Soundness of Unravelings for Deterministic Conditional Term Rewriting Systems via Ultra-Properties Related to Linearity

Naoki Nishida[1], Masahiko Sakai[1], and Toshiki Sakabe[1]

1 **Graduate School of Information Science, Nagoya University**
Furo-cho, Chikusa-ku, 4648603 Nagoya, Japan
`{nishida,sakai,sakabe}@is.nagoya-u.ac.jp`

Abstract

Unravelings are transformations from a conditional term rewriting system (CTRS, for short) over an original signature into an unconditional term rewriting systems (TRS, for short) over an extended signature. They are not *sound* for every CTRS *w.r.t. reduction*, while they are complete w.r.t. reduction. Here, soundness w.r.t. reduction means that every reduction sequence of the corresponding unraveled TRS, of which the initial and end terms are over the original signature, can be simulated by the reduction of the original CTRS. In this paper, we show that an optimized variant of Ohlebusch's unraveling for deterministic CTRSs is sound w.r.t. reduction if the corresponding unraveled TRSs are left-linear or both right-linear and non-erasing. We also show that soundness of the variant implies that of Ohlebusch's unraveling.

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1 Introduction

Unravelings are transformations from a conditional term rewriting system (CTRS, for short) over an original signature into an unconditional term rewriting system (TRS, for short) over an extended signature, that are complete for every CTRS w.r.t. the simulation of reduction sequences of the CTRS [11], i.e., every reduction sequence of the CTRS can be simulated by the reduction of the corresponding unraveled TRS. The unraveled TRSs are approximations of the original CTRSs and they are useful in analyzing properties of the CTRSs, such as syntactic properties, modularity and operational termination, since TRSs are much easier to handle than CTRSs. Marchiori has proposed unravelings for *join* and *normal* CTRSs in order to analyze *ultra-properties* and *modularity* of the CTRSs [11], and he has also proposed an unraveling for *deterministic* CTRSs (DCTRS, for short) [12]. Ohlebusch has presented an improved variant of Marchiori's unraveling for DCTRSs in order to analyze termination of logic programs [20]. Termination of the unraveled TRSs is a practical sufficient-condition for proving *operational termination* of the original CTRSs [10]. A variant of Ohlebusch's unraveling for DCTRSs has been proposed in [14] and [4] (cf. [19, 18, 3]). This variant is sometimes called *optimized* in the sense that the variable-carrying arguments of *U symbols* introduced via the application of the unraveling are optimized.

Although the mechanism of unconditional rewriting is much simpler than that of conditional rewriting, the reduction of the unraveled TRSs has never been used as an alternative

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to that of the original CTRSs in order to simulate reduction sequences of the CTRSs. This is because unravelings are not *sound* for every CTRS *w.r.t. reduction* [11, 20]. Here, *soundness w.r.t. reduction* (simply, *soundness*) means that every reduction sequence of the unraveled TRSs, of which the initial and end terms are over the original signatures, can be simulated by the reduction of the original CTRSs [11]. It has been shown that unravelings are sound if the unraveled TRSs satisfy some syntactic properties or if appropriate reduction strategies are introduced to the reduction of the unraveled TRSs. Marchiori has shown in [11] that his unravelings for join and normal CTRSs are sound for left-linear ones, and he has also shown in [12] that his unraveling for DCTRSs is sound for *semi-linear* DCTRSs. Nishida et al. have shown in [19] that the combined reduction restriction of the *membership* [25] and *context-sensitive* [9] conditions that are determined via the application of the optimized unraveling is sufficient for soundness. Schernhammer and Gramlich have shown in [23, 22] that a similar context-sensitive restriction is sufficient for soundness of Ohlebusch's unraveling. Gmeiner et al. have shown in [5] that Marchiori's unraveling for normal CTRSs is sound for confluent, non-erasing or weakly left-linear ones. They have also given a discussion what properties are necessary or sufficient for soundness.

In this paper, we show two sufficient syntactic-conditions of DCTRSs for soundness of the optimized unraveling. One is *ultra-left-linearity w.r.t. the unraveling*, i.e., that the unraveled TRSs are left-linear. The other is the combination of *ultra-right-linearity* and *ultra-non-erasingness w.r.t. the unraveling*, i.e., that the unraveled TRSs are right-linear and non-erasing. We also provide necessary and sufficient syntactic-conditions of DCTRSs in which the corresponding unraveled TRSs are left-linear, right-linear and non-erasing, respectively. Finally, we show that soundness of the optimized unraveling implies that of Ohlebusch's unraveling, i.e., if the optimized one is sound for a DCTRS, then Ohlebusch's one is also sound for the DCTRS. A main difference to the preliminary version [17] is the result on the relationship with Ohlebusch's unraveling.

The optimized unraveling in this paper is employed in the *inversion compilers* for constructor TRSs [14, 19, 18]. The compilers transform a constructor TRS into a DCTRS defining inverses of functions defined in the constructor TRS and then unravel it into a TRS (see Example 3.3). The resulting TRS may have extra variables since the intermediate DCTRS may have extra variables that occur in the right-hand side but not in the conditional part. For this reason, this paper allows TRSs to have extra variables (called *EV-TRS*). It is allowed to instantiate extra variables with arbitrary terms in applying rewrite rules. Since many instantiated terms of extra variables are meaningless and sometimes cause non-termination, we focus on meaningful derivations by giving a restriction to reduction sequences of the resulting TRS. The restriction, called *EV-basicness* [16, 14, 17], is a relaxed variant of the *basicness* property [7, 13] of reduction sequences: if a TRS has extra variables, then any redex introduced by extra variables is not reduced anywhere in reduction sequences. Roughly speaking, in applying the inversion compilers, the resulting TRS is often right-linear (left-linear, resp.) if the input constructor TRS is left-linear (right-linear, resp.). Moreover, the resulting TRS is usually non-erasing if the target function is injective. Note that injective functions are the most interesting targets of program inversion. For these reasons, the sufficient conditions shown in this paper are very practical because they guarantee that the resulting TRS is definitely an inverse of the given constructor TRS.

As described above, Ohlebusch's unraveling is sound for any DCTRS if we introduce the particular context-sensitive restriction to the reduction of the corresponding unraveled TRSs. However, characterizing sufficient syntactic-properties for soundness without the restriction to the reduction is important for the use of the unraveled TRSs instead of the original CTRSs since the context-sensitivity makes the reduction more complicated than the ordinary reduction. Moreover, if the unraveling is sound for the resulting TRS obtained by the inversion compilers [14, 19, 18] without context-sensitivity, then we can apply the restricted *completion* [15] to the resulting TRS in order to make it convergent. Note that when the target of the inversion compilers are injective functions, convergence of the resulting TRSs is desirable. For these reasons, soundness of unravelings without any restriction to the reduction is important in employing the reduction of the unraveled TRSs instead of that of the original CTRSs.

Finally, we briefly describe a related work that is not mentioned above. Serbanuta and Rosu have proposed a sound and complete transformation of left-linear or ground-confluent DCTRSs into TRSs where function symbols in the original signatures are completely extended, increasing their arities [24]. Their transformation is based on Viry's approach [26] that is another direction of developing transformations of CTRSs to TRSs. Rules produced by this transformation are much more complicated than those produced by unravelings. Moreover, it is not easy to know if DCTRSs are ground-confluent.

This paper is organized as follows. In Section 2, we review basic notions and notations of term rewriting. In Section 3, we review unravelings for DCTRSs and syntactic properties related to DCTRSs and the corresponding unraveled TRSs. In Section 4, we show the main results of this paper, i.e., the optimized unraveling for DCTRSs is sound if the corresponding unraveled TRSs are left-linear or both right-linear and non-erasing. We also show that these results hold for Ohlebusch's unraveling. In Section 5, we briefly describe future work on soundness of unravelings.

2 Preliminaries

In this section, we review basic notions and notations of term rewriting [2, 21].

Throughout the paper, we use $\mathcal{V}$ as a countably infinite of *variables*. Let $\mathcal{F}$ be a *signature*, a finite set of *function symbols* each of which has its own fixed arity that is denoted by $\mathsf{ar}(f)$ for a function symbol f. The set of *terms* over $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathcal{T}(\mathcal{F},\mathcal{V})$, and the set of variables appearing in any of terms $t_1,\cdots,t_n$ is denoted by $\mathcal{V}ar(t_1,\cdots,t_n)$. The *identity* of terms s and t is written by $s \equiv t$. A term is called *linear* if any variable occurs in the term at most once. The set of *positions* of a term t is denoted by $\mathcal{P}os(t)$. The sets of positions for function symbols and for variables in t are denoted by $\mathcal{P}os_{\mathcal{F}}(t)$ and $\mathcal{P}os_{\mathcal{V}}(t)$, respectively. For a term t and a position p of t, the notation $t|_p$ represents the *subterm* of t at the position p. The function symbol at the *root* position ε of term t is denoted by $\mathsf{root}(t)$. Given an n-hole *context* $C[\ ]$ with parallel positions $p_1,\cdots,p_n$, the notation $C[t_1,\cdots,t_n]_{p_1,\cdots,p_n}$ represents the term obtained by replacing each occurrence of hole $\Box$ at position p_i with term t_i for all $1 \leq i \leq n$. We may omit the subscription $_{p_1,\cdots,p_n}$. For positions p and p' of a term, we write $p' \geq p$ if p is a prefix of p' (i.e., there exists a q' such that $pq = p'$). Moreover, we write p' $> p$ if p is a proper prefix of p'.

The *domain* and *range* of a *substitution* σ are denoted by $\mathcal{D}om(\sigma)$ and $\mathcal{R}an(\sigma)$, respectively. We may denote σ by $\{x_1 \mapsto t_1,\cdots,x_n \mapsto t_n\}$ if $\mathcal{D}om(\sigma) = \{x_1,\cdots,x_n\}$ and $\sigma(x_i)$ $\equiv t_i$ for all $1 \leq i \leq n$. For a signature $\mathcal{F}$, the set of *substitutions* whose domains are over $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathcal{S}ub(\mathcal{F},\mathcal{V})$: $\mathcal{S}ub(\mathcal{F},\mathcal{V}) = \{\sigma \mid \mathcal{R}an(\sigma) \subseteq \mathcal{T}(\mathcal{F},\mathcal{V})\}$. The application $\sigma(t)$ of a substitution σ to a term t is abbreviated to $t\sigma$. Given a set X of variables, $\sigma|_X$ denotes the restricted substitution of σ w.r.t. X: $\sigma|_X = \{x \mapsto x\sigma \mid x \in \mathcal{D}om(\sigma) \cap X\}$. The *composition* $\sigma\theta$ of substitutions σ and θ is defined as $x\sigma\theta = (x\sigma)\theta$.

An *oriented conditional rewrite rule* over a signature $\mathcal{F}$ is a triple (l, r, c), denoted by

$l \to r \Leftarrow c$, such that the *left-hand side* l is a non-variable term in $\mathcal{T}(\mathcal{F},\mathcal{V})$, the *right-hand side* r is a term in $\mathcal{T}(\mathcal{F},\mathcal{V})$, and the *conditional part* c is a sequence $s_1 \twoheadrightarrow t_1; \cdots; s_k \twoheadrightarrow t_k$ ($k \geq 0$) where all of $s_1, t_1 \cdots, s_k, t_k$ are terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$. In particular, the rewrite rule is called *unconditional* if the conditional part is the empty sequence (i.e., $k = 0$), and we may abbreviate it to $l \to r$. The rewrite rule is called *extended* if the condition "$l \notin \mathcal{V}$" is not imposed. We sometimes attach a unique label ρ to the rewrite rule $l \to r \Leftarrow c$ by denoting $\rho : l \to r \Leftarrow c$, and we use the label to refer to the rewrite rule. The set of variables in c and in ρ are denoted by $\mathcal{V}ar(c)$ and $\mathcal{V}ar(\rho)$, respectively: $\mathcal{V}ar(s_1 \twoheadrightarrow t_1; \cdots; s_k \twoheadrightarrow t_k) = \mathcal{V}ar(s_1, t_1, \cdots, s_k, t_k)$ and $\mathcal{V}ar(\rho) = \mathcal{V}ar(l, r, c)$. A variable occurring in r or c is called an *extra variables of the rule* ρ if it does not occur in l. The set of extra variables of ρ is denoted by $\mathcal{EV}ar(\rho)$: $\mathcal{EV}ar(\rho) = \mathcal{V}ar(r, c) \setminus \mathcal{V}ar(l)$.

