A Category Theoretic View of Nondeterministic Recursive Program Schemes

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Abstract

Deterministic recursive program schemes (RPS's) have a clear category theoretic semantics presented by Ghani et al. and by Milius and Moss. Here we extend it to nondeterministic RPS's. We provide a category theoretic notion of guardedness and of solutions. Our main result is a description of the canonical greatest solution for every guarded nondeterministic RPS, thereby giving a category theoretic semantics for nondeterministic RPS's. We show how our notions and results are connected to classical work.

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1 Introduction

Deterministic and nondeterministic recursive program schemes (RPS's) were investigated in the 1970's and 80's by several authors (see related work below). Different semantics for RPS's were proposed and relationships between them were proved. More recently, a category theoretic semantics for deterministic RPS's has been developed by Ghani et al. [9] and by Milius and Moss [16]. There are clear advantages of this semantics: it applies to a considerably generalized notion of RPS and it requires less assumptions than classical semantics, which need order or metric structures.

However, no category theoretic semantics for nondeterministic RPS's has been presented so far. The present paper bridges this gap: it provides a category theoretic notion of nondeterministic RPS, of guardedness and of solutions. As our main result, a semantics of the guarded nondeterministic RPS's is given by proving them to have a canonical greatest solution.

Technically this turns out to be a challenging task: parts of the techniques known from [9, 16] are not available in the nondeterministic case. Thus large technical parts of our work reflect the effort it takes to adjust the category theoretic methods to the nondeterministic case. Nevertheless, this pays off in the end: besides obtaining a semantics for a generalized notion of a nondeterministic RPS, our approach has a clear structure and is easily related to classical semantics of nondeterministic RPS's as well as to the deterministic category theoretic semantics of [9, 16]. Moreover, several generalizations and extensions can be considered (see future work in Section 6).

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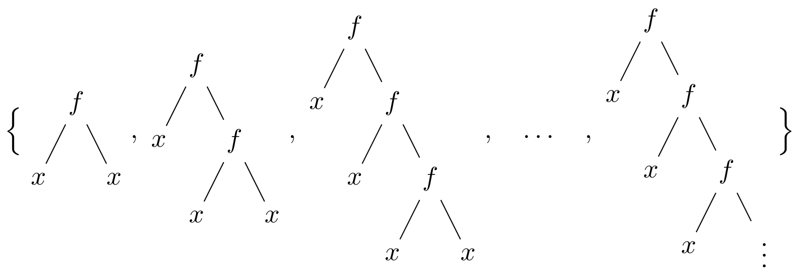
We illustrate the topic of this paper on the nondeterministic RPS

$$\phi(x) = f(x, x) \text{ or } f(x, \phi(x))\,. \tag{1}$$

Here ϕ is a new function symbol of arity one, f a given function symbol of arity two and x a variable.

A semantics of a nondeterministic RPS attaches to every new function symbol a set of possibly infinite trees with nodes labeled by given function symbols or variables which "solves" the system of equations. In our example this is the following infinite set



(2)

of trees where the right-hand tree is infinite. Substituting every element on the right-hand side of (1) and interpreting the operation or as nondeterministic choice we get back this set. But removing the infinite tree from this set also gives such a "solution"—unlike for deterministic RPS we must make a decision when giving a semantics to nondeterministic RPS's. In our main result we prove that there always is a greatest solution that can be chosen canonically.

Structure of the Paper

Section 2 provides several notions and results needed in this paper. In Section 3 we show how to obtain a canonical distributive law of the free completely iterative monad T^H on a Set-functor H over the nonempty powerset monad $\mathcal{P}^+$. It is used in Section 4 to prove that $\mathcal{P}^+T^H$ is a weakly completely iterative monad; moreover, in this section we consider a functor $\bar{\mathcal{H}}$ on a certain Kleisli category derived from H, and show an $\bar{\mathcal{H}}$-coalgebra to be weakly final. In Section 5 we give a category theoretic notion of a nondeterministic RPS and prove our main result, namely that every guarded nondeterministic RPS has a canonical greatest solution, using the technical results from the previous section. We compare our work with an existing category theoretic notion of a deterministic RPS and with classical work on nondeterministic RPS's. Finally we give a brief summary and discuss several directions for future work in Section 6.

Nearly all proofs are omitted; they can be found in the full version on the author's web page[1].

Related Work

Different semantics of RPS's have been investigated in the 1970's and 80's: for deterministic RPS's see for example Courcelle [8], Guessarian [10] and Nivat [19]; for nondeterministic RPS's we mention Boudol [7], Arnold and Nivat [4] and Poigné [20]. In Section 5 we compare our work in particular with [4] to see how we cover the classical definitions and results. A

[1] http://www.tu-braunschweig.de/iti/mitarbeiter/ehemalige/schwencke

category theoretic approach to deterministic RPS's is given by Ghani, Lüth and de Marchi [9] and by Milius and Moss [16]. We make use of several techniques from [16]. This paper is also loosely related to our previous work [17] where we used distributive laws of the same kind to bring recursion (but on the level of equations for variables) together with effects like non-determinism.

2 Preliminaries

We assume that the reader is familiar with the basic notions of category theory such as category, functor, natural transformation and commutative diagram; we shall also need coproducts. Moreover we assume basic knowledge about algebras and coalgebras for a functor; in particular we use free algebras as well as (weakly) final coalgebras.

Monads and Distributive Laws

▶ **Definition 2.1.** A *monad* (M, η, μ) on a category $\mathcal{A}$ is an endofunctor $M : \mathcal{A} \to \mathcal{A}$ together with natural transformations $\eta : \mathrm{Id} \to M$ (called the *unit* of the monad) and $\mu : MM \to M$ (called the *multiplication* of the monad) such that the *unit laws* $\mu \cdot \eta M = \mu \cdot M\eta = \mathrm{id}$ and the *multiplication law* $\mu \cdot M\mu = \mu \cdot \mu M$ hold.
A *monad morphism* between monads (M, η^M, μ^M) and (N, η^N, μ^N) on $\mathcal{A}$ is a natural transformation $\theta : M \to N$ such that $\theta \cdot \eta^M = \eta^N$ and $\theta \cdot \mu^M = \mu^N \cdot N\theta \cdot \theta M$.

▶ **Example 2.2.** The most important example of a monad in this paper is the *nonempty powerset monad* $(\mathcal{P}^+, \eta^+, \mu^+)$ on the category Set of sets and functions:

- the functor $\mathcal{P}^+ : \mathsf{Set} \to \mathsf{Set}$ assigns to a set X the set of all nonempty subsets of X; on maps $f : X \to Y$ it is defined by $(\mathcal{P}^+ f)(X') = f[X']$ where $X' \in \mathcal{P}^+ X$;
- the X-component of the unit $\eta^+ : \mathrm{Id} \to \mathcal{P}^+$ assigns to an element $x \in X$ the singleton set $\{x\} \in \mathcal{P}^+ X$;
- the X-component of the multiplication $\mu^+ : \mathcal{P}^+\mathcal{P}^+ \to \mathcal{P}^+$ performs the union of subsets of X.

▶ **Definition 2.3.** A free monad on an endofunctor H on a category $\mathcal{A}$ is a monad (F^H, η^H, μ^H) together with a natural transformation $\kappa^H : H \to F^H$ such that for every monad (M, η^M, μ^M) on $\mathcal{A}$ together with a natural transformation $\alpha : H \to M$ there exists a unique monad morphism $\alpha^\# : F^H \to M$ such that $\alpha^\# \cdot \kappa^H = \alpha$.

