_{ij}).$$

For every $d \in D$ we use a fresh propositional letter P_d. λ_0 says that each world contains a domino piece, λ^H_{ij} and λ^V_{ij} say that pairs of elements satisfying horizontal and vertical adjacency relations respect D_H and D_V, respectively.

$$\lambda^H_{ij} = \bigwedge_{d \in D} ((P_d \wedge P_{ij}) \to \Box(P_{i+1,j} \to \bigvee_{d' : (d,d') \in D_H} P_{d'})),$$

$$\lambda^V_{ij} = \bigwedge_{d\in D}((P_d \wedge P_{ij}) \to \Box(P_{i,j+1} \to \bigvee_{d':(d,d')\in D_V} P_{d'})).$$

The following lemma establishes the undecidability of the global satisfiability and the global finite satisfiability problems for modal logic over $\mathcal{K}_\Gamma$.

▶ **Lemma 5.** *Let $\mathcal{D}$ be a domino system.*

(i) $\mathcal{D}$ tiles $\mathbb{N}\times\mathbb{N}$ iff there exists $\mathfrak{M}\in\mathcal{K}_\Gamma$ such that $\mathfrak{M}\models\tau\wedge\lambda^{\mathcal{D}}$.
(ii) $\mathcal{D}$ tiles some $\mathbb{Z}_k\times\mathbb{Z}_k$ iff there exists a finite $\mathfrak{M}\in\mathcal{K}_\Gamma$ such that $\mathfrak{M}\models\tau\wedge\lambda^{\mathcal{D}}$.

Proof. As in the case of symbols P_{ij}, when referring to $\tau^{\Box}_{ij}$ or $\tau^{\Diamond}_{ij}$ we assume that subscripts are taken modulo 3.

Part (i), ⇒ Let t be a tiling of $\mathbb{N}\times\mathbb{N}$. We construct $\mathfrak{M}$ by expanding $\mathfrak{G}_{\mathbb{N}}$ in such a way that for every $i,j\in\mathbb{N}$ the element (i,j) satisfies $P_{t(i,j)}$. It is readily checked that $\mathfrak{M}$ is as required.

Part (i), ⇐ We explain how to construct a function $f:\mathbb{N}\times\mathbb{N}\to M$, such that for every $i,j\in\mathbb{N}$: (a) $\mathfrak{M}\models P_{ij}(f(i,j))$, (b) $\mathfrak{M}\models f(i,j)Rf(i+1,j)$, (c) $\mathfrak{M}\models f(i,j)Rf(i,j+1)$.

First we show how to define f on $\mathbb{N}\times\{0\}$. Let $f(0,0)=c$ for an arbitrary element c of M satisfying P_{00}. Such c exists owing to τ_0 and $\tau^{\Diamond}_{ij}$. Assume that for some $i>0$ we have defined $f(i-1,0)=a$, and let a_h be an R-successor of a satisfying P_{i0}. Such a_h exists owing to $\tau^{\Diamond}_{i-1,0}$. Define $f(i,0)=a_h$.

Assume now that f is defined for $\mathbb{N}\times[0,\dots,j-1]$ for some $j>0$. We extend this definition to $\mathbb{N}\times\{j\}$. Let $f(0,j-1)=a$. By the inductive assumption a satisfies $P_{0,j-1}$. Choose a_v to be an R-successor of a satisfying P_{0j}. Such a_v exists by $\tau^{\Diamond}_{0,j-1}$. Set $f(0,j)=a_v$.

Assume that we have defined $f(i-1,j-1)=a$, $f(i-1,j)=a_v$, and $f(i,j-1)=a_h$. By the inductive assumptions $\mathfrak{M}\models P_{i-1,j-1}(a)\wedge P_{i-1,j}(a_v)\wedge P_{i,j-1}(a_h)\wedge aRa_h\wedge aRa_v$. Choose a_d to be an R-successor of a satisfying P_{ij}. Such a_d exists by $\tau^{\Diamond}_{i-1,j-1}$. By $\tau^{\Box}_{ij}$, a_h, a_v and a_d cannot be connected to a, so Γ enforces R-connections between a_h and a_d, and between a_v and a_d. Since τ^{A}_{ij} forbids connection from a_d to a_h, and from a_d to a_v, it has to be that $\mathfrak{M}\models a_hRa_d\wedge a_vRa_d$. This finishes definition of f with the desired properties.