An *oriented conditional term rewriting system* (*CTRS*, for short) over a signature $\mathcal{F}$ is a finite set of oriented conditional rewrite rules over $\mathcal{F}$. In particular, a CTRS is called an *EV-TRS* if all of its rules are unconditional, and called an *extended CTRS* (*eCTRS*, for short) if the condition "$l \notin \mathcal{V}$" of conditional rewrite rules is not imposed. Moreover, a CTRS is called an (*unconditional*) *term rewriting system* (*TRS*, for short) if every rule $l \to r \Leftarrow c$ in it is unconditional and satisfies $\mathcal{V}ar(l) \supseteq \mathcal{V}ar(r)$. Note that an eCTRS is called an *eTRS* if all of its rules are unconditional. For an eCTRS R, the *n-level reduction relation* $\to_{(n),R}$ of R is defined as follows: $\to_{(0),R} = \emptyset$, and $\to_{(i+1),R} = \to_{(i),R} \cup \{(C[l\sigma]_p, C[l\sigma]_p) \mid \rho : l \to r \Leftarrow s_1 \twoheadrightarrow t_1; \cdots; s_k \twoheadrightarrow t_k \in R,\ s_1\sigma \to^*_{(i),R} t_1\sigma,\ \cdots,\ s_k\sigma \to^*_{(i),R} t_k\sigma\}$ where $i \geq 0$ and $\to^*_{(i),R}$ is the reflexive and transitive closure of $\to_{(i),R}$. The *reduction relation* of R is defined as $\to_R = \bigcup_{n \geq 0} \to_{(n),R}$. To specify the applied rule ρ and the position p, we may write $\to_{\rho,p,R}$ or $\to_{p,R}$ instead of $\to_R$. Moreover, we may write $\to_{>\varepsilon,R}$ instead of $\to_{\rho,p,R}$ or $\to_{p,R}$ if $p > \varepsilon$. The *join* relation $\downarrow_R$ is defined as $\downarrow_R = \{(s,t) \mid \exists u.\ s \to^*_R u \wedge t \to^*_R u\}$. The *parallel reduction* $\rightrightarrows_R$ is defined as $\rightrightarrows_R = \{(C[s_1, \cdots, s_n]_{p_1,\cdots,p_n}, C[t_1, \cdots, t_n]_{p_1,\cdots,p_n}) \mid s_1 \to_R t_1,\ \cdots,\ s_n \to_R t_n\}$. We may write $\rightrightarrows_{>\varepsilon,R}$ instead of $\rightrightarrows_R$ if $p_i > \varepsilon$ for all $1 \leq i \leq p_n$.

An (extended) conditional rewrite rule $\rho : l \to r \Leftarrow c$ is called *left-linear* (*LL*, for short) if l is linear, called *right-linear* (*RL*, for short) if r is linear, called *non-erasing* (*NE*, for short) if $\mathcal{V}ar(l) \subseteq \mathcal{V}ar(r)$, called *non-collapsing* if the right-hand side r is not a variable, and called *non-left-variable* (*non-LV*, for short) if l is not a variable. An eCTRS is called *left-linear* (*right-linear*, *non-erasing*, *non-collapsing*, *non-left-variable*, resp.) if all of its rules are left-linear (right-linear, non-erasing, non-collapsing and non-LV, resp.). Note that a non-LV eCTRS is a CTRS (i.e., it is not an extended one).

An (extended) conditional rewrite rule $\rho : l \to r \Leftarrow s_1 \twoheadrightarrow t_1; \cdots; s_k \twoheadrightarrow t_k$ is called *deterministic* if $\mathcal{V}ar(s_i) \subseteq \mathcal{V}ar(l, t_1, \cdots, t_{i-1})$ for all $1 \leq i \leq k$. An eCTRS is called *deterministic* (*eDCTRS*, for short) if all of its rules are deterministic. The rule ρ is classified according to the distribution of variables in the rule as follows: *Type 1* if $\mathcal{V}ar(r, s_1, t_1, \cdots, s_k, t_k) \subseteq \mathcal{V}ar(l)$, *Type 2* if $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$, *Type 3* if $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l, s_1, t_1, \cdots, s_k, t_k)$, and *Type 4* otherwise. An e(D)CTRS is called a *1-e(D)CTRS* (*2-e(D)CTRS 3-e(D)CTRS*, and *4-e(D)CTRS*, resp.) if all of its rules are Type 1 (Type 2, Type 3 and Type 4, resp.).

3 Unraveling for DCTRSs

In this section, we first recall an unraveling for DCTRSs proposed by Ohlebusch and its optimized variant. Then, we show some syntactic properties related to the unraveled TRSs. The unravelings and some results are extended to eDCTRSs.

A computable transformation U from eCTRSs into eTRSs is called an *unraveling* if for every eCTRS R, $\downarrow_R \subseteq \downarrow_{U(R)}$ and $U(T \cup R) = T \cup U(R)$ whenever T is an eTRS [11, 12].

Note that a sufficient condition for $\downarrow_R \subseteq \downarrow_{U(R)}$ is $\to_R \subseteq \to^*_{U(R)}$. For an eDCTRS R over a signature $\mathcal{F}$, the unraveling U is called *sound w.r.t. reduction* (*simulation-sound* [17, 19], or simply *sound*) if $\to^*_{U(R)} \subseteq \to^*_R$ on $\mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V})$ (i.e., for any terms s and t in $\mathcal{T}(\mathcal{F},\mathcal{V})$, $s \to^*_{U(R)} t$ implies $s \to^*_R t$).

For a finite set $A = \{a_1, \cdots, a_n\}$, $\overrightarrow{A}$ denotes the unique sequence $a_1, \cdots, a_n$ of elements in A, following some fixed ordering $\prec$ such that $a_1 \prec \cdots \prec a_n$.

▶ **Definition 3.1** (unraveling $\mathbb{U}$ [20]). Let R be an eDCTRS over a signature $\mathcal{F}$. For every conditional rule $\rho : l \to r \Leftarrow s_1 \twoheadrightarrow t_1; \cdots ; s_k \twoheadrightarrow t_k$ in R, we prepare k fresh function symbols $U^\rho_1, \cdots, U^\rho_k$, called *U symbols*, that do not appear in $\mathcal{F}$. We transform ρ into a set $\mathbb{U}(\rho)$ of $k+1$ unconditional rewrite rules as follows:

$$\mathbb{U}(\rho) = \{\ l \to U^\rho_1(s_1, \overrightarrow{X_1}),\ U^\rho_1(t_1, \overrightarrow{X_1}) \to U^\rho_2(s_2, \overrightarrow{X_2}),\ \cdots,\ U^\rho_k(t_k, \overrightarrow{X_k}) \to r\ \}$$

where $X_i = \mathcal{V}ar(l, t_1, \cdots, t_{i-1})$. Note that $\mathbb{U}(l' \to r') = \{l' \to r'\}$. $\mathbb{U}$ is extended to eDCTRSs (i.e., $\mathbb{U}(R) = \bigcup_{\rho \in R} \mathbb{U}(\rho)$) and $\mathbb{U}(R)$ is an eTRS over the extended signature $\mathcal{F}_{\mathbb{U}(R)}$ where $\mathcal{F}_{\mathbb{U}(R)} = \mathcal{F} \cup \{U^\rho_i \mid \rho : l \to r \Leftarrow s_1 \twoheadrightarrow t_1; \cdots ; s_k \twoheadrightarrow t_k \in R,\ 1 \le i \le k\}$.

It is clear that $\to_R \subseteq \to^*_{\mathbb{U}(R)}$ and $\mathbb{U}(T \uplus R) = T \cup \mathbb{U}(R)$ if T is unconditional. Thus, $\mathbb{U}$ is an unraveling for eDCTRSs.

▶ **Definition 3.2** (optimized unraveling $\mathbb{U}_{\mathrm{opt}}$ [14, 4]). Let R be an eDCTRS over a signature $\mathcal{F}$. Introducing U symbols $U^\rho_1, \cdots, U^\rho_k$ again, we transform $\rho : l \to r \Leftarrow s_1 \twoheadrightarrow t_1; \cdots ; s_k \twoheadrightarrow t_k$ into a set $\mathbb{U}_{\mathrm{opt}}(\rho)$ of $k+1$ unconditional rewrite rules as follows:

$$\mathbb{U}_{\mathrm{opt}}(\rho) = \{\ l \to U^\rho_1(s_1, \overrightarrow{X_1}),\ U^\rho_1(t_1, \overrightarrow{X_1}) \to U^\rho_2(s_2, \overrightarrow{X_2}),\ \cdots,\ U^\rho_k(t_k, \overrightarrow{X_k}) \to r\ \}$$

where $X_i = \mathcal{V}ar(l, t_1, \cdots, t_{i-1}) \cap \mathcal{V}ar(r, t_i, s_{i+1}, t_{i+1}, \cdots, s_k, t_k)$. Note that $\mathbb{U}_{\mathrm{opt}}(l' \to r') = \{l' \to r'\}$. $\mathbb{U}_{\mathrm{opt}}$ is extended to eDCTRSs (i.e., $\mathbb{U}_{\mathrm{opt}}(R) = \bigcup_{\rho \in R} \mathbb{U}_{\mathrm{opt}}(\rho)$) and $\mathbb{U}_{\mathrm{opt}}(R)$ is an eTRS over the extended signature $\mathcal{F}_{\mathbb{U}_{\mathrm{opt}}(R)}$ where $\mathcal{F}_{\mathbb{U}_{\mathrm{opt}}(R)} = \mathcal{F}_{\mathbb{U}(R)}$.

It is clear that $\mathbb{U}_{\mathrm{opt}}$ is also an unraveling for eDCTRSs. Note that X_i in Definition 3.2 is the set of variables which appear in any of $l, t_1, \cdots, t_{i-1}$ and also appear in any of $r, t_i, s_{i+1}, t_{i+1}, \cdots, s_k, t_k$, i.e., every variable in X_i is referred after s_i is considered. On the other hand, X_i in Definition 3.1 is used for carrying all the variables that already appear. This is the only difference between $\mathbb{U}$ and $\mathbb{U}_{\mathrm{opt}}$ and the reason why $\mathbb{U}_{\mathrm{opt}}$ is sometimes called an *optimized* variant of $\mathbb{U}$. Note that all of the following are equivalent: R is in Type 3, $\mathbb{U}(R)$ has no extra variable, and $\mathbb{U}_{\mathrm{opt}}(R)$ has no extra variable.