▶ **Theorem 2.4** ([5]). *If for every object X of $\mathcal{A}$ the free H-algebras $\phi^H_X : HF^H X \to F^H X$ on X exist, the free monad on H is given objectwise by these algebras, and the free algebra maps form a natural transformation ϕ^H such that $\mu^H \cdot \phi^H F^H = \phi^H \cdot H\mu^H$ and $\phi^H = \mu^H \cdot \kappa^H F^H$.*

▶ **Definition 2.5.** The *Kleisli category* $\mathcal{A}_M$ of a monad (M, η, μ) on a category $\mathcal{A}$ is given as follows:

- the objects of $\mathcal{A}_M$ are the same objects as the ones of $\mathcal{A}$;
- the morphisms of $\mathcal{A}_M$ between X and Y are all morphisms $X \to MY$ from $\mathcal{A}$;
- the identity morphism on X is $\eta_X : X \to MX$;
- composition of $f : X \to MY$ and $g : Y \to MZ$ is given by

$$X \xrightarrow{f} MY \xrightarrow{Mg} MMZ \xrightarrow{\mu_Z} MZ\,.$$

Furthermore, there is a canonical inclusion functor $J : \mathcal{A} \to \mathcal{A}_M$ given as the identity on objects and by $Jf = \eta_Y \cdot f : X \to MY$ on morphisms $f : X \to Y$.

▶ **Definition 2.6.** A *distributive law of a functor H over a monad* (M, η^M, μ^M) is a natural transformation $\lambda : HM \to MH$ such that $\lambda \cdot H\eta^M = \eta^M H$ and $\lambda \cdot H\mu^M = \mu^M H \cdot M\lambda \cdot \lambda M$. A *distributive law of monads N and M* is a natural transformation $\lambda : NM \to MN$ such that in addition to the two laws for a distributive law of a functor over a monad (with H replaced by N) the laws $\lambda \cdot \eta^N M = M\eta^N$ and $\lambda \cdot \mu^N M = M\mu^N \cdot \lambda N \cdot N\lambda$ hold.

Throughout the paper, we denote parallel composition $G\alpha' \cdot \alpha F' = \alpha G' \cdot F\alpha' : FF' \to GG'$ of natural transformations $\alpha : F \to G$ and $\alpha' : F' \to G'$ by $\alpha * \alpha'$.

▶ **Lemma 2.7** ([6]). *Given a distributive law $\lambda : NM \to MN$ of monads, $(MN, \eta^M N \cdot \eta^N, (\mu^M * \mu^N) \cdot M\lambda N)$ is again a (composite) monad.*

Completely Iterative Algebras and Complete Elgot Algebras

Now (and for the rest of the paper) assume the category $\mathcal{A}$ to have binary coproducts, and let $H : \mathcal{A} \to \mathcal{A}$ be a functor. We denote coproduct injections by $\mathsf{inl} : X \to X + Y$ and $\mathsf{inr} : Y \to X + Y$ and use the notation can for the canonical morphism $[H\mathsf{inl}, H\mathsf{inr}] : HX + HY \to H(X + Y)$.

▶ **Definition 2.8.** A *flat equation morphism* in an object A (of parameters) is a morphism $e : X \to HX + A$.
A *solution* of e in an H-algebra $a : HA \to A$ is a morphism $e^\dagger : X \to A$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ HX + A \arrow[r, "He^\dagger + A"'] & HA + A \arrow[u, "{[a, A]}"'] \end{tikzcd}$$

commutes.
A *completely iterative H-algebra* is an H-algebra $a : HA \to A$ in which every flat equation morphism has a unique solution.
A *complete Elgot algebra for H* is an H-algebra $a : HA \to A$ together with a function $(-)^\dagger$ assigning to each flat equation morphism e a solution $e^\dagger$ in a such that $(-)^\dagger$ is functorial and compositional (see Definition 2.10 below).

▶ **Notation 2.9.** Let $e : X \to HX + Y$ and $g : Y \to HY + A$ be flat equation morphisms and let $f : Y \to Z$ be any morphism. We denote by $f \bullet e$ the flat equation morphism $(HX + f) \cdot e : X \to HX + Z$, and we denote by $g \blacksquare e$ the flat equation morphism $(\mathsf{can} + A) \cdot (HX + g) \cdot [e, \mathsf{inr}] : X + Y \to H(X + Y) + A$.

▶ **Definition 2.10.** A function $(-)^\dagger$ assigning to each flat equation morphism e a solution $e^\dagger$ in an algebra $a : HA \to A$ is called *functorial* if for every homomorphism $h : X \to Y$ between flat equation morphisms $e : X \to HX + A$ and $g : Y \to HY + A$ (i. e. $(Hh + A) \cdot e = g \cdot h$) we have $e^\dagger = g^\dagger \cdot h$. This is, $(-)^\dagger$ is a functor between the category of all flat equation morphisms in A and their homomorphisms and the comma category of A. $(-)^\dagger$ is called *compositional* if for any equation morphisms $e : X \to HX + Y$ and $g : Y \to HY + A$ we have $(g \blacksquare e)^\dagger \cdot \mathsf{inl} = (g^\dagger \bullet e)^\dagger$.
A morphism $h : A \to B$ between complete Elgot algebras $(a : HA \to A, (-)^\dagger)$ and $(b : HB \to B, (-)^\ddagger)$ is called *solution preserving* if for all flat equation morphisms $e : X \to HX + A$ the equation $h \cdot e^\dagger = (h \bullet e)^\ddagger$ holds.

All complete Elgot algebras for H and solution preserving H-algebra homomorphisms between them form a category. It follows from [15, 2] that all completely iterative H-algebras and H-algebra homomorphisms between them form a full subcategory.

▶ **Theorem 2.11** ([15, 2]). *The following are equivalent:*

1. $\tau_X : HT^H X \to T^H X$ *is the free completely iterative H-algebra on X with universal arrow* $\eta_X : X \to T^H X$*;*
2. $\tau_X : HT^H X \to T^H X$ *is the free complete Elgot algebra for H on X with universal arrow* $\eta_X : X \to T^H X$*;*
3. $[\tau_X, \eta_X]^{-1} : T^H X \to HT^H X + X$ *is the final $H(-) + X$-coalgebra.*

Completely Iterative Monads

▶ **Definition 2.12.** Let (T, η, μ) be a monad on $\mathcal{A}$. A *T-module* (F, ν) is an endofunctor $F : \mathcal{A} \to \mathcal{A}$ together with a natural transformation $\nu : FT \to F$ such that the following diagrams commute:

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, equal] & FT \arrow[d, "\nu"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} FTT \arrow[r, "F\mu"] \arrow[d, "\nu T"'] & FT \arrow[d, "\nu"] \\ FT \arrow[r, "\nu"'] & F \end{tikzcd}$$

A *module homomorphism* between T-modules (F, ν^F) and (G, ν^G) is a natural transformation $\vartheta : F \to G$ such that $\vartheta \cdot \nu^F = \nu^G \cdot \vartheta T$.

▶ Remark 2.13. For every monad (T, η, μ), (T, μ) is a T-module.