We define a tiling $t:\mathbb{N}\times\mathbb{N}$ by setting $t(i,j)=d$ for such d that $f(i,j)$ satisfies P_d (there is at least one such d owing to λ_0). Properties (a), (b), (c) of f and the formulas $\lambda^H_{ij},\lambda^V_{ij}$ imply that t is a correct tiling.

Part (ii) ⇒ Let $l=3k$ for some $k\in\mathbb{Z}$. We define $\mathfrak{G}_l$ to be the quotient of $\mathfrak{G}_{\mathbb{N}}$ by the relation $\approx$: $(i,j)\approx(i',j')$ iff $i\equiv i' \mod l$ and $j\equiv j' \mod l$. $\mathfrak{G}_l$ can be seen as an expansion of the standard grid on $\mathbb{Z}_l\times\mathbb{Z}_l$ torus. It is readily checked that for every $k\in\mathbb{N}$ we have $\mathfrak{G}_{3k}\models\Gamma$ and $\mathfrak{G}_{3k}\models\tau$.

If $\mathcal{D}$ tiles $\mathbb{Z}_k\times\mathbb{Z}_k$ then it also tiles $\mathbb{Z}_{3k}\times\mathbb{Z}_{3k}$. Let t be a tiling of $\mathbb{Z}_{3k}\times\mathbb{Z}_{3k}$. We construct $\mathfrak{M}$ by expanding $\mathfrak{G}_{3k}$ in such a way that for every $i,j\in\mathbb{Z}_{3k}$ the element (i,j) satisfies $P_{t(i,j)}$. Again, checking that $\mathfrak{M}$ is as required is straightforward.

Part (ii) ⇐ We want to define for some $k,l\in\mathbb{Z}$ a function $f:\mathbb{Z}_k\times\mathbb{Z}_l\to M$ satisfying: (a) $\mathfrak{M}\models P_{ij}(f(i,j))$, (b) $\mathfrak{M}\models f(i,j)Rf(i+1 \mod k, j)$, (c) $\mathfrak{M}\models f(i,j)Rf(i,j+1 \mod l)$.

We define f as a partial function on $\mathbb{N}\times\mathbb{N}$ and then restrict it to an appropriate domain. We first define f on $\mathbb{N}\times\{0\}$, exactly as in the proof of Part (i), ⇐. Since $\mathfrak{M}$ is finite this time, it has to be that $f(k,0)=f(k',0)$ for some $k>k'$. To simplify the presentation we assume $k'=0$, but this assumption is not relevant. Observe that for $i\in[0,k)$ we have $\mathfrak{M}\models f(i,0)Rf(i+1 \mod k,0)$. We extend the definition of f to $[0,k)\times\mathbb{N}$ inductively. Assume that f is defined on $[0,k)\times\{0,\dots,j-1\}$. We define it on $[0,k)\times\{j\}$. For each $i\in[0,k)$ we find an element a_d^i in M such that $\mathfrak{M} \models P_{i+1,j}(a_d^i) \wedge f(i,j-1)Ra_d^i$. Such a_d^i exists owing to $\tau^{\Diamond}_{i,j-1}$. We set $f(i+1 \mod k, j) = a_d^i$. Now Γ and formulas of the type $\tau^{\Box}$ enforce for all $i \in [0,k)$ that $\mathfrak{M} \models f(i,j-1)Rf(i,j)$, and $\mathfrak{M} \models f(i,j)Rf(i+1 \mod k, j)$.

Finiteness of $\mathfrak{M}$ implies now that for some $l > l'$ we have $f\restriction[0,k) \times \{l\} = f\restriction[0,k) \times \{l'\}$. Again for simplicity we assume that $l' = 0$. Observe that at this moment f is as desired on $\mathbb{Z}_k \times \mathbb{Z}_l$. We define a tiling $t : \mathbb{Z}_k \times \mathbb{Z}_l$ by setting $t(i,j) = d$ for such d that $f(i,j)$ satisfies P_d (there is at least one such d owing to λ_0). Properties (a), (b), (c) of f and the formulas λ^H_{ij} and λ^V_{ij} imply that t is a correct tiling of $\mathbb{Z}_k \times \mathbb{Z}_l$. This implies that there exists also a correct tiling of $\mathbb{Z}_m \times \mathbb{Z}_m$ for $m = gcd(k,l)$.