▶ **Example 3.3.** Consider the following TRS defining addition and multiplication of natural numbers encoded as $\mathsf{0}, \mathsf{s}(\mathsf{0}), \mathsf{s}(\mathsf{s}(\mathsf{0})), \cdots$:

$$R_1 = \left\{ \begin{array}{lll} \mathsf{0} + y \to y & \mathsf{s}(x) + y \to \mathsf{s}(x+y) & \\ \mathsf{0} \times y \to \mathsf{0} & x \times \mathsf{0} \to \mathsf{0} & \mathsf{s}(x) \times \mathsf{s}(y) \to \mathsf{s}((x \times \mathsf{s}(y)) + y) \end{array} \right\}$$

This TRS is inverted to the following 4-DCTRS R_2 [14, 19, 18] where $+^{-1}$ and $\times^{-1}$ are function symbols that define the inverse relation of $+$ and $\times$, respectively (i.e., $+^{-1}(\mathsf{s}^{m+n}(\mathsf{0})) \to^*_{R_2} \mathsf{tp}_2(\mathsf{s}^m(\mathsf{0}), \mathsf{s}^n(\mathsf{0}))$ and $\times^{-1}(\mathsf{s}^{m \times n}(\mathsf{0})) \to^*_{R_2} \mathsf{tp}_2(\mathsf{s}^m(\mathsf{0}), \mathsf{s}^n(\mathsf{0})))$ and tp_2 is a constructor for representing tuples of two terms:

$$R_2 = \left\{ \begin{array}{c} +^{-1}(y) \to \mathsf{tp}_2(\mathsf{0}, y) \qquad +^{-1}(\mathsf{s}(z)) \to \mathsf{tp}_2(\mathsf{s}(x), y) \Leftarrow +^{-1}(z) \twoheadrightarrow \mathsf{tp}_2(x, y) \\ \times^{-1}(\mathsf{0}) \to \mathsf{tp}_2(\mathsf{0}, y) \qquad \times^{-1}(\mathsf{0}) \to \mathsf{tp}_2(x, \mathsf{0}) \\ \times^{-1}(\mathsf{s}(z)) \to \mathsf{tp}_2(\mathsf{s}(x), \mathsf{s}(y)) \Leftarrow +^{-1}(z) \twoheadrightarrow \mathsf{tp}_2(w, y);\ \times^{-1}(w) \twoheadrightarrow \mathsf{tp}_2(x, \mathsf{s}(y)) \end{array} \right\}$$

This DCTRS is unraveled by $\mathbb{U}$ and $\mathbb{U}_{\mathrm{opt}}$ as follows:

$$\mathbb{U}(R_2) = \left\{ \begin{array}{r} +^{-1}(y) \to \mathsf{tp}_2(0,y) \\ +^{-1}(\mathsf{s}(z)) \to \mathsf{U}_1(+^{-1}(z),z) \\ \mathsf{U}_1(\mathsf{tp}_2(x,y),z) \to \mathsf{tp}_2(\mathsf{s}(x),y) \\ \times^{-1}(0) \to \mathsf{tp}_2(0,y) \\ \times^{-1}(0) \to \mathsf{tp}_2(x,0) \\ \times^{-1}(\mathsf{s}(z)) \to \mathsf{U}_2(+^{-1}(z),z) \\ \mathsf{U}_2(\mathsf{tp}_2(w,y),z) \to \mathsf{U}_3(\times^{-1}(w),z,w,y) \\ \mathsf{U}_3(\mathsf{tp}_2(x,\mathsf{s}(y)),z,w,y) \to \mathsf{tp}_2(\mathsf{s}(x),\mathsf{s}(y)) \end{array} \right\} \quad \mathbb{U}_{\mathrm{opt}}(R_2) = \left\{ \begin{array}{r} +^{-1}(y) \to \mathsf{tp}_2(0,y) \\ +^{-1}(\mathsf{s}(z)) \to \mathsf{U}_1(+^{-1}(z)) \\ \mathsf{U}_1(\mathsf{tp}_2(x,y)) \to \mathsf{tp}_2(\mathsf{s}(x),y) \\ \times^{-1}(0) \to \mathsf{tp}_2(0,y) \\ \times^{-1}(0) \to \mathsf{tp}_2(x,0) \\ \times^{-1}(\mathsf{s}(z)) \to \mathsf{U}_2(+^{-1}(z)) \\ \mathsf{U}_2(\mathsf{tp}_2(w,y)) \to \mathsf{U}_3(\times^{-1}(w),y) \\ \mathsf{U}_3(\mathsf{tp}_2(x,\mathsf{s}(y)),y) \to \mathsf{tp}_2(\mathsf{s}(x),\mathsf{s}(y)) \end{array} \right\}$$

Unravelings are not sound for every target (e)CTRS. The CTRS shown in the following example is a counterexample against soundness of an unraveling proposed in [11], and also of both $\mathbb{U}$ and $\mathbb{U}_{\mathrm{opt}}$.

▶ **Example 3.4.** Consider the following 3-DCTRS and its unraveled TRS:

$$R_3 = \left\{ \begin{array}{lllllll} \mathsf{f}(x) \to x \Leftarrow x \twoheadrightarrow \mathsf{e} & \mathsf{g}(\mathsf{d},x,x) \to \mathsf{A} & \mathsf{a} \to \mathsf{c} & \mathsf{b} \to \mathsf{c} & \mathsf{c} \to \mathsf{e} & \mathsf{k} \to \mathsf{l} \\ \mathsf{h}(x,x) \to \mathsf{g}(x,x,\mathsf{f}(\mathsf{k})) & \mathsf{d} \to \mathsf{m} & \mathsf{a} \to \mathsf{d} & \mathsf{b} \to \mathsf{d} & \mathsf{c} \to \mathsf{l} & \mathsf{k} \to \mathsf{m} \end{array} \right\}$$

$$\mathbb{U}(R_3) = \mathbb{U}_{\mathrm{opt}}(R_3) = \{\ \mathsf{f}(x) \to \mathsf{U}_4(x,x) \quad \mathsf{U}_4(\mathsf{e},x) \to x \quad \cdots\ \}$$

We have a reduction sequence of $\mathbb{U}(R_3)$ from $\mathsf{h}(\mathsf{f}(\mathsf{a}),\mathsf{f}(\mathsf{b}))$ to A but not a reduction sequence of R_3. Thus, neither $\mathbb{U}$ nor $\mathbb{U}_{\mathrm{opt}}$ is sound for R_3. We will observe the detail of the reduction sequence in Subsection 4.1.

Soundness of $\mathbb{U}$ can be recovered by restricting the reduction of the unraveled TRSs to the *context-sensitive* one [9] with the *replacement mapping* determined via the application of $\mathbb{U}$: $\mathbb{U}$ is sound for a 3-DCTRS R if the reduction of $\mathbb{U}(R)$ is restricted to context-sensitive rewriting with the replacement mapping μ such that $\mu(U_i^\rho) = \{1\}$ for any U symbol U_i^ρ [23, 22]. This holds for $\mathbb{U}_{\mathrm{opt}}$ by restricting the context-sensitive reduction to one with the *membership constraints* [25] that $x \in \mathcal{T}(\mathcal{F},\mathcal{V})$ for any variable x appearing in the left-hand sides of rules in $\mathbb{U}_{\mathrm{opt}}(R)$ [19]. Soundness of $\mathbb{U}_{\mathrm{opt}}$ requires a more complicated restriction than $\mathbb{U}$ requires. From this viewpoint, $\mathbb{U}_{\mathrm{opt}}$ does not look an "optimized" variant of $\mathbb{U}$.

To analyze syntactic relationships between eDCTRS and the corresponding unraveled eTRSs, we recall *ultra-properties* of DCTRSs [11, 12], extending them to eDCTRSs.

▶ **Definition 3.5** (ultra-property [11, 12])**.** Let P be a property on (extended) conditional rewrite rules, and U be an unraveling. An (extended) conditional rewrite rule ρ is said to be *ultra-P w.r.t. U* (U-P, for short) if all the rules in $U(\rho)$ satisfy the property P. An eDCTRS R is said to be *ultra-P w.r.t. U* if all the rules in R are ultra-P.

For example, $\mathbb{U}$-*LL*, $\mathbb{U}$-*RL* and $\mathbb{U}$-*NE* denote *ultra-left-linear w.r.t.* $\mathbb{U}$, *ultra-right-linear w.r.t.* $\mathbb{U}$ and *ultra-non-erasing w.r.t.* $\mathbb{U}$, respectively, and $\mathbb{U}_{\mathrm{opt}}$-*LL*, $\mathbb{U}_{\mathrm{opt}}$-*RL* and $\mathbb{U}_{\mathrm{opt}}$-*NE* denote *ultra-left-linear w.r.t.* $\mathbb{U}_{\mathrm{opt}}$, *ultra-right-linear w.r.t.* $\mathbb{U}_{\mathrm{opt}}$ and *ultra-non-erasing w.r.t.* $\mathbb{U}_{\mathrm{opt}}$, respectively. Note that ultra-left-linearity w.r.t. $\mathbb{U}$ is the same as the *semi-linearity* in [12].

▶ **Example 3.6.** The DCTRS R_2 in Example 3.3 is non-LV and non-collapsing w.r.t. both $\mathbb{U}$ and $\mathbb{U}_{\mathrm{opt}}$ but R_2 is not $\mathbb{U}$-LL, $\mathbb{U}$-RL or $\mathbb{U}$-NE, while R_2 is $\mathbb{U}_{\mathrm{opt}}$-RL and $\mathbb{U}_{\mathrm{opt}}$-NE but not $\mathbb{U}_{\mathrm{opt}}$-LL.

The ultra-LL, ultra-RL and ultra-NE properties w.r.t. $\mathbb{U}_{\mathrm{opt}}$ are characterized by syntactic properties of DCTRSs as follows.

▶ **Lemma 3.7** ([14, 18]). *Let $\rho : l \to r \Leftarrow s_1 \twoheadrightarrow t_1; \cdots ; s_k \twoheadrightarrow t_k$ be an extended deterministic conditional rewrite rule. Then, all of the following hold:*

- *ρ is $\mathbb{U}_{\mathrm{opt}}$-LL iff all of $l, t_1, \cdots, t_k$ are linear and $\mathcal{V}ar(t_i) \cap \mathcal{V}ar(l, t_1, \cdots, t_{i-1}) = \emptyset$ for all $1 \leq i \leq k$,*
- *ρ is $\mathbb{U}_{\mathrm{opt}}$-RL iff all of $r, s_1, \cdots, s_k$ are linear and $\mathcal{V}ar(s_i) \cap \mathcal{V}ar(r, t_i, s_{i+1}, t_{i+1}, \cdots, s_k, t_k)$ $= \emptyset$ for all $1 \leq i \leq k$, and*
- *ρ is $\mathbb{U}_{\mathrm{opt}}$-NE iff $\mathcal{V}ar(l) \subseteq \mathcal{V}ar(r, s_1, \cdots, s_k)$ and $\mathcal{V}ar(t_i) \subseteq \mathcal{V}ar(r, s_{i+1}, \cdots, s_k,)$ for all $1 \leq i \leq k$.*

The sufficient and necessary condition for the $\mathbb{U}_{\mathrm{opt}}$-NE property in Lemma 3.7 is equivalent to the one shown in [14, 18] that $\mathcal{V}ar(l) \subseteq \mathcal{V}ar(r, s_1, t_1, \cdots, s_k, t_k)$ and $\mathcal{V}ar(t_i) \subseteq \mathcal{V}ar(r, s_{i+1}, t_{i+1}, \cdots, s_k,\ t_k)$ for all $1 \leq i \leq k$. Neither of the second nor third claims in Lemma 3.7 holds for $\mathbb{U}$ (cf. Example 3.6), while the first one holds for $\mathbb{U}$. Quite restricted variants of the second and third claims hold for $\mathbb{U}$.