▶ **Definition 2.14.** An *idealized monad* $(T, \eta, \mu, \bar{T}, \bar{\mu}, \vartheta)$ on $\mathcal{A}$ is a monad (T, η, μ) on $\mathcal{A}$ together with a T-module $(\bar{T}, \bar{\mu})$ and a module homomorphism $\vartheta : (\bar{T}, \bar{\mu}) \to (T, \mu)$.
An *ideal natural transformation* is a natural transformation $\alpha : F \to T$ into an idealized monad which factors

$$\alpha \equiv (\, F \xrightarrow{\bar{\alpha}} \bar{T} \xrightarrow{\vartheta} T \,)\,.$$

An *idealized monad morphism* $(\theta, \bar{\theta})$ between idealized monads $(T, \eta^T, \mu^T, \bar{T}, \bar{\mu}^T, \vartheta^T)$ and $(S, \eta^S, \mu^S, \bar{S}, \bar{\mu}^S, \vartheta^S)$ is a monad morphism $\theta : T \to S$ together with a natural transformation $\bar{\theta} : \bar{T} \to \bar{S}$ such that the following diagrams commute:

$$\begin{tikzcd} \bar{T}T \arrow[r, "\bar{\theta} * \theta"] \arrow[d, "\bar{\mu}^T"'] & \bar{S}S \arrow[d, "\bar{\mu}^S"] \\ \bar{T} \arrow[r, "\bar{\theta}"'] & \bar{S} \end{tikzcd} \qquad \begin{tikzcd} \bar{T} \arrow[r, "\bar{\theta}"] \arrow[d, "\vartheta^T"'] & \bar{S} \arrow[d, "\vartheta^S"] \\ T \arrow[r, "\theta"'] & S \end{tikzcd}$$

▶ Remark 2.15. Every monad (T, η, μ) can be canonically completed to an idealized monad $(T, \eta, \mu, T, \mu, \mathrm{id})$. In general, there are other ways to complete T to an idealized monad as we shall see in Theorem 2.17 below.

▶ **Definition 2.16.** Let $(T, \eta, \mu, \bar{T}, \bar{\mu}, \vartheta)$ be an idealized monad. An *equation morphism* is a morphism $e : X \to T(X + Y)$. It is called *guarded* if it factors

$$e \equiv (\, X \xrightarrow{e'} \bar{T}(X + Y) + Y \xrightarrow{[\vartheta_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}]} T(X + Y)\,)$$

for some e'. A *solution* of e is a morphism $e^\dagger : X \to TY$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & TY \\
T(X+Y) \arrow[r, "T{[e^\dagger, \eta_Y]}"'] & TTY \arrow[u, "\mu_Y"']
\end{tikzcd}$$

commutes. An idealized monad is called *completely iterative* if every guarded equation morphism has a unique solution. It is called *weakly completely iterative* if every guarded equation morphism has a solution.

All idealized monads on $\mathcal{A}$ together with the idealized monad morphisms form a category. In particular, we are interested in the free completely iterative monads on functors $H : \mathcal{A} \to \mathcal{A}$, albeit not in their freeness property.

▶ **Theorem 2.17** ([15]). *Let $H : \mathcal{A} \to \mathcal{A}$ be a functor such that for every object X of $\mathcal{A}$ the final $H(-) + X$-coalgebra exists. The free completely iterative monad on H is given by $(T^H, \eta^H, \mu^H, HT^H, H\mu^H, \tau^H)$ with universal ideal natural transformation $\kappa^H : H \to T^H$ where*

- *T^H is defined on objects X as the free completely iterative H-algebra $T^H X$ on X and on morphisms $f : X \to Y$ as the unique homomorphism between the free completely iterative H-algebras on X and Y extending $\eta_Y \cdot f : X \to T^H Y$;*
- *η^H_X is given by the universal arrow of the free completely iterative H-algebra on X;*
- *μ^H_X is given as the unique homomorphism between the free completely iterative H-algebras on $T^H X$ and on X extending $\mathrm{id}_{T^H X}$;*
- *τ^H_X is given by the structure of the free completely iterative H-algebra on X; and*
- *κ^H_X is given by $\tau^H_X \cdot H\eta^H_X$.*

▶ Remark 2.18. By the definition of μ in Theorem 2.17 we have $\mu^H_X \cdot \tau^H_{T^H X} = \tau^H_X \cdot H\mu^H_X$ for every X, and from the same theorem we know that μ^H and τ^H are natural transformations. Consequently it holds $\mu^H \cdot \tau^H T^H = \tau^H \cdot H\mu^H$.

▶ **Lemma 2.19.** *It holds $\tau^H = \mu^H \cdot \kappa^H T^H$.*

Proof. Consider the diagram

$$\begin{tikzcd}
HT^H \arrow[r, "\kappa^H T^H"] \arrow[dr, "H\eta^H T^H"'] \arrow[drr, bend right, equal] & T^H T^H \arrow[r, "\mu^H"] & T^H \\
& HT^H T^H \arrow[u, "\tau^H T^H"'] \arrow[r, "H\mu^H"] & HT^H \arrow[u, "\tau^H"']
\end{tikzcd}$$

The triangle is the definition of κ, the lower part is one of the monad unit laws, and for the right-hand square see Remark 2.18. Thus the desired outer square commutes. ◀

3 Canonical Distributive Laws over $\mathcal{P}^+$

In this section we provide canonical distributive laws of polynomial Set-functors H and the corresponding free completely iterative monads T^H on H over the nonempty powerset monad $\mathcal{P}+$ and prove some properties of them. These distributive laws are an integral part of our category theoretic approach to nondeterministic computations since they formalize the idea of non-determinism that all possible choices are considered.

▶ **Definition 3.1.** A *polynomial* Set*-functor* H is a Set-functor of the form $HX = \coprod_{\sigma\in\Sigma} X^{n_\sigma}$, where Σ is a signature of (possibly infinitely many) operation symbols σ with (finite) arities n_σ. We write H_Σ for the polynomial Set-functor associated with the signature Σ; elements from $H_\Sigma X$ are denoted by $\sigma(x_1, \dots, x_n)$ where $\sigma \in \Sigma$ and $x_1, \dots, x_n \in X$.

▶ **Lemma 3.2** ([11]). *There exist canonical distributive laws $\lambda : HM \to MH$ of every polynomial* Set*-functor H over every commutative monad M on* Set.

▶ Remarks 3.3. **1.** We do not state the definition of a commutative monad here, but only mention that $\mathcal{P}^+$ is commutative which is sufficient for our purposes. For more details, see Kock's papers [13, 14].

2. Our work in [17] extends Lemma 3.2 to the wider class of analytic functors; for $\mathcal{P}^+$ there even exist canonical distributive laws $\lambda : H\mathcal{P}^+ \to \mathcal{P}^+H$ for every weak pullback preserving functor H, see e. g. [12].

▶ **Example 3.4.** For every polynomial functor H_Σ, the canonical distributive law $\lambda : H_\Sigma\mathcal{P}^+ \to \mathcal{P}^+H_\Sigma$ is given by $\lambda_X(\sigma(X_1, \dots, X_n)) = \{\sigma(x_1, \dots, x_n) \mid x_i \in X_i, 1 \le i \le n\}$ for every n-ary operation symbol $\sigma \in \Sigma$ and $X_1, \dots, X_n \in \mathcal{P}^+X$.

If H is a polynomial Set-functor, so is $H(-) + X$ for every set X. The final coalgebra of $H(-) + X$ is carried by the set T^HX of all finite and infinite trees with nodes labeled by operation symbols from the signature corresponding to H or by constant elements from X, where the number of children is given by the arity of the operation symbols labeling a node (see [1], Example 2.7). Whenever trees are mentioned in this paper, such trees are meant. We shall also refer to the elements of T^HX as finite and infinite terms built from operation symbols from the signature corresponding to H over variables from X.