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3.5 Local satisfiability

Observe that our proof of the undecidability of global satisfiability over $\mathcal{K}_\Gamma$ works for the subclass of reflexive models. This allows us to use the trick from [8] to cover also the case of local satisfiability. We enforce by a modal formula the existence of an irreflexive world and, by a first-order formula, we make it connected to all reflexive worlds. Such a *universal world* can be then used to reach all relevant elements in the model. The class of structures is defined by a formula Γ', which says that each world with an incoming edge is reflexive and has an incoming edge from all irreflexive worlds, and enforces Γ for all reflexive worlds:

$$\Gamma' = \forall xyz.((xRy \wedge \neg zRz) \to (yRy \wedge zRy)) \wedge$$
$$((xRx \wedge yRy \wedge zRz) \to (\neg xRy \vee yRx \vee \neg xRz \vee zRx \vee yRz \vee zRy)).$$

In the modal formula we use a fresh symbol P_U to distinguish an irreflexive world. Now, for a given domino system $\mathcal{D}$ we can show that $P_U \wedge \Box\neg P_U \wedge \Diamond\top \wedge \Box(\tau \wedge \lambda^D)$ is locally (finitely) satisfiable over $\mathcal{K}_{\Gamma'}$ iff $\mathcal{D}$ covers $\mathbb{N} \times \mathbb{N}$ (some $\mathbb{Z}_k \times \mathbb{Z}_k$). This proves Theorem 1.

See subsection 5.6 of [8] for details of the outlined trick.

4 Decidability

In this section, we prove Theorem 2. The general idea of the proof is standard: we are going to show that for every UHF^3 formula Φ and every modal formula φ, if φ is $\mathcal{K}_\Phi$-satisfiable then it is also $\mathcal{K}_\Phi$-satisfiable in a "nice" model.

We start from an arbitrary model $\mathfrak{M} \models \varphi$ based on a frame from $\mathcal{K}_\Phi$ and unravel it into a model $\mathfrak{M}_0$ whose frame is a tree with the degree of its nodes bounded by $|\varphi|$. Clearly the frame of $\mathfrak{M}_0$ is not necessarily a member of $\mathcal{K}_\Phi$. In the next step we add to $\mathfrak{M}_0$ the edges implied by the Horn clauses of Φ. This is performed in countably many stages, until the least fixed point is reached. We observe that the resulting structure, $\mathfrak{M}_\infty$, is still a model of φ, and its frame belongs to $\mathcal{K}_\Phi$.

Then we show that every model which can be obtained in the described way falls into one of the four classes, which we call the class of semi-trees, transitive-trees, clique-unions, and tripartitions.[1] Moreover, for a given UHF^3 formula Φ there exists a single class of models, such that every $\mathcal{K}_\Phi$-satisfiable modal formula φ has a model from this class.

Finally, we argue that for a given modal formula φ, checking if it has a model from one of our four classes is decidable. If φ is $\mathcal{K}_\Phi$-satisfiable in a clique-union or in a tripartition it

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[1] We choose such names for simplicity. In fact, in transitive trees transitivity may fail near the end of a path, and clique-unions may have *heads* and *tails*. See Definition 7.

can be shown that it is also $\mathcal{K}_\Phi$-satisfiable in a clique-union or a tripartition of polynomially bounded size, so we can simply guess such a small model and verify it; if φ is $\mathcal{K}_\Phi$-satisfiable in a semi-tree or in a transitive-tree then we use some adaptations of the standard techniques for satisfiability of modal logics over the class of all frames, and over the class of transitive frames, respectively.