▶ **Lemma 3.8.** *Let $\rho : l \to r \Leftarrow s_1 \twoheadrightarrow t_1; \cdots ; s_k \twoheadrightarrow t_k$ be an extended deterministic conditional rewrite rule. Then, all of the following hold:*

- *ρ is $\mathbb{U}$-LL iff $l, t_1, \cdots, t_k$ are linear and $\mathcal{V}ar(t_i) \cap \mathcal{V}ar(l, t_1, \cdots, t_{i-1}) = \emptyset$ for all $1 \leq i \leq k$, (i.e., ρ is $\mathbb{U}$-LL iff ρ is $\mathbb{U}_{\mathrm{opt}}$-LL),*
- *ρ is $\mathbb{U}$-RL iff r is linear and all of $s_1, \cdots, s_k$ are ground, and*
- *ρ is $\mathbb{U}$-NE iff $\mathcal{V}ar(l, t_1, \cdots, t_k) \subseteq \mathcal{V}ar(r)$.*

By definition of $\mathbb{U}(\rho)$, it is clear that Lemma 3.8 holds. Due to Lemmas 3.7 and 3.8, we have the following relationship between the ultra-RL and ultra-NE properties w.r.t. $\mathbb{U}$ and $\mathbb{U}_{\mathrm{opt}}$.

▶ **Corollary 3.9.** *The $\mathbb{U}$-RL and $\mathbb{U}$-NE properties imply $\mathbb{U}_{\mathrm{opt}}$-RL and $\mathbb{U}_{\mathrm{opt}}$-NE, resp.*

As for the non-collapsing and non-LV properties, we have the following relationships between eDCTRSs and the corresponding unraveled eTRSs.

▶ **Lemma 3.10.** *Let U be either $\mathbb{U}$ or $\mathbb{U}_{\mathrm{opt}}$, and ρ be an (extended) conditional rewrite rule. Then, ρ is non-collapsing (non-LV, resp.) iff $U(\rho)$ is non-collapsing (non-LV, resp.). Thus, an eDCTRS R is non-collapsing (non-LV, resp.) iff $U(R)$ is non-collapsing (non-LV, resp.).*

By definition, it is clear that Lemma 3.10 holds. It follows from Lemma 3.10 that for both $\mathbb{U}$ and $\mathbb{U}_{\mathrm{opt}}$, the non-LV and non-collapsing properties are equivalent to the ultra-non-LV and ultra-non-collapsing properties, respectively.

4 Soundness without Context-Sensitivity

In this section, we first show that the unraveling $\mathbb{U}_{\mathrm{opt}}$ is sound for a $\mathbb{U}_{\mathrm{opt}}$-LL DCTRS if the reduction of the corresponding unraveled EV-TRS is restricted to *EV-basic* ones (see Definition 4.2). Then, we show that $\mathbb{U}_{\mathrm{opt}}$ is sound for DCTRSs that are both $\mathbb{U}_{\mathrm{opt}}$-RL and $\mathbb{U}_{\mathrm{opt}}$-NE. Finally, we show that these claims also hold for the unraveling $\mathbb{U}$. In the rest of this paper, we may write the terminology "*RL-NE*" for "right-linear and non-erasing", and may also write the terminology "*ultra-RL-NE w.r.t. an unraveling U*" (*U-RL-NE*, for short) for "ultra-RL and ultra-NE w.r.t. U".

4.1 Observation of Unsoundness

To begin with, we discuss why $\mathbb{U}_{\mathrm{opt}}$ is not sound for R_3 in Example 3.4. Consider the detail of the derivation $\mathsf{h}(\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b})) \to^*_{\mathbb{U}_{\mathrm{opt}}(R_3)} \mathsf{A}$:

$$\begin{aligned}\mathsf{h}(\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b})) &\to^*_{\mathbb{U}_{\mathrm{opt}}(R_3)} \mathsf{h}(\mathsf{U}_4(\mathsf{c}, \mathsf{d}), \mathsf{U}_4(\mathsf{c}, \mathsf{d})) \to^*_{\mathbb{U}_{\mathrm{opt}}(R_3)} \mathsf{g}(\mathsf{U}_4(\mathsf{c}, \mathsf{d}), \mathsf{U}_4(\mathsf{c}, \mathsf{d}), \mathsf{f}(\mathsf{k}))\\ &\to^*_{\mathbb{U}_{\mathrm{opt}}(R_3)} \mathsf{g}(\mathsf{d}, \mathsf{U}_4(\mathsf{l}, \mathsf{m}), \mathsf{U}_4(\mathsf{l}, \mathsf{m})) \to_{\mathbb{U}_{\mathrm{opt}}(R_3)} \mathsf{A}\end{aligned}$$

To succeed in this derivation, the following subderivations are necessary:

- to apply the rule $\mathsf{g}(\mathsf{d}, x, x) \to \mathsf{A}$, the subterm $\mathsf{f}(\mathsf{a})$ in the initial term is reduced to d,
- to apply the rule $\mathsf{h}(x, x) \to \mathsf{g}(x, x, \mathsf{f}(\mathsf{k}))$, both the subterms $\mathsf{f}(\mathsf{a})$ and $\mathsf{f}(\mathsf{b})$ in the initial term are reduced to the same term, and
- to apply the rule $\mathsf{g}(\mathsf{d}, x, x) \to \mathsf{A}$, both the subterm $\mathsf{f}(\mathsf{b})$ in the initial term and the term $\mathsf{f}(\mathsf{k})$ derived from the application of $\mathsf{h}(x, x) \to \mathsf{g}(x, x, \mathsf{f}(\mathsf{k}))$ are reduced to the same term.

In summary, all of the terms $\mathsf{f}(\mathsf{a})$, $\mathsf{f}(\mathsf{b})$ and $\mathsf{f}(\mathsf{k})$ have to be reduced to the same term d. However, this is impossible on the reduction of R_3. Nevertheless, in the above derivation, $\mathsf{h}(x, x) \to \mathsf{g}(x, x, \mathsf{f}(\mathsf{k}))$ is applied after reducing $\mathsf{f}(\mathsf{a})$ and $\mathsf{f}(\mathsf{b})$ to $\mathsf{U}_4(\mathsf{c}, \mathsf{d})$; one of $\mathsf{U}_4(\mathsf{c}, \mathsf{d})$ that comes from $\mathsf{f}(\mathsf{a})$ is reduced to d, and the other $\mathsf{U}_4(\mathsf{c}, \mathsf{d})$ that comes from $\mathsf{f}(\mathsf{b})$ is reduced to $\mathsf{U}_4(\mathsf{l}, \mathsf{m})$ in order to be the same with $\mathsf{f}(\mathsf{k})$; finally, $\mathsf{g}(\mathsf{d}, x, x) \to \mathsf{A}$ is applied. These undesired subderivations must be caused by the non-right-linear rule $\mathsf{h}(x, x) \to \mathsf{g}(x, x, \mathsf{f}(\mathsf{k}))$ and the erasing rule $\mathsf{g}(\mathsf{d}, x, x) \to \mathsf{A}$ in $\mathbb{U}_{\mathrm{opt}}(R_3)$. This is because

- the application of $\mathsf{h}(x, x) \to \mathsf{g}(x, x, \mathsf{f}(\mathsf{k}))$ to $\mathsf{h}(\mathsf{U}_4(\mathsf{c}, \mathsf{d}), \mathsf{U}_4(\mathsf{c}, \mathsf{d}))$ keeps two occurrences of $\mathsf{U}_4(\mathsf{c}, \mathsf{d})$ that are intermediate states of evaluating $\mathsf{f}(\mathsf{a})$ and $\mathsf{f}(\mathsf{b})$, respectively, and each of them has a capability to be reduced to a different term later though they should be the same, and
- $\mathsf{g}(\mathsf{d}, x, x) \to \mathsf{A}$ erases the two occurrences of $\mathsf{U}_4(\mathsf{l}, \mathsf{m})$ as if they come from the same term (in fact, they come from the terms $\mathsf{f}(\mathsf{b})$ and $\mathsf{f}(\mathsf{k})$, respectively, that should be reduced to different terms).

Viewed in this light, it is conjectured that the combination of right-linearity and non-erasingness of the unraveled TRSs is a sufficient condition for soundness of $\mathbb{U}_{\mathrm{opt}}$.

On the other hand, left-linearity of the unraveled TRSs also seems a sufficient condition for soundness of $\mathbb{U}_{\mathrm{opt}}$. A positive witness is that the unravelings for *join* and *normal* CTRSs are sound for left-linear CTRSs [11, 5] and Marchiori's unraveling for 3-DCTRSs is sound for $\mathbb{U}_{\mathrm{opt}}$-LL ones [12]. In addition, left-linearity of the unraveled TRSs seems another solution to avoid the problem mentioned above. Thus, it is conjectured that left-linearity of the unraveled TRSs is a sufficient condition for soundness of $\mathbb{U}_{\mathrm{opt}}$.

In the next two subsections, we will prove these two conjectures above. We first show the case of left-linearity since the other case can be reduced to soundness under the left-linearity case, by transforming a DCTRS into the inverted one. The key features are that the inverted one is $\mathbb{U}_{\mathrm{opt}}$-LL if the DCTRS is $\mathbb{U}_{\mathrm{opt}}$-RL, and that the unraveled TRS of the inverted one is equivalent to the inverted unraveled TRS of the DCTRS if the DCTRS is $\mathbb{U}_{\mathrm{opt}}$-NE. The converse of this approach is impossible since the second key feature needs the $\mathbb{U}_{\mathrm{opt}}$-NE property (i.e., every $\mathbb{U}_{\mathrm{opt}}$-LL DCTRS does not imply the $\mathbb{U}_{\mathrm{opt}}$-NE property of the corresponding inverted DCTRS).

4.2 Soundness on Ultra-Left-Linearity

In this subsection, we show that the optimized unraveling $\mathbb{U}_{\mathrm{opt}}$ is sound for $\mathbb{U}_{\mathrm{opt}}$-LL DCTRSs if the reduction of the unraveled TRSs is restricted to the *EV-basic* one (see Definition 4.2). Roughly speaking, in an EV-basic reduction sequence, any redex introduced via extra variables at the application of rewrite rules is never reduced anywhere. Note that for eTRSs having no extra variables, the EV-basic property is not a restriction at all, since all of their reduction sequences are EV-basic. In practical cases (e.g., *inverse* TRSs [14, 19, 18, 16]), extra variables are instantiated with constructor terms. At the application of rewrite rules, extra variables in the unraveled eTRSs may introduce undesired terms, e.g., terms rooted by U symbols that are not reachable from terms over the original signature. As a consequence,

$\mathbb{U}_{\rm opt}$ is not always sound w.r.t. non-EV-basic reduction sequences of the unraveled eTRSs (see Example 4.7).

We first prepare a technical lemma to help us to prove the main lemma. Let X be a finite set of variables, σ and θ be substitutions, and $\to$ be a binary relation on terms. Then, we write $X\sigma \to X\theta$ if $x\sigma \to x\theta$ for any $x \in X$.