Since for a polynomial Set-functor H final coalgebras $H(-) + X$ exist for every set X, the free completely iterative monad $(T^H, \eta^H, \mu^H, HT^H, H\mu^H, \tau^H)$ on H together with the universal natural transformation κ^H exists and is given as in Theorem 2.17. Explicitly, for a polynomial Set-functor H the natural transformations involved act as follows:

- $\eta^H_X : X \to T^HX$ considers a variable as a singleton tree;
- $\mu^H_X : T^HT^HX \to T^HX$ considers a tree with leaves labeled by trees with leaves labeled by variables from X as a tree with leaves labeled by variables from X by using the leaf labels as subtrees;
- $\tau^H_X : HT^HX \to T^HX$ acts similar as μ^H_X but for a flat tree (i. e. one of depth one) with leaves labeled by trees (of arbitrary depth); and
- $\kappa^H_X : HX \to T^HX$ considers a flat tree as a tree.

We shall leave out the superscript H when the functor H is clear from the context.

For the following proposition recall Definition 2.8 of a complete Elgot algebra.

▶ **Proposition 3.5.** *For the canonical distributive law λ of a polynomial* Set*-functor H over the monad $\mathcal{P}^+$, $\mathcal{P}^+\tau_Y \cdot \lambda_{T^HY} : H\mathcal{P}^+T^HY \to \mathcal{P}^+T^HY$ is a complete Elgot algebra for H for every set Y.*

Proof (sketch). For every set Y we define a function $(-)^\dagger$ which assigns to each flat equation morphism $e : X \to HX + \mathcal{P}^+T^HY$ a morphism $e^\dagger : X \to \mathcal{P}^+T^HY$. For example, let e be given by the system

$$x_0 = \sigma(x_0, x_1)$$
$$x_1 = \{t_0, t_1\}$$

of equations where σ is an operation symbol from the signature associated with the polynomial functor H, $x_0, x_1 \in X$ and $t_0, t_1 \in T^H Y$. Then we let $e^\dagger(x_0)$ consist of the unique solutions in the free completely iterative H-algebra τ_Y on Y of all variables of all flat equation morphisms $\bar{e} : \bar{X} \to H\bar{X} + T^H Y$ which are "over" x_0: for example $\bar{x}_0$ and $\bar{x}_1$ in the system

$$\begin{aligned}
\bar{x}_0 &= \sigma(\bar{x}_1, \bar{x}_2)\\
\bar{x}_1 &= \sigma(\bar{x}_0, \bar{x}_3)\\
\bar{x}_2 &= t_0\\
\bar{x}_3 &= t_1
\end{aligned}$$

are "over" x_0 since there is a function $\bar{X} \to X$ mapping $\bar{x}_0$ and $\bar{x}_1$ to x_0 (and $\bar{x}_2$ and $\bar{x}_3$ to x_1) which is homomorphic for equations with right-hand sides from HX and otherwise relates variables whose right-hand sides are in the containment relation $\in$. Similarly we define $e^\dagger(x_1)$ which is easily seen to be $\{t_0, t_1\}$. Then $e^\dagger$ can be shown to be a greatest solution of e w.r.t. to subset inclusion on $\mathcal{P}^+ T^H Y$. Equivalently, $e^\dagger$ can be obtained as a greatest fixed point of an operator corresponding to the solution diagram in Definition 2.8 with $(A, a) = (\mathcal{P}^+ T^H Y, \mathcal{P}^+ \tau_Y \cdot \lambda_{T^H Y})$. This enables us to use the dual of the proof of Proposition 3.5 from [2] in order to show that $(-)^\dagger$ is functorial and compositional which concludes the current proof. ◀

▶ **Definition 3.6.** Given the canonical distributive law λ of a polynomial Set-functor H over the monad $\mathcal{P}^+$, we define for every set Y the map $\lambda'_Y : T^H \mathcal{P}^+ Y \to \mathcal{P}^+ T^H Y$ to be the unique homomorphism between the free complete Elgot algebra $\tau_{\mathcal{P}^+ Y}$ on $\mathcal{P}^+ Y$ and the complete Elgot algebra $\mathcal{P}^+ \tau_Y \cdot \lambda_{T^H Y}$ (see Proposition 3.5) extending $\mathcal{P}^+ \eta^H_Y$, i.e. λ'_Y is uniquely determined by the following commutative diagrams:

$$\begin{tikzcd}
HT^H\mathcal{P}^+Y \arrow[rr, "H\lambda'_Y"] \arrow[dd, "\tau_{\mathcal{P}^+Y}"'] & & H\mathcal{P}^+T^HY \arrow[d, "\lambda_{T^HY}"] \\
 & & \mathcal{P}^+HT^HY \arrow[d, "\mathcal{P}^+\tau_Y"] \\
T^H\mathcal{P}^+Y \arrow[rr, "\lambda'_Y"] & & \mathcal{P}^+T^HY \\
 & \mathcal{P}^+Y \arrow[ul, "\eta^H_{\mathcal{P}^+Y}"] \arrow[ur, "\mathcal{P}^+\eta^H_Y"'] &
\end{tikzcd} \tag{3}$$

▶ **Lemma 3.7.** *The maps $\lambda'_Y : T^H \mathcal{P}^+ Y \to \mathcal{P}^+ T^H Y$ from Definition 3.6 act as follows: given a tree $t \in T^H \mathcal{P}^+ Y$ where leaves may be labeled with nonempty subsets of Y, $\lambda'_Y(t)$ is the set of all trees obtained by choosing in each of these leaves one element from the labeling set.*

▶ **Proposition 3.8.** *The canonical distributive law $\lambda : H\mathcal{P}^+ \to \mathcal{P}^+ H$ of a polynomial* Set*-functor H over the monad $\mathcal{P}^+$ extends to a distributive law $\lambda' : T^H \mathcal{P}^+ \to \mathcal{P}^+ T^H$ of monads.*

Proof (sketch). We prove that, given the canonical distributive law λ of a polynomial Set-functor H over the monad $\mathcal{P}^+$, the maps λ'_Y from Definition 3.6 form a distributive law of monads. Since one of the axioms for a distributive law of monads is already given by the lower triangle in diagram (3), we need to prove naturality of λ' and the three remaining axioms. The proof uses freeness of the complete Elgot algebras $\tau_Y : HT^H Y \to T^H Y$ (see Theorem 2.11) for naturality and one of the remaining axioms, and the concrete description of λ' from Lemma 3.7 for the remaining two axioms. ◀

▶ **Lemma 3.9.** *For a distributive law λ' obtained from λ according to Proposition 3.8 we have $\lambda' \cdot \kappa\mathcal{P}^+ = \mathcal{P}^+\kappa \cdot \lambda$.*

Proof. The lemma is an easy consequence of the definitions of κ (Theorem 2.17) and λ' (diagram (3)) and therefore left to the reader. ◀

4 A Weakly Final Coalgebra

Milius and Moss ([16], Theorem 6.5) proved guarded deterministic RPS's to have unique solutions by exploiting the finality of the coalgebra $[\tau^H, \eta^H]^{-1}$ for some functor $\mathcal{H}$. As we have seen in the introduction, in the nondeterministic case solutions need not be unique. However, as our main result we shall provide in Section 5 canonical greatest solutions of nondeterministic RPS's. There we exploit weak finality of the coalgebra $J[\tau^H, \eta^H]^{-1}$ for a lifting $\bar{\mathcal{H}}$ of $\mathcal{H}$ to a suitable Kleisli category with inclusion functor J, which is proved in the present section.

▶ **Definition 4.1.** Given a distributive law $\delta : NM \to MN$ of monads, a *δ-distributive law* of an N-module $(\bar{N}, \bar{\mu}^N)$ over the monad M is a natural transformation $\bar{\delta} : \bar{N}M \to M\bar{N}$ such that the first two laws from Definition 2.6 (with H replaced by $\bar{N}$ and λ replaced by $\bar{\delta}$) and the law $\bar{\delta} \cdot \bar{\mu}^N M = M\bar{\mu}^N \cdot \bar{\delta}N \cdot \bar{N}\delta$ hold.