4.1 Minimal tree-based models

We say that an edge (w_1, w_2) is a *consequence* of Φ in $\langle W, R\rangle$ if for some $w_3 \in W$ and $\Psi_1 \Rightarrow \Psi_2 \in \Phi$ we have $R \models \Psi_1(w_1, w_2, w_3)$, and $\Psi_2(w_1, w_2, w_3) = w_1 R w_2$. We define the *consequence operator* as follows.

$$\textsc{Cons}_{\Phi,W}(R) = R \cup \{(w_1, w_2) : (w_1, w_2) \text{ is a consequence of } \Phi \text{ in } \langle W, R\rangle\}$$

We are going to use this operator in stages, starting from a tree and adding edges required by Φ. We define the *closure operator* as the least fixed-point of *Cons*:

$$\textsc{Closure}_{\Phi,W}(R) = \bigcup_{i>0} \textsc{Cons}^i_{\Phi,W}(R)$$

For a tree $\mathcal{T} = \langle W, R\rangle$, we now define the *minimal* T-*based model of* Φ as $\mathfrak{C}_\Phi(\mathcal{T}) = \langle W, Closure_{\Phi,W}(R)\rangle$. Note that $\mathfrak{C}_\Phi(\mathcal{T})$ is the smallest model of Φ^p containing all edges from R.

▶ **Lemma 6.** *Let φ be a modal formula and let $\Phi \in \mathsf{UHF}^3$. If φ is $\mathcal{K}_\Phi$-satisfiable, then there exists a tree $\mathcal{T}$ in which the degree of its nodes is bounded by $|\varphi|$, such that φ has a model based on the frame $\mathfrak{C}_\Phi(\mathcal{T})$.*

Proof. Let $\mathfrak{M} = \langle W, R, \pi\rangle$, $u_0 \in W$ be such that $\mathfrak{M} \models \Phi$ and $\mathfrak{M}, u_0 \models \varphi$.

We construct $\mathfrak{M}_0 = \langle W_0, R_0, \pi_0\rangle$ by an unraveling of $\mathfrak{M}$ as follows. W_0 is a subset of the set of finite sequences of elements of W. We define W_0 and R_0 inductively. Initially, we put $(u_0) \in W_0$. Assume that $(u_0, \ldots, u_k) \in W_0$. Let $\Diamond\psi_1, \ldots, \Diamond\psi_s$ be all the formulas of the form $\Diamond\psi$ from $tp_{\mathfrak{M}}(u_k)$. There exist $u^1_{k+1}, \ldots, u^s_{k+1} \in W$, such that for every $i \in \{1, \ldots, s\}$ we have $\mathfrak{M} \models u_k R u^i_{k+1}$ and $\psi_i \in tp_{\mathfrak{M}}(u^i_{k+1})$. For each such i we put $(u_0, \ldots, u_k, u^i_{k+1})$ into W_0 and add $((u_0, \ldots, u_k), (u_0, \ldots, u_k, u^i_{k+1}))$ to R_0. We define π_0 as $\pi_0((u_0, \ldots, u_k)) = \pi(u_k)$. Observe that $\mathcal{M}_0 = \langle W_0, R_0\rangle$ is a tree in which the degree of the nodes is bounded by $|\varphi|$.

Let $f : W_0 \to W$ be defined as $f((u_0, \ldots, u_k)) = u_k$. By a straightforward induction the reader may verify that, for every $\vec{u} \in W_0$ we have $tp_{\mathfrak{M}_0}(\vec{u}) = tp_{\mathfrak{M}}(f(\vec{u}))$. This implies that $\mathfrak{M}_0, (u_0) \models \varphi$.

Now, in countably many stages we add to $\mathcal{M}_0$ the edges implied by Φ. We define a sequence of frames $(\mathcal{M}_i)_{i>0}$ and models $(\mathfrak{M}_i)_{i>0}$ sharing the same universe W_0 and mapping π_0. For $K > 0$ let $\mathcal{M}_K = \langle W_0, \textsc{Cons}^K_{\Phi,W_0}(R_0)\rangle$, $\mathfrak{M}_K = \langle \mathcal{M}_K, \pi_0\rangle$. Let $\mathfrak{M}_\infty$ be the natural limit $\mathfrak{M}_\infty = \langle \mathfrak{C}_\Phi(\mathcal{M}_0), \pi_0\rangle$.