▶ **Lemma 4.1.** *Let R be an eDCTRS, $\rho : l \to r \Leftarrow s_1 \twoheadrightarrow t_1; \cdots ; s_k \twoheadrightarrow t_k$ be a $\mathbb{U}_{\rm opt}$-LL conditional rewrite rule in R, $\sigma_1, \cdots, \sigma_{k+1}$ be substitutions, and $X_i = \mathcal{V}ar(l, t_1, \cdots, t_{i-1}) \cap \mathcal{V}ar(r, t_i, s_{i+1}, t_{i+1}, \cdots, s_k, t_k)$ for all $1 \leq i \leq k$. If $s_i\sigma_i \to_R^* t_i\sigma_{i+1}$ and $X_i\sigma_i \to_R^* X_i\sigma_{i+1}$ for all $1 \leq i \leq k$, then $l\sigma_1 \to_R^+ r\sigma_{k+1}$.*

Proof. Let σ be the substitution $\sigma_1|_{\mathcal{V}ar(l)} \cup \sigma_2|_{X_1 \setminus \mathcal{V}ar(l)} \cup \cdots \cup \sigma_k|_{X_k \setminus X_{k-1}} \cup \sigma_{k+1}|_{\mathcal{V}ar(t_i,r) \setminus X_k}$. Then, we have that $l\sigma \equiv l\sigma_1$. It follows from $X_i\sigma_i \to_R^* X_i\sigma_{i+1}$ that $X_i\sigma \to_R^* X_i\sigma_{i+1}$ for all $1 \leq i \leq k$. It follows from the $\mathbb{U}_{\rm opt}$-LL property and Lemma 3.7 that $\mathcal{V}ar(t_i) \cap \big(\mathcal{D}om(\sigma_1|_{\mathcal{V}ar(l)}) \cup \cdots \cup \mathcal{D}om(\sigma_{i-1}|_{X_{i-1} \setminus X_{i-2}})\big) = \emptyset$ for all $1 \leq i \leq k$, and hence $t_i\sigma_i \equiv t_i\sigma$ for all $1 \leq i \leq k$. Thus we have that $s_i\sigma \to_R^* s_i\sigma_i \to_R^* t_i\sigma_{i+1} \equiv t_i\sigma$. Similarly, we have that $r\sigma \to_R^* r\sigma_{k+1}$. Therefore, we have that $l\sigma_1 \equiv l\sigma \to_R r\sigma \to_R^* r\sigma_{k+1}$. ◀

Next we define the notion of *EV-basic* (*EV-safe* [16, 14, 17]) reduction sequences of eTRSs. Roughly speaking, in an EV-basic reduction sequences, any redex introduced via extra variables are not reduced anywhere. This notion can be formalized by relaxing the notion of *basic* reduction sequences [7, 13].

▶ **Definition 4.2** (EV-basic reduction [16]). Let R be an eTRS and $\rho_i : l_i \to r_i \in R$ for all $i \geq 1$. Let $t_0 \to_{\rho_1,p_1,R} t_2 \to_{\rho_2,p_2,R} \cdots$ be a reduction sequence of R, and $B_0 \subseteq \mathcal{P}os_{\mathcal{F}}(t_0)$ such that B_0 is prefix closed (i.e., if $p < q$ and $q \in B_0$ then $p \in B_0$). We define the sets $B_1, B_2, \cdots$ of positions from the sequence and B_0 inductively as $B_i = (B_{i-1} \setminus \{q \in B_{i-1} \mid q \geq p_i\}) \cup \{p_iq \mid q \in \mathcal{P}os_{\mathcal{F}}(r_i)\} \cup \{p_ip'q \mid p_ipq \in B_{i-1},\ p \in \mathcal{P}os_{\mathcal{V}}(l_i),\ l_i|_p \equiv r_i|_{p'}\}$ for all $i \geq 1$. Note that $B_1, B_2, \cdots$ are prefix closed. For all $i \geq 0$, positions in B_i are referred as *basic positions of t_i w.r.t. extra variables*. The reduction sequence above is said to be *based on B_0 w.r.t. extra variables* if $p_i \in B_{i-1}$ for all $i \geq 1$. If the sequence is finite with length n, then we denote it by $B_0 : t_0 \xrightarrow[\mathsf{evb}]{}{}^*_R B_n : t_n$ or $B_0 : t_0 \xrightarrow[\mathsf{evb}]{}{}^*_R t_n$. In particular, the reduction sequence is called *basic w.r.t. extra variables* (*EV-basic*, for short) if $B_0 = \mathcal{P}os_{\mathcal{F}}(t_0)$. If the EV-basic sequence is finite with length n, then we denote it by $t_0 \xrightarrow[\mathsf{evb}]{}{}^*_R t_n$.

Note that EV-basicness is different from *basicness* [7, 13] in the sense that all the basic positions are propagated at the application of rewrite rules but none of the positions for extra variables are added to basic positions. A typical instance of EV-basic reduction sequences is a reduction sequence obtained by substituting a normal form for each extra variable when applying rewrite rules.

To specify a set of terms with which extra variables are possibly instantiated at the rule application, we introduce the notion of *EV-instantiation on sets of terms.* Let R be an eTRS and T be a set of terms. A derivation $t_0 \to_{\rho_1,p_1,R} t_1 \to_{\rho_2,p_2,R} \cdots$ is called *EV-instantiated on T* if any extra variable of $\rho_i : l_i \to r_i$ is instantiated by a term in T, i.e., $t_i|_{p_iq} \in T$ for any $q \in \mathcal{P}os_{\mathcal{V}}(r_i)$ such that $r_i|_q \in \mathcal{EV}ar(\rho_i)$. By the same token, the notion of the EV-instantiation property is defined for the parallel reduction of eTRSs. For any of the unraveled eTRSs, its EV-basic reduction sequences have the following property related to EV-instantiation on the set of terms over the original signature.

▶ **Lemma 4.3.** *Let R be a $\mathbb{U}_{\rm opt}$-LL eDCTRS over a signature $\mathcal{F}$, and s,t be terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$. If $s \xrightarrow[\mathsf{evb}]{}{}^*_{\mathbb{U}_{\rm opt}(R)} t$ then there exists a derivation $s \to^*_{\mathbb{U}_{\rm opt}(R)} t$ that is EV-instantiated on $\mathcal{T}(\mathcal{F},\mathcal{V})$.*

Proof. It can be proved by induction on the term structure that for a term s, a linear term l with U-symbol-free proper subterms, and substitutions θ, σ, η such that $\theta \in \mathcal{S}ub(\mathcal{F}, \mathcal{V})$ and $\mathsf{root}(x\delta)$ is a U symbol for any $x \in \mathcal{D}om(\eta)$, if $s\theta\eta \equiv l\sigma$, then there exists a substitution σ' such that $s\theta \equiv l\sigma'$ and $l\sigma \equiv l\sigma'\eta$.

To prove this lemma, it suffices to show that for terms $s \in \mathcal{T}(\mathcal{F}_{\mathbb{U}_{\mathrm{opt}}(R)}, \mathcal{V})$ and $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ and for substitutions θ and η such that $\theta \in \mathcal{S}ub(\mathcal{F}_{\mathbb{U}_{\mathrm{opt}}(R)}, \mathcal{V})$ and $\mathsf{root}(x\eta)$ is a U symbol for any $x \in \mathcal{D}om(\eta)$, if $\mathcal{P}os_{\mathcal{F}}(s\theta) : s\theta\eta \xrightarrow[\mathsf{evb}\ \mathbb{U}_{\mathrm{opt}}(R)]{n} t$ then there exists a substitution $\sigma \in \mathcal{S}ub(\mathcal{F}, \mathcal{V})$ such that $s\theta\sigma \to^{n}_{\mathbb{U}_{\mathrm{opt}}(R)} t$ and the derivation is EV-instantiated on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. We prove this claim by induction on n.

Suppose that $\mathcal{P}os_{\mathcal{F}}(s\theta) : s\theta\eta \xrightarrow[\mathsf{evb}\ \mathbb{U}_{\mathrm{opt}}(R)]{n} t$. From the EV-basic property of the derivation and the above claim, we can assume w.l.o.g. that $s\theta$ is of the form $C[s']_p$, $s\theta\eta \equiv C\theta\eta[s'\eta]$ $\equiv C\theta\eta[l\delta\eta]_p \to_{\rho,p,\mathbb{U}_{\mathrm{opt}}(R)} C\theta\eta[r\delta\eta] \to^{n-1}_{\mathbb{U}_{\mathrm{opt}}(R)} t$, $\delta \in \mathcal{S}ub(\mathcal{F}, \mathcal{V})$, $s' \equiv l\delta$, and $p \in \mathcal{P}os_{\mathcal{F}}(s\theta)$, where ρ is $l \to r$, $\mathcal{V}ar(l, r) \cap \mathcal{V}ar(s\theta) = \emptyset$, the set B of EV-basic positions in $C\theta\eta[r\delta\eta]$ is $(\mathcal{P}os_{\mathcal{F}}(s\theta) \setminus \{q \in \mathcal{P}os_{\mathcal{F}}(s\theta) \mid p \leq q\}) \cup \{pq \mid q \in \mathcal{P}os_{\mathcal{F}}(r)\} \cup \{pp'q \mid pp''q \in \mathcal{P}os_{\mathcal{F}}(s\theta),\ p'' \in \mathcal{P}os_{\mathcal{V}}(l),\ l|_{p''} \equiv r|_{p'}\}$, and $B : C\theta\eta[r\delta\eta] \xrightarrow[\mathsf{evb}\ \mathbb{U}_{\mathrm{opt}}(R)]{n-1} t$. Let δ' and δ'' be substitutions such that $\delta' \in \mathcal{S}ub(\mathcal{F}, \mathcal{V})$, $\delta|_{\mathcal{EV}ar(\rho)} = \delta'\eta$, $\mathcal{D}om(\delta'') \cap (\mathcal{V}ar(l, r) \cup \mathcal{D}om(\eta)) = \emptyset$, and $\mathsf{root}(x\delta'')$ is a U symbol for any $x \in \mathcal{D}om(\delta'')$.

Let $\theta' = \theta|_{\mathcal{V}ar(C[\,])} \cup \delta_{\mathcal{V}ar(l)} \cup \delta'|_{\mathcal{EV}ar(\rho)}$ and $\eta' = \eta \cup \delta''$. Then, θ' and η' are substitutions such that $C\theta\eta[r\delta] \equiv (C[r])\theta'\eta'$. By the definition of the EV-basic property, we have that B $= \mathcal{P}os_{\mathcal{F}}((C[r])\theta')$. Thus, by the induction hypothesis, we have that there exists a substitution σ in $\mathcal{S}ub(\mathcal{F}, \mathcal{V})$ such that $(C[r])\theta'\sigma \to^{n-1}_{\mathbb{U}_{\mathrm{opt}}(R)} t$ and the derivation is EV-instantiated on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Now we have that $s\theta\sigma \equiv (C\theta[s'])\sigma \equiv (C\theta[l\delta])\sigma \equiv (C\theta'[l\theta'])\sigma \to_{\mathbb{U}_{\mathrm{opt}}(R)} (C\theta'[r\theta'])\sigma$ $\equiv (C[r])\theta'\sigma' \to^{n-1}_{\mathbb{U}_{\mathrm{opt}}(R)} t$. Since θ and σ are in $\mathcal{S}ub(\mathcal{F}, \mathcal{V})$, any extra variables in r is instantiated by a term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Therefore, this derivation is EV-instantiated on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. ◀

The soundness result of this subsection is a consequence of the following key lemma.

▶ **Lemma 4.4.** *Let R be a $\mathbb{U}_{\mathrm{opt}}$-LL eDCTRS over a signature $\mathcal{F}$, s be a term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$, t be a linear term in $\mathcal{T}(\mathcal{F}_{\mathbb{U}_{\mathrm{opt}}(R)}, \mathcal{V})$, and σ be a substitution in $\mathcal{S}ub(\mathcal{F}_{\mathbb{U}_{\mathrm{opt}}(R)}, \mathcal{V})$. Suppose that R is non-LV or non-collapsing. If $s \rightrightarrows^{n}_{\mathbb{U}_{\mathrm{opt}}(R)} t\sigma$ for some $n \geq 0$ and the derivation is EV-instantiated on $\mathcal{T}(\mathcal{F}, \mathcal{V})$, then there exists a substitution θ in $\mathcal{S}ub(\mathcal{F}, \mathcal{V})$ such that $s \to^{*}_{R} t\theta \rightrightarrows^{m}_{\mathbb{U}_{\mathrm{opt}}(R)} t\sigma$ and the derivation $t\theta \rightrightarrows^{m}_{\mathbb{U}_{\mathrm{opt}}(R)} t\sigma$ is EV-instantiated on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ for some $m \leq n$ such that if $t\sigma \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $m = 0$.*

Proof. We prove this lemma by induction on the lexicographic product (n, s) of n and the structure of s. Suppose that $s \rightrightarrows^{n}_{\mathbb{U}_{\mathrm{opt}}(R)} t\sigma$. Since the case that s is a variable is trivial, we only consider the remaining case that s is rooted by a function symbol.