▶ **Lemma 4.2.** *Let $\delta : NM \to MN$ be a distributive law of the idealized monad $(N, \eta^N, \mu^N, \bar{N}, \bar{\mu}^N, \vartheta)$ over the monad (M, η^M, μ^M), and let $\bar{\delta} : \bar{N}M \to M\bar{N}$ be a δ-distributive law such that $M\vartheta \cdot \bar{\delta} = \delta \cdot \vartheta M$. Then the composite monad induced by δ is an idealized monad $(MN, \eta^M N \cdot \eta^N, (\mu^M * \mu^N) \cdot M\delta N, M\bar{N}, (\mu^M * \bar{\mu}^N) \cdot M\bar{\delta}N, M\vartheta)$.*

Specializing to $M = \mathcal{P}^+$ and $N = T^H$, we can now prove the following

▶ **Theorem 4.3.** *Let H be a polynomial* Set*-functor. For the extension $\lambda' : T^H\mathcal{P}^+ \to \mathcal{P}^+T^H$ of the canonical distributive law $\lambda : H\mathcal{P}^+ \to \mathcal{P}^+H$ (cf. Proposition 3.8),*

$$(\mathcal{P}^+T^H, \eta^+T^H \cdot \eta^H, (\mu^+ * \mu^H) \cdot \mathcal{P}^+\lambda' T^H, \mathcal{P}^+HT^H, (\mu^+ * H\mu^H) \cdot \mathcal{P}^+\lambda T^H T^H \cdot \mathcal{P}^+H\lambda' T^H, \mathcal{P}^+\tau)$$

is a weakly completely iterative monad (see Definition 2.16).

Proof (sketch). We know from Theorem 2.17 that $(T^H, \eta^H, \mu^H, HT^H, H\mu^H, \tau)$ is the free completely iterative monad on H, i.e. in particular, it is an idealized monad. Moreover, $\lambda T^H \cdot H\lambda' : HT^H\mathcal{P}^+ \to \mathcal{P}^+HT^H$ is easily seen to be a λ'-distributive law such that $\mathcal{P}^+\tau \cdot \lambda T^H \cdot H\lambda' = \lambda' \cdot \tau\mathcal{P}^+$. Thus we can apply Lemma 4.2 to see that the six-tuple in the statement of the theorem is an idealized monad. We still have to check that every guarded equation morphism has a solution. This is done by deriving deterministic guarded equation morphisms from the given nondeterministic one (similar to the proof of Proposition 3.5) and showing that the (unique) solutions of the former constitute a solution of the latter. ◀

▶ Remark 4.4. The part of the proof of Theorem 4.3 showing that all guarded equation morphisms have a solution even works for (non-guarded) equation morphisms $e : X \to \mathcal{P}^+T^H(X+Y)$ that factor

$$e = (X \xrightarrow{e'} \mathcal{P}^+(HX+Y) \xrightarrow{\mathcal{P}^+(\kappa^H_X + \eta^H_Y)} \mathcal{P}^+(T^HX + T^HY) \xrightarrow{\mathcal{P}^+\mathsf{can}} \mathcal{P}^+T^H(X+Y)).$$

The reason is that although the equation morphism e is not necessarily guarded, the derived deterministic equation morphisms always are; the rest of the proof remains the same.

Observe that for every set Y, $\mathcal{P}^+Y$ carries the partial order $\subseteq$ given by subset inclusion. This extends elementwise to a partial order $\leq$ on all sets $\mathsf{Set}(X, \mathcal{P}^+Y)$ of functions from some set X into $\mathcal{P}^+Y$, i. e. $f \leq g \Leftrightarrow \forall x \in X : f(x) \subseteq g(x)$ for functions $f, g \in \mathsf{Set}(X, \mathcal{P}^+Y)$. In this sense we use the term "greatest solution/homomorphism" in the following lemma, in Lemma 4.11 below and in Section 5.

▶ **Lemma 4.5.** *The canonical solutions $e^\dagger$ of (guarded) equation morphisms e from the proof of Theorem 4.3 and Remark 4.4 are greatest solutions; moreover, for all solutions s of a (guarded) equation morphism e the sets of all finite cuttings of the trees from $e^\dagger(x)$ and $s(x)$ are the same for every $x \in X$.*

Let us denote by $[\mathcal{A}, \mathcal{A}]$ the category of all $\mathcal{A}$-endofunctors and natural transformations between them. Any functor $H : \mathcal{A} \to \mathcal{A}$ gives rise to a functor $\mathcal{H} : [\mathcal{A}, \mathcal{A}] \to [\mathcal{A}, \mathcal{A}]$ defined on objects (i. e. functors F) and morphisms (i. e. natural transformations $\alpha : F \to G$) by $\mathcal{H}F = HF + \mathrm{Id}$ and $\mathcal{H}\alpha = H\alpha + \mathrm{id}$.
And any monad (M, η^M, μ^M) on $\mathcal{A}$ gives rise to a monad $(\mathcal{M}, \eta^{\mathcal{M}}, \mu^{\mathcal{M}})$ on $[\mathcal{A}, \mathcal{A}]$ as follows: the functor $\mathcal{M}$ is defined by $\mathcal{M}F = MF$ and $\mathcal{M}\alpha = M\alpha$, and the F-components of unit and multiplication are given by $\eta^{\mathcal{M}}_F = \eta^M F$ and $\mu^{\mathcal{M}}_F = \mu^M F$. The monad laws follow straight from the ones for (M, η^M, μ^M).

▶ **Lemma 4.6.** *Any distributive law λ of a functor H over a monad M on $\mathcal{A}$ induces a distributive law Λ of the functor $\mathcal{H}$ over the monad $\mathcal{M}$ on $[\mathcal{A}, \mathcal{A}]$.*

Proof. For every object from $[\mathcal{A}, \mathcal{A}]$ (i. e. every functor $F : \mathcal{A} \to \mathcal{A}$) we define $\Lambda_F = \mathsf{can} \cdot (\lambda F + \eta^M)$. Naturality of Λ is proved by the commutative diagram

$$\begin{tikzcd}
\mathcal{H}\mathcal{M}F = HMF + \mathrm{Id} \arrow[r, "\lambda F + \eta^M"] \arrow[rr, bend left, "\Lambda_F"] \arrow[d, "\mathcal{H}\mathcal{M}\alpha = HM\alpha + \mathrm{id}"'] & MHF + M \arrow[r, "\mathsf{can}"] \arrow[d, "MH\alpha + M\mathrm{id}"] & M(HF + \mathrm{Id}) = \mathcal{M}\mathcal{H}F \arrow[d, "M(H\alpha + \mathrm{id}) = \mathcal{M}\mathcal{H}\alpha"] \\
\mathcal{H}\mathcal{M}G = HMG + \mathrm{Id} \arrow[r, "\lambda G + \eta^M"'] \arrow[rr, bend right, "\Lambda_G"'] & MHG + M \arrow[r, "\mathsf{can}"'] & M(HG + \mathrm{Id}) = \mathcal{M}\mathcal{H}G
\end{tikzcd}$$

for every morphism from $[\mathcal{A}, \mathcal{A}]$ (i. e. every natural transformation $\alpha : F \to G$ from $\mathcal{A}$): the left-hand part commutes by naturality of λ, and the right-hand part by naturality of can. The two axioms for Λ are easily checked componentwise for every object from $[\mathcal{A}, \mathcal{A}]$ (i. e. for every functor F) in