We show by induction over K, that for each $\vec{u}_1, \vec{u}_2 \in W_0$ if $\mathfrak{M}_K \models \vec{u}_1 R \vec{u}_2$, then $\mathfrak{M} \models f(\vec{u}_1) R f(\vec{u}_2)$. It follows that for each $\vec{u}_1, \vec{u}_2 \in W_0$ if $\mathfrak{M}_\infty \models \vec{u}_1 R \vec{u}_2$, then $\mathfrak{M} \models f(\vec{u}_1) R f(\vec{u}_2)$. For $K = 0$ the conclusion is a straightforward consequence of the definition of $\mathfrak{M}_0$. Assume that $\mathfrak{M}_K$ satisfies the inductive hypothesis. For each $\vec{u}_1, \vec{u}_2 \in W_0$, if $\mathfrak{M}_{K+1} \models \vec{u}_1 R \vec{u}_2$, then either $\mathfrak{M}_K \models \vec{u}_1 R \vec{u}_2$ and by the inductive assumption $\mathfrak{M} \models f(\vec{u}_1) R f(\vec{u}_2)$, or for some $\vec{u_3} \in W_0$ and $\Psi_1 \Rightarrow \Psi_2 \in \Phi$, we have $\mathfrak{M}_K \models \Psi_1(\vec{u_1}, \vec{u_2}, \vec{u_3})$, and $\Psi_2(\vec{u_1}, \vec{u_2}, \vec{u_3}) = \vec{u_1} R \vec{u_2}$. In this case, $\mathfrak{M}_K \models \Psi_1(\vec{u_1}, \vec{u_2}, \vec{u_3})$ implies by the inductive assumption that $\mathfrak{M} \models \Psi_1(f(\vec{u_1}), f(\vec{u_2}), f(\vec{u_3}))$. Since $\mathfrak{M} \models \Psi_1 \Rightarrow \Psi_2$, we have $\mathfrak{M} \models f(\vec{u}_1) R f(\vec{u}_2)$.

Let $\mathfrak{M}_\infty = \langle W_0, R_\infty, \pi_0\rangle$. The structures $\mathfrak{M}_0$ and $\mathfrak{M}_\infty$ have the same carrier and $R_0 \subseteq R_\infty$. We show that for each $\vec{u} \in W_0$ we have $tp_{\mathfrak{M}_\infty}(\vec{u}) = tp_{\mathfrak{M}_0}(\vec{u})$. It implies that

$\mathfrak{M}_\infty, (u_0) \models \varphi$. Since the labeling of the worlds is the same, it is enough to show that in $\mathfrak{M}_0$ and $\mathfrak{M}_\infty$ each world is connected with the worlds that satisfy the same subformulas. We show that by induction.

Clearly, for every edge $(\vec{u}, \vec{v})$ from $R_\infty \setminus R_0$ and a subformula $\Diamond\psi$ of φ, if a world $\vec{v}$ satisfies ψ in $\mathfrak{M}_\infty$, then by the inductive assumption we have that $\psi \in tp_{\mathfrak{M}_0}(\vec{v}) = tp_{\mathfrak{M}}(f(\vec{v}))$, and since $\mathfrak{M} \models f(\vec{u})Rf(\vec{v})$ we have that $\Diamond\psi \in tp_{\mathfrak{M}}(f(\vec{u})) = tp_{\mathfrak{M}_0}(\vec{u})$. See the full version of this paper for a detailed proof.

Finally, we have to prove that $\mathfrak{C}_\Phi(\mathcal{M}_0) \models \Phi$. By definition $\mathfrak{C}_\Phi(\mathcal{M}_0)$ satisfies every $\Psi_1 \Rightarrow \Psi_2 \in \Phi^p$. Suppose that $\mathfrak{C}_\Phi(\mathcal{M}_0)$ does not satisfy $\Psi \Rightarrow \bot \in \Phi$. For some $\vec{w_1}, \vec{w_2}, \vec{w_3}$ we have $\mathfrak{C}_\Phi(\mathcal{M}_0) \models \Psi(\vec{w_1}, \vec{w_2}, \vec{w_3})$, but then $\mathfrak{M} \models \Psi(f(\vec{w_1}), f(\vec{w_2}), f(\vec{w_3}))$. This contradicts the assumption that $\mathfrak{M} \models \Phi$. ◀

4.2 Catalogue of models

A well known result shows that every satisfiable modal formula is satisfied in a finite tree. This *tree-model property* is crucial for the robust decidability of modal logics. Standard restrictions of classes of frames lead to similar results, stating that some "nice" models exist for all satisfiable formulas. For example, every formula satisfiable over transitive structures has a model which is a transitive tree.