We first consider the case that $s \rightrightarrows^{n}_{\mathbb{U}_{\mathrm{opt}}(R)} t\sigma$ does not contain any reduction step at the root position. Let s be of the form $f(s_1, \cdots, s_k)$. Then, we have that $s \equiv f(s_1, \cdots, s_k)$ $\rightrightarrows^{n}_{\mathbb{U}_{\mathrm{opt}}(R)} f(t_1, \cdots, t_k)\sigma \equiv t\sigma$ and thus $s_i \rightrightarrows^{n_i}_{\mathbb{U}_{\mathrm{opt}}(R)} t_i\sigma$, where $n_1 + \cdots + n_k = n$. By the induction hypothesis, there exists a substitution $\theta_i \in \mathcal{S}ub(\mathcal{F}, \mathcal{V})$ such that $s_i \to^{*}_{R} t_i\theta_i$ $\rightrightarrows^{m_i}_{\mathbb{U}_{\mathrm{opt}}(R)} t_i\sigma$ and the derivation $t_i\theta_i \rightrightarrows^{m_i}_{\mathbb{U}_{\mathrm{opt}}(R)} t_i\sigma$ is EV-instantiated on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ for some m_i $\leq n_i$ such that $m_i = 0$ if $t_i\sigma \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Let $\theta = \theta_1|_{\mathcal{V}ar(t_1)} \cup \cdots \cup \theta_k|_{\mathcal{V}ar(t_k)}$. Then, it follows from the linearity of t that θ is a substitution in $\mathcal{S}ub(\mathcal{F}, \mathcal{V})$. We have that $s \equiv f(s_1, \cdots, s_k)$ $\to^{*}_{R} f(t_1, \cdots, t_k)\theta \equiv t\theta \rightrightarrows^{m}_{\mathbb{U}_{\mathrm{opt}}(R)} t\sigma$ and the derivation $t\theta \rightrightarrows^{m}_{\mathbb{U}_{\mathrm{opt}}(R)} t\sigma$ is EV-instantiated on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. where $m = m_1 + \cdots + m_k \leq n$ such that if $t\sigma \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $m = 0$.

Next we consider the remaining case. To simplify the proof, we assume w.l.o.g. that any rule has two conditions of the form $s_1 \twoheadrightarrow t_1; s_2 \twoheadrightarrow t_2$. Then, we can assume that

$$\begin{aligned} s \rightrightarrows^{n_0}_{\mathbb{U}_{\rm opt}(R)} l\sigma_1 \to_{\varepsilon,\mathbb{U}_{\rm opt}(R)} u_1\sigma_1 \rightrightarrows^{n_1}_{>\varepsilon,\mathbb{U}_{\rm opt}(R)} u'_1\sigma_2 \\ \to_{\varepsilon,\mathbb{U}_{\rm opt}(R)} u_2\sigma_2 \rightrightarrows^{n_2}_{>\varepsilon,\mathbb{U}_{\rm opt}(R)} u'_2\sigma_3 \to_{\varepsilon,\mathbb{U}_{\rm opt}(R)} r\sigma_3 \rightrightarrows^{n_3}_{\mathbb{U}_{\rm opt}(R)} t_3\sigma_3, \end{aligned}$$

where $s \rightrightarrows^{n_0}_{>\varepsilon,\mathbb{U}_{\rm opt}(R)} l\sigma_1$ if R is non-LV, and $r\sigma_3 \rightrightarrows^{n_3}_{>\varepsilon,\mathbb{U}_{\rm opt}(R)} r\sigma$ otherwise (i.e., if R is non-collapsing), $\rho : l \to r \Leftarrow s_1 \twoheadrightarrow t_1; s_2 \twoheadrightarrow t_2 \in R$, $u_i \equiv U^\rho_i(s_i, \overrightarrow{X_i})$, $u'_i \equiv U^\rho_i(t_i, \overrightarrow{X_i})$, $X_1 = \mathcal{V}ar(l) \cap \mathcal{V}ar(r, t_1, s_2, t_2)$, $X_2 = \mathcal{V}ar(l, t_1) \cap \mathcal{V}ar(r)$, and $t\sigma$ is a term between $u_1\sigma$ to $t_3\sigma_3$. We only consider the case that $t\sigma$ is $t_3\sigma_3$ since this case is the most complicated. For this reason, we assume that $t_3\sigma_3 \equiv t\sigma$ and $n_0 + n_1 + n_2 + n_3 + 3 = n$.

By the induction hypothesis, there exists a substitution $\theta_1 \in \mathcal{S}ub(\mathcal{F}, \mathcal{V})$ such that $s \to^*_R l\theta \rightrightarrows^{m_0}_{\mathbb{U}_{\rm opt}(R)} l\sigma_1$ and the derivation $l\theta \rightrightarrows^{m_0}_{\mathbb{U}_{\rm opt}(R)} l\sigma_1$ is EV-instantiated on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Let $\theta'_1 = \theta|_{\mathcal{V}ar(l)} \cup \sigma_1|_{\mathcal{EV}ar(l \to u_1)}$. Then, θ'_1 is a substitution in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Moreover, it follows from the standard property of the parallel reduction that $u_1\theta'_1 \rightrightarrows^{m_1}_{>\varepsilon,\mathbb{U}_{\rm opt}(R)} u_1\sigma_1 \rightrightarrows^{n_2}_{>\varepsilon,\mathbb{U}_{\rm opt}(R)} u'_1\sigma_2$. Thus, we have that $s_1\theta'_1 \rightrightarrows^{m'_1}_{\mathbb{U}_{\rm opt}(R)} t_1\sigma_2$ and $X_1\theta'_1 \rightrightarrows^{m''_1}_{\mathbb{U}_{\rm opt}(R)} X_1\sigma_2$ where m''_1 is the summation of reduction steps and $m'_1 + m''_1 = m_1 + n_2$. Since the $\mathbb{U}_{\rm opt}$-LL property provides $\mathcal{V}ar(t_1) \cap X_1 = \emptyset$, by the induction hypothesis, there exists a substitution $\theta_2 \in \mathcal{S}ub(\mathcal{F}, \mathcal{V})$ such that $s_1\theta'_1 \to^*_R t_1\theta_2 \rightrightarrows^{m_2}_{\mathbb{U}_{\rm opt}(R)} t_1\sigma_2$ and $X_1\theta'_1 \to^*_R X_1\theta_2 \rightrightarrows^{j_2}_{\mathbb{U}_{\rm opt}(R)} X_1\sigma_2$ where j_2 is the summation of reduction steps and $m_2 + j_2 \le m_1 + n_2$.

In the same way, we obtain substitutions θ_3 and θ in $\mathcal{S}ub(\mathcal{F}, \mathcal{V})$ such that $s_2\theta_2 \to^*_R t_2\theta_3$, $X_2 \to^*_R X_2\theta_3$, $r\theta_3 \to^*_R t\theta \rightrightarrows^{m}_{\mathbb{U}_{\rm opt}(R)} t\sigma$, where $m \le n$ such that if $t\sigma \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $m = 0$. It follows from Lemma 4.1 that $l\theta_1 \to^*_R r\theta_3$. Therefore, we have that $s \to^*_R r\theta_3 \to^*_R t\theta \rightrightarrows^{m}_{\mathbb{U}_{\rm opt}(R)} t\sigma$ where $m \le n$ such that if $t\sigma \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $m = 0$. ◀

As a consequence of Lemma 4.4, we show the main theorem of this subsection.

▶ **Theorem 4.5.** $\mathbb{U}_{\rm opt}$ *is sound for a* $\mathbb{U}_{\rm opt}$*-LL eDCTRSs* R *over a signature* $\mathcal{F}$ *if* R *is non-LV or non-collapsing and if the reduction of* $\mathbb{U}_{\rm opt}(R)$ *is restricted to the EV-basic one (i.e., for any terms* s *and* t *in* $\mathcal{T}(\mathcal{F}, \mathcal{V})$*, if* $s \xrightarrow[\mathsf{evb}]{}{}^*_{\mathbb{U}_{\rm opt}(R)} t$ *then* $s \to^*_R t$*).*

Proof. Suppose that $s \xrightarrow[\mathsf{evb}]{}{}^*_{\mathbb{U}_{\rm opt}(R)} t$ and $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Then, it follows from Lemma 4.3 that there is a derivation $s \to^*_{\mathbb{U}_{\rm opt}(R)} t$ that is EV-instantiated on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Since a single step of $\to_{\mathbb{U}_{\rm opt}(R)}$ can be considered as a single step of the parallel reduction, we have the derivation $s \rightrightarrows^*_{\mathbb{U}_{\rm opt}(R)} t$ that is EV-instantiated on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Let x be a variable and σ be a substitution such that $x\sigma \equiv t$. Then, it follows from Lemma 4.4 that $s \to^*_R x\sigma \equiv t$. ◀

It is clear that for a 3-eDCTRS R, any reduction sequence of R is EV-basic. Therefore, $\mathbb{U}_{\rm opt}$ is sound for $\mathbb{U}_{\rm opt}$-LL 3-eDCTRSs.

▶ **Corollary 4.6.** $\mathbb{U}_{\rm opt}$ *is sound for* $\mathbb{U}_{\rm opt}$*-LL 3-eDCTRSs that are non-LV or non-collapsing.*

Due to the technical proof of Lemma 4.4, we assumed that eDCTRS is non-LV or non-collapsing. It is not known yet that this assumption can be relaxed (or removed). However, this assumption is not so restrictive since every DCTRS (not an extended one) is non-LV. Corollary 4.6 is not a direct consequence of the result in [12] on soundness for $\mathbb{U}_{\rm opt}$-LL 3-DCTRSs since U symbols introduced by $\mathbb{U}_{\rm opt}$ have less arguments than those introduced by Marchiori's unraveling for DCTRSs.

Finally, we show a counterexample against Theorem 4.5 without the EV-basic property.

▶ **Example 4.7.** Consider the following DCTRS and its unraveled EV-TRS:

$$R_4 = \{\ \mathsf{e} \to \mathsf{f}(x) \Leftarrow \mathsf{l} \twoheadrightarrow \mathsf{d}, \quad \mathsf{A} \to \mathsf{h}(x,x)\ \}$$
$$\mathbb{U}_{\mathrm{opt}}(R_4) = \mathbb{U}(R_4) = \{\ \mathsf{e} \to \mathsf{U}_5(\mathsf{l}), \quad \mathsf{U}_5(\mathsf{d}) \to \mathsf{f}(x), \quad \mathsf{A} \to \mathsf{h}(x,x)\ \}$$

We have the derivation $\mathsf{A} \to_{\mathbb{U}_{\mathrm{opt}}(R_4)} \mathsf{h}(\mathsf{U}_5(\mathsf{d}), \mathsf{U}_5(\mathsf{d})) \to^*_{\mathbb{U}_{\mathrm{opt}}(R_4)} \mathsf{h}(\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}))$ that is not EV-basic: the term $\mathsf{U}_5(\mathsf{d})$ introduced by instantiating the extra variable x in the applied rule $\mathsf{A} \to \mathsf{h}(x,x)$ is reduced. However, A cannot be reduced by R_4 to $\mathsf{h}(\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}))$. Therefore, $\mathbb{U}_{\mathrm{opt}}$ is not sound for R_4. Note that $\mathbb{U}$ is not sound either.