$$\Lambda_F \cdot \mathcal{H}\eta^{\mathcal{M}}_F = \mathsf{can} \cdot (\lambda F + \eta^M) \cdot (H\eta^M F + \mathrm{Id}) = \mathsf{can} \cdot (\eta^M HF + \eta^M) = \eta^M (HF + \mathrm{Id}) = \eta^{\mathcal{M}}_{\mathcal{H}F}$$

and

$$\begin{aligned}
\Lambda_F \cdot \mathcal{H}\mu^{\mathcal{M}}_F &= \mathsf{can} \cdot (\lambda F + \eta^M) \cdot (H\mu^M F + \mathrm{Id}) \\
&= \mathsf{can} \cdot (\mu^M HF + \mu^M) \cdot (M\lambda F + M\eta^M) \cdot (\lambda MF + \eta^M) \\
&= \mu^M (HF + \mathrm{Id}) \cdot M\mathsf{can} \cdot \mathsf{can} \cdot (M\lambda F + M\eta^M) \cdot (\lambda MF + \eta^M) \\
&= \mu^M (HF + \mathrm{Id}) \cdot M\mathsf{can} \cdot M(\lambda F + \eta^M) \cdot \mathsf{can} \cdot (\lambda MF + \eta^M) \\
&= \mu^{\mathcal{M}}_{\mathcal{H}F} \cdot \mathcal{M}\Lambda_F \cdot \Lambda_{\mathcal{M}F} .
\end{aligned}$$

◀

Now recall Definition 2.5 (Kleisli category).

▶ **Proposition 4.7** ([18]). *For any functor $H : \mathcal{A} \to \mathcal{A}$ and monad M on $\mathcal{A}$ the following are equivalent:*
1. *there is a distributive law $\lambda : HM \to MH$ of the functor over the monad;*
2. *H lifts to a functor $\bar{H}$ on $\mathcal{A}_M$.*

▶ Remark 4.8. We do not state the definition of a lifting of a functor here; let us only remark that in the proof of Proposition 4.7 for a given distributive law $\lambda : HM \to MH$ the corresponding functor $\bar{H}$ on $\mathcal{A}_M$ is given by $\bar{H}X = HX$ on objects X of $\mathcal{A}_M$ and by $\bar{H}f = \lambda_Y \cdot Hf : HX \to MHY$ on morphisms $f : X \to MY$ of $\mathcal{A}_M$.

▶ **Corollary 4.9.** *$\mathcal{H}$ lifts to a functor $\bar{\mathcal{H}}$ on $[\mathcal{A}, \mathcal{A}]_{\mathcal{M}}$.*

Explicitly $\bar{\mathcal{H}}$ is given on objects (i. e. functors F) and morphisms (i. e. natural transformations $\alpha : F \to MG$) by

$$\bar{\mathcal{H}}F = HF + \mathrm{Id} \quad \text{and} \quad \bar{\mathcal{H}}\alpha = \mathsf{can} \cdot (\lambda G + \eta^M) \cdot (H\alpha + \mathrm{id}).$$

Let us come back to the setting where $H : \mathsf{Set} \to \mathsf{Set}$ is polynomial, $M = \mathcal{P}^+$ and $\lambda : H\mathcal{P}^+ \to \mathcal{P}^+H$ is canonical.

▶ **Theorem 4.10.** *$J[\tau, \eta]^{-1} : T^H \to \mathcal{P}^+(HT^H + \mathrm{Id})$ is a weakly final $\bar{\mathcal{H}}$-coalgebra.*

Proof (sketch). The components of every $\bar{\mathcal{H}}$-coalgebra give rise to equation morphisms whose canonical solutions from the proof of Theorem 4.3 can be shown to form a homomorphism h into the $\bar{\mathcal{H}}$-coalgebra $J[\tau, \eta]^{-1}$. ◀

▶ **Lemma 4.11.** *The $\bar{\mathcal{H}}$-coalgebra homomorphisms $h : F \to \mathcal{P}^+T^H$ into the weakly final $\bar{\mathcal{H}}$-coalgebra from the proof of Theorem 4.10 are (componentwise) the greatest such homomorphisms; moreover, for every $\bar{\mathcal{H}}$-coalgebra homomorphism $\alpha : F \to \mathcal{P}^+T^H$ the sets of all finite cuttings of trees from $\alpha_X(z)$ and $h_X(z)$ are the same for every set X and every $z \in FX$.*

Proof. This follows from Lemma 4.5 and the proof of Theorem 4.10. ◀

5 Nondeterministic Recursive Program Schemes

In this section, we present our category theoretic notion of a nondeterministic RPS. We compare this notion with the one of a deterministic RPS from Milius and Moss [16] and with the classical notion of a nondeterministic RPS as given by Arnold and Nivat [4]. Using the technical results from the previous section, we prove our main theorem giving a semantics to nondeterministic RPS's.

▶ **Definition 5.1.** Let H and V be polynomial Set-functors. A *nondeterministic recursive program scheme* (or *NDRPS*, for short) is a natural transformation $e : V \to \mathcal{P}^+F^{H+V}$. It is called *guarded* if it factors

$$e \equiv (V \xrightarrow{e'} \mathcal{P}^+HF^{H+V} \xrightarrow{\mathcal{P}^+\mathsf{inl}F^{H+V}} \mathcal{P}^+(H+V)F^{H+V} \xrightarrow{\mathcal{P}^+\phi^{H+V}} \mathcal{P}^+F^{H+V}).$$

An *uninterpreted solution* of e is a natural transformation $e^\dagger : V \to \mathcal{P}^+T^H$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "e^\dagger"] \arrow[d, "e"'] & \mathcal{P}^+T^H \\ \mathcal{P}^+F^{H+V} \arrow[r, "\mathcal{P}^+[\eta^+T^H\cdot\kappa^H, e^\dagger]^\#"'] & \mathcal{P}^+\mathcal{P}^+T^H \arrow[u, "\mu^+T^H"'] \end{tikzcd} \tag{4}$$

commutes.

▶ Remark 5.2. Notice that $[\eta^+T^H \cdot \kappa^H, e^\dagger]^\#$ in Definition 5.1 is the unique monad morphism such that $[\eta^+T^H \cdot \kappa^H, e^\dagger]^\# \cdot \kappa^{H+V} = [\eta^+T^H \cdot \kappa^H, e^\dagger]$. It exists since F^{H+V} is the free monad on $H+V$ with universal natural transformation $\kappa^{H+V} : H+V \to F^{H+V}$ (cf. Definition 2.3) and $\mathcal{P}^+T^H$ is a monad by Theorem 4.3. Explicitly, for polynomial functors H and V and any set X, $F^{H+V}X$ is the set of all finite trees or terms built from the operation symbols from the signatures associated with H and V and the variables from X (similar to our description of T^HX above Proposition 3.5). $[\eta^+T^H \cdot \kappa^H, e^\dagger]^\#_X$ performs a nondeterministic variant of second-order substitution in trees (cf. [16], Section 4.1).