Here we generalize those results. We introduce four classes of models and show that for each formula Φ all formulas satisfiable over $\mathcal{K}_\Phi$ have models in one of those classes.

▶ **Definition 7.** We say that a graph $\langle W, R\rangle$ is

- a *semi-tree* if and only if there exists $R_0 \subseteq R$ such that $\langle W, R_0\rangle$ is a tree and R is contained in the reflexive, symmetric closure of R_0.
- a *transitive-tree* if and only if there exists $R_0 \subseteq R$ such that $\langle W, R_0\rangle$ is a tree, R is contained in the reflexive, transitive closure of R_0, and for each directed path $(u_0, u_1, \ldots, u_k)$ in $\langle W, R\rangle$ and each $2 \leq i \leq j \leq k-2$ we have an edge from u_i to u_j.
- a *tripartition* if and only if W can be partitioned into three independent sets I_1, I_2, I_3 such that for $d \in I_i$ and $e \in I_j$ we have that $dRe \iff j = i+1 \mod 3$.
- a *clique-union* if and only if W can be partitioned into $Head, Tails, C_1, \ldots, C_k$, where $C_1, \ldots, C_k$ are disjoint cliques, $Heads$ is a semi-tree of height at most 2, $Tails$ is a forest of semi-trees of height at most 2, and there are no edges from $C_1 \cup \ldots \cup C_k$ to $Head$ and from $Tails$ to $Head \cup C_1 \cup \ldots \cup C_k$.

▶ **Lemma 8.** *Let $\Phi \in \mathsf{UHF}^3$. One of the following conditions holds:*

- *For each tree $\mathcal{T}$, the structure $\mathfrak{C}_\Phi(\mathcal{T})$ is a semi-tree.*
- *For each tree $\mathcal{T}$, the structure $\mathfrak{C}_\Phi(\mathcal{T})$ is a transitive tree.*
- *For each tree $\mathcal{T}$, the structure $\mathfrak{C}_\Phi(\mathcal{T})$ is a tripartition.*
- *For each tree $\mathcal{T}$, the structure $\mathfrak{C}_\Phi(\mathcal{T})$ is a clique-union.*

This lemma, together with Lemma 6 imply that for every $\Phi \in \mathsf{UHF}^3$ modal logic has one of: *semi-tree model property, transitive tree model property, tripartite model property, clique-union model property* over $\mathcal{K}_\Phi$, i.e. every satisfiable formula has a model in one particular class. The proof of Lemma 8 starts from the analysis of the possible shapes of $\mathfrak{C}_\Phi(\mathcal{I})$, for the four-element tree $\mathcal{I}$ consisting of a root a, its two children b, d and a child c of b. It appears that studying what happens on this simple tree allows to see what can happen on arbitrary trees. The whole proof goes by a careful analysis of cases. Details are given in the full version of this paper. Here we only show some examples.

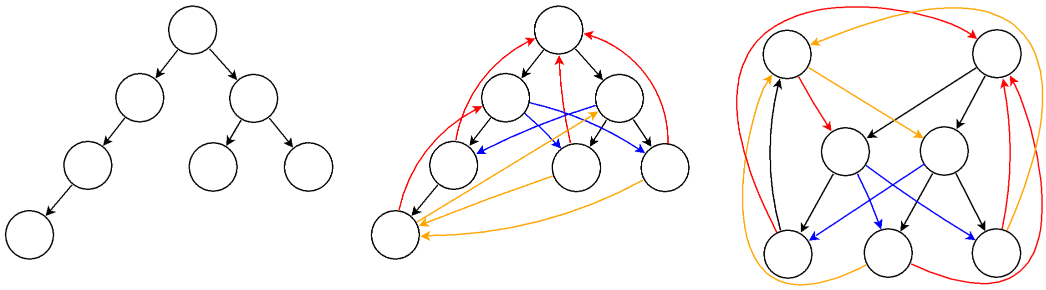

Figure 3 A closure for $\Phi = \{xRz \wedge zRy \Rightarrow yRx\}$ – three independent sets.