4.3 Soundness on Ultra-Right-Linear-Non-Erasing Property

In this subsection, we show that $\mathbb{U}_{\mathrm{opt}}$ is sound for DCTRSs that are ultra-RL-NE w.r.t. $\mathbb{U}_{\mathrm{opt}}$. To prove it, we reduce the soundness to that of $\mathbb{U}_{\mathrm{opt}}$ for ultra-LL DCTRSs.

We first define the operation to transform an eDCTRS into an eDCTRS that defines the inverse relation of the former eDCTRS. Note that the "inverse" has a different meaning from the sense of program inversion.

▶ **Definition 4.8.** Let $\rho : l \to r \Leftarrow s_1 \twoheadrightarrow t_1; \cdots ; s_k \twoheadrightarrow t_k$ be an (extended) conditional rewrite rule. We define the operation $(\ )^{-1}$ as $(l \to r \Leftarrow s_1 \twoheadrightarrow t_1; \cdots ; s_k \twoheadrightarrow t_k)^{-1} = r \to l \Leftarrow t_k \twoheadrightarrow s_k; \cdots ; t_1 \twoheadrightarrow s_1$. This operation is extended to eDCTRSs as $(R)^{-1} = \{(\rho)^{-1} \mid \rho \in R\}$. Moreover, for a binary relation $\to$, we denote the inverse relation of $\to$ by $(\to)^{-1}$: $(\to)^{-1} = \{(t,s) \mid s \to t\}$.

For an eCTRS R, the inverse relation of $\to_R$ is equivalent to the reduction of $(R)^{-1}$.

▶ **Theorem 4.9.** *Let R be an eCTRS. Then, $(\to_R)^{-1} = \to_{(R)^{-1}}$.*

Proof. It suffices to show that $(\to_{(n),R})^{-1} = \to_{(n),(R)^{-1}}$ for all $n \geq 0$. This can be proved by induction on n. ◀

Regarding the operation $(\ )^{-1}$ and the $\mathbb{U}_{\mathrm{opt}}$-NE property, there are dual relationships between $\mathbb{U}_{\mathrm{opt}}$-LL and $\mathbb{U}_{\mathrm{opt}}$-RL and between the non-LV and non-collapsing properties.

▶ **Lemma 4.10.** *Let $\rho : l \to r \Leftarrow s_1 \twoheadrightarrow t_1; \cdots ; s_k \twoheadrightarrow t_k$ be an extended deterministic rewrite rule. Then all of the following hold:*
1. *$\mathcal{V}ar(t_i) \subseteq \mathcal{V}ar(r, s_{i+1}, \cdots, s_k)$ for all $1 \leq i \leq k$ iff $(\rho)^{-1}$ is deterministic,*
2. *$\mathcal{V}ar(l) \subseteq \mathcal{V}ar(r, s_1, \cdots, s_k)$ iff $(\rho)^{-1}$ is in Type 3,*
3. *if $\mathcal{V}ar(t_i) \subseteq \mathcal{V}ar(r, s_{i+1}, \cdots, s_k)$ for all $1 \leq i \leq k$, then*
   - **a.** *$\mathbb{U}_{\mathrm{opt}}((\rho)^{-1}) = (\mathbb{U}_{\mathrm{opt}}(\rho))^{-1}$ up to the renaming of U symbols, and*
   - **b.** *ρ is $\mathbb{U}_{\mathrm{opt}}$-LL ($\mathbb{U}_{\mathrm{opt}}$-RL, resp.) iff $(\rho)^{-1}$ is $\mathbb{U}_{\mathrm{opt}}$-RL ($\mathbb{U}_{\mathrm{opt}}$-LL, resp.),*
4. *ρ is non-LV (non-collapsing, resp.) iff $(\rho)^{-1}$ is non-collapsing (non-LV, resp.).*

Proof. The claims 1, 2 and 4 are trivial. Consider $\mathbb{U}_{\mathrm{opt}}(\rho)$ in Definition 3.2. We can assume w.l.o.g. that $\mathbb{U}_{\mathrm{opt}}((\rho)^{-1}) = \{r \to U^\rho_k(t_k, \overrightarrow{Y_k}), \cdots, U^\rho_2(s_2, \overrightarrow{Y_2}) \to U^\rho_1(t_1, \overrightarrow{Y_1}), U^\rho_1(s_1, \overrightarrow{Y_1}) \to l\}$ where $Y_i = \mathcal{V}ar(r, s_k, \cdots, s_{i+1}) \cap \mathcal{V}ar(l, s_i, t_{i-1}, s_{i-1}, \cdots, t_1, s_1)$. Since ρ is deterministic, we have that $\mathcal{V}ar(s_i) \subseteq \mathcal{V}ar(l, t_1, \cdots, t_{i-1})$ for all $1 \leq i \leq k$.

To prove the claim 3-a, it suffices to show that $X_i = Y_i$ for all $1 \leq i \leq k$. It follows from $\mathcal{V}ar(s_i) \subseteq \mathcal{V}ar(l, t_1, \cdots, t_{i-1})$ that $\mathcal{V}ar(l, s_1, t_1, \cdots, s_{i-1}, t_{i-1}, s_i) = \mathcal{V}ar(l, t_1, \cdots, t_{i-1})$, and hence $Y_i = \mathcal{V}ar(r, s_{i+1}, \cdots, s_k) \cap \mathcal{V}ar(l, t_1, \cdots, t_{i-1})$. Moreover, it follows from $\mathcal{V}ar(t_i) \subseteq \mathcal{V}ar(r, s_{i+1}, \cdots, s_k)$ that $\mathcal{V}ar(r, t_i, s_{i+1}, t_{i+1}, \cdots, s_k, t_k) = \mathcal{V}ar(r, s_{i+1}, \cdots, s_k)$, and hence $X_i = \mathcal{V}ar(l, t_1, \cdots, t_{i-1}) \cap \mathcal{V}ar(r, s_{i+1}, \cdots, s_k)$. Therefore, $X_i = Y_i$ for all $1 \leq i \leq k$.

Finally, we prove the claim 3-b. Suppose that ρ is $\mathbb{U}_{\mathrm{opt}}$-LL. Then, it follows from Lemma 3.7 that l, $U_1^\rho(t_1, \overrightarrow{X_1}), \cdots, U_k^\rho(t_k, \overrightarrow{X_k})$ are linear. Thus, $\mathbb{U}_{\mathrm{opt}}((\rho)^{-1})$ is right-linear and hence $(\rho)^{-1}$ is $\mathbb{U}_{\mathrm{opt}}$-RL. Suppose that ρ is $\mathbb{U}_{\mathrm{opt}}$-RL. Then, it follows from Lemma 3.7 that $r, U_1^\rho(s_1, \overrightarrow{X_1}), \cdots, U_k^\rho(s_k, \overrightarrow{X_k})$ are linear. Thus, $\mathbb{U}_{\mathrm{opt}}((\rho)^{-1})$ is left-linear and hence $(\rho)^{-1}$ is $\mathbb{U}_{\mathrm{opt}}$-LL. The *if* part is similar to the *only-if* part above. ◀

Note that neither the claims 3-a nor 3-b in Lemma 4.10 holds for $\mathbb{U}$.

▶ **Corollary 4.11.** *Let R be an eDCTRS. Then all of the following hold:*
- *R is $\mathbb{U}_{\mathrm{opt}}$-NE iff $(R)^{-1}$ is a 3-eDCTRS,*
- *if R is $\mathbb{U}_{\mathrm{opt}}$-NE, then*
  - *$\mathbb{U}_{\mathrm{opt}}((R)^{-1}) = (\mathbb{U}_{\mathrm{opt}}(R))^{-1}$ up to the renaming of U symbols, and*
  - *R is $\mathbb{U}_{\mathrm{opt}}$-LL ($\mathbb{U}_{\mathrm{opt}}$-RL, resp.) iff $(R)^{-1}$ is $\mathbb{U}_{\mathrm{opt}}$-RL ($\mathbb{U}_{\mathrm{opt}}$-LL, resp.), and*
- *R is non-LV (non-collapsing, resp.) iff $(R)^{-1}$ is non-collapsing (non-LV, resp.).*

Finally, we show soundness of $\mathbb{U}_{\mathrm{opt}}$ for a $\mathbb{U}_{\mathrm{opt}}$-RL-NE eDCTRS R by reducing it to soundness for the $\mathbb{U}_{\mathrm{opt}}$-LL eDCTRS $(R)^{-1}$.

▶ **Theorem 4.12.** *$\mathbb{U}_{\mathrm{opt}}$ is sound for $\mathbb{U}_{\mathrm{opt}}$-RL-NE eDCTRSs that are non-LV or non-collapsing.*

Proof. Let R be a $\mathbb{U}_{\mathrm{opt}}$-RL-NE eDCTRS over a signature $\mathcal{F}$. Then, it follows from Corollary 4.11 that $(R)^{-1}$ is a $\mathbb{U}_{\mathrm{opt}}$-LL 3-eDCTRS that is non-collapsing or non-LV. Thus, it follows from Corollary 4.6 that $\mathbb{U}_{\mathrm{opt}}$ is sound for $(R)^{-1}$, i.e., $\to^*_{\mathbb{U}_{\mathrm{opt}}((R)^{-1})} \subseteq \to^*_{(R)^{-1}}$ holds over $\mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V})$. It follows from Corollary 4.11 that $\mathbb{U}_{\mathrm{opt}}((R)^{-1}) = (\mathbb{U}_{\mathrm{opt}}(R))^{-1}$, and hence $\to^*_{\mathbb{U}_{\mathrm{opt}}((R)^{-1})} = \to^*_{(\mathbb{U}_{\mathrm{opt}}(R))^{-1}}$. By Theorem 4.9, we have that $\to^*_{(\mathbb{U}_{\mathrm{opt}}(R))^{-1}} = (\to^*_{\mathbb{U}_{\mathrm{opt}}(R)})^{-1}$ and $\to^*_{(R)^{-1}} = (\to^*_R)^{-1}$. Thus, we have that $(\to^*_{\mathbb{U}_{\mathrm{opt}}(R)})^{-1} \subseteq (\to^*_R)^{-1}$ (i.e., $\to^*_{\mathbb{U}_{\mathrm{opt}}(R)} \subseteq \to^*_R$) over $\mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V})$. ◀

▶ **Example 4.13.** Consider R_2 in Example 3.3 again. The eDCTRS R_2 is non-LV, $\mathbb{U}_{\mathrm{opt}}$-RL-NE but neither $\mathbb{U}$-RL nor $\mathbb{U}$-LL. Thanks to Theorem 4.12, $\mathbb{U}_{\mathrm{opt}}$ is sound for R_2 and thus $\mathbb{U}_{\mathrm{opt}}(R_2)$ can be used for simulating the reduction of R_2.

Ultra-right-linearity is not a soundness condition for $\mathbb{U}_{\mathrm{opt}}$.

▶ **Example 4.14.** Consider the 3-DCTRS $(R_4)^{-1}$ and the unraveled TRS $(\mathbb{U}_{\mathrm{opt}}(R_4))^{-1}$ obtained from Example 4.7. $(R_4)^{-1}$ is $\mathbb{U}_{\mathrm{opt}}$-RL but not $\mathbb{U}_{\mathrm{opt}}$-NE. We have the derivation $\mathsf{h}(\mathsf{f}(\mathsf{a}),\mathsf{f}(\mathsf{b})) \to^*_{\mathbb{U}_{\mathrm{opt}}((R_4)^{-1})} \mathsf{A}$. However, $\mathsf{h}(\mathsf{f}(\mathsf{a}),\mathsf{f}(\mathsf{b}))$ cannot be reduced by $(R_4)^{-1}$ to A. Therefore, $\mathbb{U}_{\mathrm{opt}}$ is not sound for $(R_4)^{-1}$. Note that $\mathbb{U}$ is sound for $(R_4)^{-1}$.