▶ Remark 5.3. In Definition 6.1 of [16] deterministic RPS's are defined as natural transformations $e : V \to T^{H+V}$ where H and V are endofunctors on any category $\mathcal{A}$ with binary coproducts such that T^H and T^{H+V} exist. They are called guarded if they factor through a natural transformation $e' : V \to HT^{H+V}$. Uninterpreted solutions are ideal natural transformations $e^\dagger : V \to T^H$ such that $e^\dagger = [\kappa^H, e^\dagger]^\$ \cdot e$ where $[\kappa^H, e^\dagger]^\$: T^{H+V} \to T^H$ is the unique idealized monad morphism extending $[\kappa^H, e^\dagger]$ induced by the freeness property of the completely iterative monad T^{H+V}. We compare these definitions with our Definition 5.1 of NDRPS's:

1. If we "eliminate" the non-determinism from Definition 5.1 by using the identity monad $(\mathrm{Id}, \mathrm{id}, \mathrm{id})$ instead of $(\mathcal{P}^+, \eta^+, \mu^+)$, we obtain a special case of deterministic RPS's as defined in [16] where we restrict to $\mathcal{A} = \mathsf{Set}$ and to finite terms on the right-hand sides of NDRPS's. More precisely, we obtain natural transformations $e : V \to F^{H+V}$ which can be viewed as RPS's $\mathsf{in} \cdot e : V \to T^{H+V}$ where $\mathsf{in} = (\kappa^{H+V})^\# : F^{H+V} \to T^{H+V}$. If e is guarded in the sense of Definition 5.1 (using Id instead of $\mathcal{P}^+$), then $\mathsf{in} \cdot e$ is guarded in the sense of [16].
2. To "eliminate" the non-determinism from our definition of an uninterpreted solution, observe that the identity monad is commutative. By Lemma 3.2 we obtain a canonical distributive law of every polynomial Set-functor H over Id which simply is $\mathrm{id} : H = H\mathrm{Id} \to \mathrm{Id}H = H$, and analogously to Proposition 3.8 this extends to a distributive law of the monads T^H and Id which simply is $\mathrm{id} : T^H = T^H\mathrm{Id} \to \mathrm{Id}T^H = T^H$. The "composite monad" $\mathrm{Id}T^H$ becomes the completely iterative monad T^H, thus the notion of an uninterpreted solution also becomes a special case of the one from [16] (as far as ideal natural transformations $e^\dagger$ are concerned) since $[\kappa^H, e^\dagger]^\$ \cdot \mathsf{in} = [\kappa^H, e^\dagger]^\# : F^{H+V} \to T^H$ by the uniqueness of such monad morphisms extending $\kappa^{H+V} : H+V \to F^{H+V}$.
3. The assumption on uninterpreted solutions of deterministic RPS's to be ideal is necessary to ensure the existence of $[\kappa^H, e^\dagger]^\$$. Working with finite terms in Definition 5.1 has the advantage that we can drop this assumption. In case of a guarded RPS $e : V \to F^{H+V}$ (using Id instead of $\mathcal{P}^+$) uninterpreted solutions automatically are ideal.
4. Technically, the restriction to finite terms on the right-hand sides of NDRPS's is due to the fact that the monad $\mathcal{P}^+T^H$ is not a completely iterative monad and we thus cannot exploit freeness of the completely iterative monad T^{H+V}. However, since by Theorem 4.3 (and Lemma 4.5) $\mathcal{P}^+T^H$ is an idealized monad together with a solution operation $(-)^\dagger$ giving canonical (greatest) solutions for guarded equation morphisms, it comes close to a completely iterative monad. In order to capture infinite terms, it would be interesting to see whether $\mathcal{P}^+T^H$ is something like a "complete Elgot monad" and whether the free completely iterative monad T^{H+V} also is the "free complete Elgot monad". But whereas the concept of Elgot monads has recently been investigated [3], there exist no results for complete Elgot monads.

▶ **Example 5.4.** Consider the NDRPS (1) from the introduction. It is formulated in the classical way using the special binary function symbol or, see e. g. [4], Section II. It can be

viewed as a natural transformation $e : V \to \mathcal{P}^+T^{H+V}$ as follows: according to the signatures of new and given function symbols, we choose the polynomial Set-functors $VX = X$ and $HX = X \times X$. We translate the right-hand term from (1) which is headed by the symbol or into the set containing the two subterms and abstract away from a concrete variable set, obtaining the natural transformation e given by $e_X(\phi(x)) = \{f(x, x), f(x, \phi(x))\}$ for every set X. The naturality states that it is invariant under renaming the variable x.
In classical terms, the NDRPS (1) is a *Greibach scheme* since every new function symbol is part of a term headed by a given function symbol, see e. g. [4], Section IV. Correspondingly, the natural transformation e is guarded since every element of the right-hand set is a term headed by a given operation symbol.
Let us denote the infinite set (2) from the introduction by S. We obtain the natural transformation $e^\dagger$ given by $e^\dagger_X(\phi(x)) = S$ for every set X. Using Remark 5.2, we see that diagram (4) commutes; thus $e^\dagger$ is an uninterpreted solution of e. Similarly, the natural transformation s given by $s_X(\phi(x)) = S \setminus \{t\}$ for every set X is an uninterpreted solution of e, where t is the only infinite tree from S (the rightmost one in (2)).

▶ Remark 5.5. More generally, every classical NDRPS in the sense of Arnold and Nivat [4] can be translated into a NDRPS in the sense of Definition 5.1, using the following ideas:

- the polynomial Set-functors V and H are chosen according to the signatures of new and given function symbols;
- every term headed by the function symbol or is translated to the set of its two subterms;
- given function symbols are distributed over sets using the canonical distributive law $\lambda : H\mathcal{P}^+ \to \mathcal{P}^+H$;
- nested sets are flattened using $\mu^+ : \mathcal{P}^+\mathcal{P}^+ \to \mathcal{P}^+$;
- for every set S occurring in a term headed by a new function symbol an additional new function symbol $\phi_S(x_1, \ldots, x_n)$ with arity according to the number n of variables in S is introduced, S is replaced by $\phi_S(x_1, \ldots, x_n)$ and the equation $\phi_S(x_1, \ldots, x_n) = S$ is added to the NDRPS;
- occurrences of single variables x_i in sets are replaced by $\pi_i(x_1, \ldots, x_n)$ where π_i is an additional given function symbol and the $x_1, \ldots, x_n$ are all variables occurring in the elements of the set (the idea is of course that π_i denotes the i-th projection);
- the natural transformation $e : V \to \mathcal{P}^+F^{H+V}$ constituting the NDRPS is given for every set X and every element from VX by the right-hand side of the equation for the corresponding new function symbol.

In order to obtain a guarded NDRPS from a classical Greibach scheme, it might be necessary to substitute some new function symbols by the right-hand sides of their equations. In conclusion, our notion of a NDRPS covers the classical one, and classical Greibach schemes translate to guarded NDRPS's. Moreover, our notion generalizes the classical one: whereas in [4] NDRPS's define finitely many new operations, and, more important, only allow for finite (nonempty) sets of finite terms on the right-hand sides of NDRPS's, our approach also captures infinitely many newly defined operations and arbitrary (nonempty) sets. It might even be possible to generalize our approach to infinite terms on the right-hand sides, see Remark 5.3(4).

We now state our main result:

▶ **Theorem 5.6.** *Every guarded NDRPS has a canonical greatest uninterpreted solution.*

Before we give the proof of Theorem 5.6, we need to establish an important lemma first. Whenever we write λ or λ' in the rest of the paper, we mean the canonical distributive laws of a polynomial Set-functor H over the monad $\mathcal{P}^+$ from Lemma 3.2 or of T^H over $\mathcal{P}^+$ from

Proposition 3.8. To simplify notation, we denote the free monad F^{H+V} by F for the rest of the paper.