▶ **Example 9.** Consider the formula $\Phi = \{xRz \wedge zRy \Rightarrow yRx\}$ and the tree $\mathcal{T} = \langle W, R\rangle$ at the left side of Fig. 3. In the middle we present $\mathfrak{C}_\Phi(\mathcal{T})$ – red edges belong to $\textsc{Cons}_\Phi(R)$, blue to $\textsc{Cons}^2_\Phi(R)$, and yellow to $\textsc{Cons}^3_\Phi(R)$. Observe that each world from the level i of the tree is connected to all the worlds from the levels $i+1$ and $i-2$. On the right side of the figure we redraw the structure in a way underlining the partition into the three independent sets.

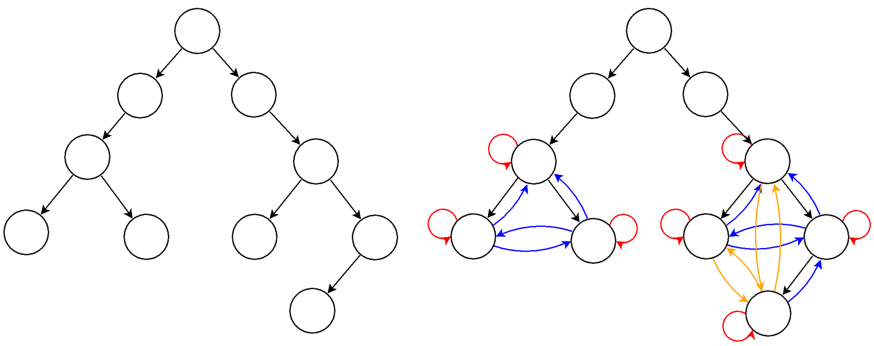

Figure 4 A closure for $\Phi = \{xRz \wedge zRy \Rightarrow yRy, xRx \wedge xRy \wedge xRz \Rightarrow yRz\}$ – a clique-union ($Tails = \emptyset$ in this example).

▶ **Example 10.** Consider the formula $\Phi = \{\varphi_1, \varphi_2\}$, where $\varphi_1 = xRz \wedge zRy \Rightarrow yRy$ and $\varphi_2 = xRx \wedge xRy \wedge xRz \Rightarrow yRz$, and the tree at the left side of Fig. 4. The formula φ_1 enforces the following property: each world that has a predecessor that has a predecessor is reflexive. The formula φ_2 makes the relation R Euclidean except for the non-reflexive worlds. As you can see at the right side of the figure, the fragment on which R is Euclidean collapses into a clique.

▶ **Example 11.** Consider the formula $\Phi = \{\varphi_1, \varphi_2\}$, where $\varphi_1 = xRy \wedge yRz \Rightarrow yRx$ and $\varphi_2 = xRy \wedge yRx \Rightarrow xRz$, and the tree at the left side of Fig. 5. The formula φ_1 enforces R to be symmetric, except for the edges that go to the worlds with no successors. The formula φ_2 enforces connections from each world symmetrically connected to some other world to all other worlds. As you can see at the right side of the figure, all worlds except for the leaves of the tree form a clique.

4.3 Decidability procedures and complexity

In this subsection we sketch procedures deciding satisfiability of modal logics over classes definable by UHF^3, and discuss the complexity. We exclude from our considerations formulas