It is possible to prove Theorem 4.12 directly [17], by using the feature that every reduction sequence of right-linear TRSs can be transformed to a *basic* one [13]. However, Theorem 4.5 cannot be proved by using Theorem 4.12. This is because $\mathbb{U}_{\mathrm{opt}}((R)^{-1}) = (\mathbb{U}_{\mathrm{opt}}(R))^{-1}$ does not hold for every $\mathbb{U}_{\mathrm{opt}}$-LL DCTRSs (see $\mathbb{U}_{\mathrm{opt}}(R_2)$ in Example 3.3).

4.4 Soundness of Ohlebusch's Unraveling

In this subsection, we show that soundness of $\mathbb{U}_{\mathrm{opt}}$ implies that of $\mathbb{U}$.

We first introduce the notion of *argument filterings* [1, 8]. An *argument filtering* over a signature $\mathcal{F}$ is a mapping π from $\mathcal{F}$ to sets of natural numbers such that $\pi(f) \subseteq \{1, \cdots, \mathsf{ar}(f)\}$ for any $f \in \mathcal{F}$. Note that this paper does not use *collapsing definitions* $\pi(f) \in \{1, \cdots, \mathsf{ar}(f)\}$. When $\pi(f)$ is not defined explicitly, we assume that $\pi(f) = \{1, \cdots, \mathsf{ar}(f)\}$. Argument filterings are extended to terms as follows: $\pi(x) = x$ for $x \in \mathcal{V}$, and $\pi(f(t_1, \cdots, t_n)) = f(\pi(t_{i_1}), \cdots, \pi(t_{i_m}))$ for $f \in \mathcal{F}$ where $\pi(f) = \{i_1, \cdots, i_m\}$ and $1 \leq i_1 < i_2 < \cdots < i_m \leq n$.

They are also extended to eTRSs as follows: $\pi(R) = \{\pi(l) \to \pi(r) \mid l \to r \in R\}$. Note that $\pi(R)$ is an eTRS. Argument filterings have the following properties.

▶ **Lemma 4.15.** *Let π be an argument filtering. Let t be a term and σ, σ_π be substitutions such that $\sigma_\pi = \{x \mapsto \pi(x\sigma) \mid x \in \mathcal{D}om(\sigma)\}$. Then, $\pi(t\sigma) \equiv (\pi(t))\sigma_\pi$. Let R be an eTRS and s, t be terms. If $s \to_R^* t$ then $\pi(s) \to_{\pi(R)}^* \pi(t)$.*

The unraveling $\mathbb{U}_{\rm opt}$ is an optimized variant of $\mathbb{U}$ in the sense that variables carried by U symbols are optimized. Thus, $\mathbb{U}_{\rm opt}(R)$ can simulate any reduction sequence of $\mathbb{U}(R)$.

▶ **Lemma 4.16.** *Let R be an eDCTRS over a signature $\mathcal{F}$, and s, t be terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. If $s \to_{\mathbb{U}(R)}^* t$, then $s \to_{\mathbb{U}_{\rm opt}(R)}^* t$.*

Proof. We assume w.l.o.g. that for every rule $\rho : l \to r \Leftarrow s_1 \twoheadrightarrow t_1; \cdots ; s_k \twoheadrightarrow t_k$ in R, the same U symbols $U_1^\rho, \cdots, U_k^\rho$ are introduced for $\mathbb{U}(R)$ and $\mathbb{U}_{\rm opt}(R)$. Let π be an argument filtering such that for every $\rho : l \to r \Leftarrow s_1 \twoheadrightarrow t_1; \cdots ; s_k \twoheadrightarrow t_k$ in R, $\pi(U_i^\rho)$ $= \{1, i_1, \cdots, i_m\}$ where $X_i = \mathcal{V}ar(l, t_1, \cdots, t_{i-1})$, $\overrightarrow{X_i}$ is a sequence $x_1, \cdots, x_n$, $Y_i = X_i \cap \mathcal{V}ar(r, t_i, s_{i+1}, t_{i+1}, \cdots, s_k, t_k)$, $\overrightarrow{Y_i}$ is a sequence $y_1, \cdots, y_m$, and $x_{i_j} \equiv y_j$ for all $1 \le j \le m$. Then, it is clear that $\pi(\mathbb{U}(R)) = \mathbb{U}_{\rm opt}(R)$. Since s, t are in $\mathcal{T}(\mathcal{F}, \mathcal{V})$, we have that $\pi(s) \equiv s$ and $\pi(t) \equiv t$. Thus, it follows from Lemma 4.15 that $s \equiv \pi(s) \to_{\mathbb{U}_{\rm opt}(R)}^* \pi(t) \equiv t$. ◀

Moreover, $\mathbb{U}_{\rm opt}(R)$ can simulate every EV-basic reduction sequence of $\mathbb{U}(R)$.

▶ **Lemma 4.17.** *Let R be an eTRS, s, t be terms, and π be an argument filtering such that $\mathcal{EV}ar(\pi(l) \to \pi(r)) \subseteq \mathcal{EV}ar(\rho)$ for every $\rho : l \to r \in R$. If $s \xrightarrow[\mathsf{evb}]{}{}_R^* t$ then $\pi(s) \xrightarrow[\mathsf{evb}]{}{}_{\pi(R)}^* \pi(t)$.*

Proof. We first define modifications for a position p of a term u and a set P of positions of u by applying an argument filtering π: $\pi_u(p) = p$ if $p = \varepsilon$; $\pi_u(p) = jp''$ if $u \equiv f(u_1, \cdots, u_n)$, $\pi(f) = \{i_1, \cdots, i_m\}$, $i_1 < \cdots < i_m$, $p = i_j p'$, and $p'' = \pi_{u_i}(p')$; $\pi_u(P) = P$ if $u \in \mathcal{V}$; $\pi_u(P) = \{\varepsilon \mid \varepsilon \in P\} \cup \left\{ jp' \mid p' \in \pi_{t_{i_j}}(\{p'' \mid i_j p'' \in P\}) \right\}$ if $u \equiv f(u_1, \cdots, u_n)$, $\pi(f) = \{i_1, \cdots, i_m\}$, $i_1 < \cdots < i_m$. We prove that if $B : s \xrightarrow[\mathsf{evb}]{}{}_R^n B' : t$ and $\pi_s(B) \subseteq B_1 \subseteq \mathcal{P}os(\pi(s))$ then $B_1 : \pi(s) \xrightarrow[\mathsf{evb}]{}{}_{\pi(R)}^* B_1' : \pi(t)$ and $\pi_t(B') \subseteq B_1'$. To prove this claim by induction on n, it suffices to show that if $B : s \xrightarrow[\mathsf{evb}]{}{}_{p,R} B' : t$ then $\pi_s(p)$ is defined and $\pi_s(B) : \pi(s) \xrightarrow[\mathsf{evb}]{}{}_{\pi_s(p),\pi(R)} \pi_t(B') : \pi(t)$. This follows from the assumption and the definitions of $\xrightarrow[\mathsf{evb}]{}$ and $\pi_t(\ )$. ◀

▶ **Lemma 4.18.** *Let R be an eDCTRS over a signature $\mathcal{F}$, and s, t be terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. If $s \xrightarrow[\mathsf{evb}]{}{}_{\mathbb{U}(R)}^* t$ then $s \xrightarrow[\mathsf{evb}]{}{}_{\mathbb{U}_{\rm opt}(R)}^* t$.*

Proof. Rules in $\mathbb{U}_{\rm opt}(R)$ that may contain extra variables are of the form $U_k^\rho(t_k, \overrightarrow{Y_k}) \to r$ where $\rho : l \to r \Leftarrow s_1 \twoheadrightarrow t_1; \cdots ; s_k \twoheadrightarrow t_k$ in R. By the definition of $\mathbb{U}$, we have that $\mathbb{U}_k^\rho(t_k, \overrightarrow{X_k}) \to r$ in $\mathbb{U}(R)$, $Y_k \subseteq X_k$, and $\mathcal{V}ar(r) \cap Y_k = \mathcal{V}ar(r) \cap X_k$. Thus, we have that $\mathcal{EV}ar(U_k^\rho(t_k, \overrightarrow{Y_k}) \to r) = \mathcal{V}ar(r) \setminus (\mathcal{V}ar(t_k) \cup Y_k) = \mathcal{V}ar(r) \setminus (\mathcal{V}ar(t_k) \cup X_k)$ $= \mathcal{EV}ar(U_k^\rho(t_k, \overrightarrow{X_k}) \to r)$. Thus, this theorem follows from Lemma 4.17. ◀

Finally, it can be said that soundness of $\mathbb{U}_{\rm opt}$ implies that of $\mathbb{U}$.

▶ **Theorem 4.19.** $\mathbb{U}$ *is sound for an eDCTRS (w.r.t. EV-basic reduction of* $\mathbb{U}(R)$*) if* $\mathbb{U}_{\rm opt}$ *is sound for the eDCTRS (w.r.t. EV-basic reduction of* $\mathbb{U}_{\rm opt}(R)$*).*

Proof. Let R be an eDCTRS over a signature $\mathcal{F}$, and s, t be terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that $s \to_{\mathbb{U}(R)}^* t$. Then, it follows from Lemma 4.16 that $s \to_{\mathbb{U}_{\rm opt}(R)}^* t$. Moreover, it follows from soundness of $\mathbb{U}_{\rm opt}$ that $s \to_R^* t$. Therefore, $\mathbb{U}$ is sound. In the same way, the case of the EV-basic reduction can be proved. ◀

▶ **Corollary 4.20.** *$\mathbb{U}$ is sound for 3-DCTRSs that are $\mathbb{U}$-LL or $\mathbb{U}_{\mathrm{opt}}$-RL-NE.*

The converse of Theorem 4.19 does not hold. For example, $\mathbb{U}$ is sound for the DCTRS $(R_4)^{-1}$ in Example 4.14 but $\mathbb{U}_{\mathrm{opt}}$ is not sound. The reason must be that U symbols introduced via the application of $\mathbb{U}$ have more variables (i.e., information) than the corresponding U symbols introduced by $\mathbb{U}_{\mathrm{opt}}$. Thus, $\mathbb{U}$ is sufficient to produce TRSs that can be used instead of the original DCTRSs. Though, $\mathbb{U}_{\mathrm{opt}}$ will be useful in investigating soundness of $\mathbb{U}$ since the unraveled TRSs obtained by $\mathbb{U}_{\mathrm{opt}}$ are simpler than those obtained by $\mathbb{U}$.

5 Conclusion

In this paper, we have shown that the unravelings for DCTRSs are sound for DCTRSs that are ultra-LL or ultra-RL-NE, and shown that Ohlebusch's unraveling is sound for a DCTRS if the optimized one is sound for the DCTRS. We have also shown necessary and sufficient syntactic conditions for ultra-LL, ultra-RL, and ultra-NE, respectively. Future work is to relax these syntactic conditions for the soundness, e.g., that each rule is ultra-LL or ultra-RL-NE.

Extending the results in [5], it is shown in [6] that $\mathbb{U}$ is sound for confluent 3-DCTRSs w.r.t. the reduction to normal forms, and $\mathbb{U}$ is sound for 3-DCTRSs that are $\mathbb{U}$-RL or weakly left-linear. For the case of $\mathbb{U}$-RL 3-DCTRSs, this result is incompatible with Theorem 4.12 (see Example 4.13). For the case of weakly left-linear 3-DCTRS, this result is strictly stronger than Corollary 4.6 since $\mathbb{U}_{\mathrm{opt}}$-LL 3-DCTRSs are weakly left-linear. Extending the results in [6] w.r.t. confluence and weak left-linearity to $\mathbb{U}_{\mathrm{opt}}$ is a further direction of this research.

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