▶ **Definition 5.7.** Given a natural transformation $e' : V \to \mathcal{P}^+HF$, we define the $\bar{\mathcal{H}}$-coalgebra p by

$$p = \Big(F \xrightarrow{[\phi^{H+V},\eta^{H+V}]^{-1}} (H+V)F + \mathrm{Id} \xrightarrow{[\eta^+HF \cdot H\eta^{H+V}, e']F + \eta^+} \mathcal{P}^+HFF + \mathcal{P}^+ \xrightarrow{\mathcal{P}^+H\mu^{H+V} + \mathcal{P}^+} \mathcal{P}^+HF + \mathcal{P}^+ \xrightarrow{\mathsf{can}} \mathcal{P}^+(HF + \mathrm{Id}) \Big). \tag{5}$$

By Theorem 4.10 (and Lemma 4.11) there is a (componentwise greatest) natural transformation h such that the diagram

$$\begin{tikzcd} F \arrow[r, "h"] \arrow[d, "p"'] & T^H \arrow[d, "J[\tau^H,\eta^H]^{-1}"] \\ HF + \mathrm{Id} \arrow[r, "\bar{\mathcal{H}}h"'] & HT^H + \mathrm{Id} \end{tikzcd} \tag{6}$$

commutes (in $[\mathsf{Set},\mathsf{Set}]_{\mathcal{M}}$ for $M = \mathcal{P}^+$). Observe that diagram (6) translates to

$$h = \mathcal{P}^+[\tau^H, \eta^H] \cdot \mu^+(HT^H + \mathrm{Id}) \cdot \mathcal{P}^+\mathsf{can} \cdot \mathcal{P}^+(\lambda T^H + \eta^+) \cdot \mathcal{P}^+(Hh + \mathrm{id}) \cdot p \tag{7}$$

in $[\mathsf{Set},\mathsf{Set}]$.

▶ **Lemma 5.8.** *The natural transformation $h : F \to \mathcal{P}^+T^H$ from Definition 5.7 is a monad morphism.*

We remark that in the proof of Lemma 5.8, Theorem 4.10 is used to prove the second monad morphism law for h.

Proof of Theorem 5.6 (sketch). The given guarded NDRPS $e : V \to \mathcal{P}^+F$ factors through a natural transformation $e' : V \to \mathcal{P}^+HF$, thus we obtain a natural transformation $h : F \to \mathcal{P}^+T^H$ as in Definition 5.7. We define

$$e^\dagger \equiv \big(V \xrightarrow{\mathsf{inr}} H + V \xrightarrow{\kappa^{H+V}} F \xrightarrow{h} \mathcal{P}^+T^H \big)$$

and prove that this is the componentwise greatest solution of e.

In a first step, one proves that $e^\dagger$ solves e using Lemma 5.8. An important part of this step is to prove that h is the unique monad morphism $[\eta^+T^H \cdot \kappa^H, e^\dagger]^\#$.

In a second step, $e^\dagger$ is proved to be the greatest solution. Here one considers any solution $s : V \to \mathcal{P}^+T^H$ of e. It suffices to show that $x = [\eta^+T^H \cdot \kappa^H, s]^\# : F \to \mathcal{P}^+T^H$ is a coalgebra homomorphism between p and the weakly final $\bar{\mathcal{H}}$-coalgebra over $[\mathsf{Set},\mathsf{Set}]_{\mathcal{M}}$ from Theorem 4.10: since h is known to be the componentwise greatest such homomorphism, it follows $h_X \geq x_X$ for every set X and we conclude

$$e^\dagger_X = \mu^+_{T^HX} \cdot \mathcal{P}^+[\eta^+T^H \cdot \kappa^H, e^\dagger]^\#_X \cdot e_X = \mu^+_{T^HX} \cdot \mathcal{P}^+h_X \cdot e \geq \mu^+_{T^HX} \cdot \mathcal{P}^+x_X \cdot e_X = s_X$$

for every set X using Definition 5.1 and monotonicity of composition in $\mathsf{Set}_{\mathcal{P}^+}$. ◀

▶ **Corollary 5.9.** *For every uninterpreted solution $s : V \to \mathcal{P}^+T^H$ of a NDRPS the sets of all finite cuttings of trees from $s_X(z)$ and $e^\dagger_X(z)$ are the same for every set X and every $z \in VX$.*

Proof. In the second part of the proof of Theorem 5.6 we prove that every solution s of every guarded NDRPS e the monad morphism $[\eta^+ T^H \cdot \kappa^H, s]^\#$ is an $\bar{\mathcal{H}}$-coalgebra homomorphism. According to Lemma 4.11, we have the desired property for this $\bar{\mathcal{H}}$-coalgebra homomorphism and the $\bar{\mathcal{H}}$-coalgebra homomorphism h; this implies that this property also holds for their respective restrictions $s = [\eta^+ T^H \cdot \kappa^H, s]^\# \cdot \kappa^{H+V} \cdot \mathsf{inr}$ and $e^\dagger = h \cdot \kappa^{H+V} \cdot \mathsf{inr}$. ◀

▶ Remark 5.10. The main result of Arnold and Nivat [4] is that greatest solutions of Greibach schemes give the "right" semantics of NDRPS's and can be computed as greatest fixed points. We confirmed the former in Theorem 5.6 and generalized it to a wider class of NDRPS's (cf. Remark 5.5). From our results we also easily recover the latter: restricting to finite sets on the right-hand sides of NDRPS's, the operator $h \mapsto \mathcal{P}^+[\tau^H, \eta^H] \cdot \mu^+(HT^H + \mathrm{Id}) \cdot \mathcal{P}^+\mathsf{can} \cdot \mathcal{P}^+(\lambda T^H + \eta^+) \cdot \mathcal{P}^+(Hh + \mathrm{id}) \cdot p$ on $\mathsf{Set}(F, \mathcal{P}^+ T^H)$ given by equation (7) or equivalently by diagram (6) is componentwise continuous; since we know from Theorem 4.10 and Lemma 4.11 that the greatest fixed point of this operator exists, the second part of Arnold's and Nivat's result follows from (the dual of) Kleene's fixed point theorem. However, the operator is no longer continuous if we allow for infinite sets on the right-hand sides of NDRPS's.

6 Conclusion

We have given a category theoretic definition and semantics of (uninterpreted) NDRPS's. This was achieved by reusing the technical core of Milius and Moss' work on a category theoretic semantics for (ordinary) RPS's [16] and by adding category theoretic concepts that capture the nondeterminism as the nonempty powerset monad and canonical distributive laws over this monad. We showed how our work is related to loc. cit. and that it extends the classical work on NDRPS's by Arnold and Nivat [4].

Although our approach is inspired by [16] and its precursor [9], the non-determinism causes various differences: it is not only more complicated to work with the additional nonempty powerset monad and the canonical distributive laws for it, but many proofs have to be carried out in a more basic setting. For example, the coalgebra functor $\mathcal{H}$ can only be considered on a more basic category, or we even need to use techniques inherent to non-determinism like "determinization" (see the proofs of Proposition 3.5 and Theorem 4.3).

Still, due to the abstract category theoretic framework there are several directions for future generalizations: instead of polynomial functors H it might be possible to use analytic or even weak pullback preserving functors; a starting point is given in Remark 3.3(2). We also suspect that our work can be applied to the environment monad $(-)^E$ instead of $\mathcal{P}^+$ giving an even stronger result (unique solutions) for E-composite RPS's. Technically, our work might be improved by the development of a theory of "complete Elgot monads" as pointed out in Remark 5.3(4). And clearly this paper leaves the question of a category theoretic semantics of interpreted NDRPS's open for future research.

Finally we mention that it is of course possible to admit the empty set in solutions of NDRPS's, i. e. to use the powerset functor $\mathcal{P}$ instead of its nonempty variant $\mathcal{P}^+$. However, this causes a shift in the results since additional least solutions are added: for example it is not difficult to see that every NDRPS where we have recursion in every element of every right-hand set, has a solution where every new function symbol is assigned the empty set. We shall consider this notion of a NDRPS elsewhere.

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