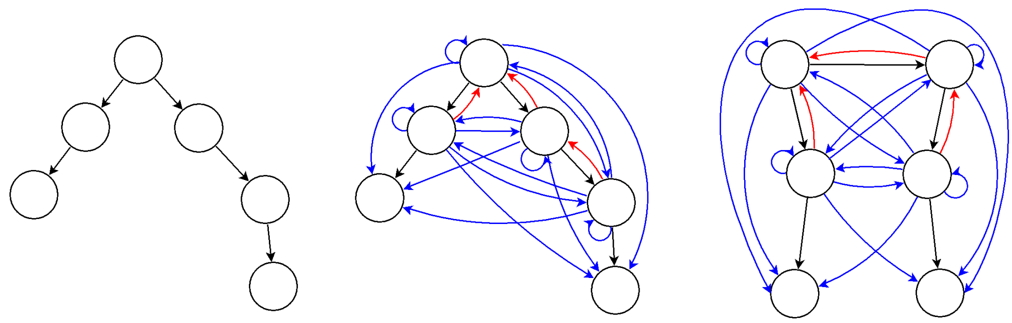

Figure 5 A closure for $\Phi = \{xRy \wedge yRz \Rightarrow yRx, xRy \wedge yRx \Rightarrow xRz\}$ – a clique with tails.

allowing only for paths of lenght bounded by a constant, e.g. $xRy \wedge yRz \to \perp$. Clearly, the satisfiability problem over classes of frames defined by such formulas is NP-complete.

Tripartitions and clique unions. It appears that in these two cases we can prove the following *polynomial model property.*

▶ **Lemma 12.** *For a given* UHF^3 *formula* Φ *and a modal formula* φ *if* φ *has a model in* $\mathcal{K}_\Phi$ *which is a tripartition or a clique union then it has a finite model of the same kind of size polynomially bounded by* $|\varphi|$.

Consider first the case of tripartitions. For every subformula $\Diamond\psi$ of φ, and every class of the partition I_i, if ψ is true at some elements of I_i, then we mark one such element. We also mark an element satisfying φ. We remove all unmarked elements. Since for a pair of classes of the partition they are either not connected or connected universally this procedure does not affect types of elements, so they still satisfy the same subformulas of φ.

The case of clique-unions is slightly more complicated. Recall that models from this class except cliques may also contain heads and tails, which cause that sometimes for a subformula $\Diamond\psi$ of φ we need more than one element satisfying ψ in a clique. However, the number of such elements may be bounded polynomially in $|\varphi|$. Similarly, we can also bound the number of cliques and tails. Technical details can be found in the full version of this paper.

In both cases the outlined arguments work for both local and global satisfiability. The decision procedure is to guess for a given formula φ a model of polynomial size and verify it. This establishes NP-upper bound. The matching lower bound follows from a trivial reduction from the boolean satisfiability problem.

Semi-trees. Here we can use standard approaches to satisfiability of modal logic over the class of all frames. In the case of local satisfiability we can bound the depth of tree-models and the degree of their nodes linearly in $|\varphi|$ and then check the existence of such models in a depth-first search manner in PSpace. (see e.g. [9][2]). The lower bound comes from the standard reduction of QBF (see also [9]).

In the case of global satisfiability we can enforce models of depth exponential with respect to the length of the formula. The existence of models can be checked by an alternating procedure which first guesses the type of the root, then guesses types of its children, and universally repeats the procedure for the children. This algorithm works in alternating polynomial space, and thus the problem is in ExpTime. A matching lower bound can be obtained as in [3].

[2] Please note that while the cited result does not consider reflexivity and symmetry, there are only some minor changes needed to cover these cases.

Table 1 Complexity of modal logics defined by consistent UHF^3 formulas.

A property implied by a formula	Global satisfiability	Local satisfiability
Polynomial model property	NP-c	NP-c
Semi-tree model property	ExpTime-c	PSpace-c
Transitive-tree model property	PSpace-c	PSpace-c

Transitive trees. This case can be treated similarly to the case of satisfiability over the class of transitive frames, i.e. the case of logic $\mathsf{K4}$ (see [9]). There are slight differences because in our case transitivity may fail at the last two elements of a path, however this detail does not cause real problems. We can also simply enforce infinite models (consider e.g. the class of irreflexive, transitive models and a modal formula $\top \wedge \Diamond\top \wedge \Box\Diamond\top$), so the length of paths cannot be bounded. However, we can bound polynomially the number of types on a path, which allows to show PSpace-completeness in both global and local cases.

Theorem 2 follows from the discussion above. The complexity results are summarized in Table 1.

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