A Reduction-Preserving Completion for Proving Confluence of Non-Terminating Term Rewriting Systems

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Abstract

We give a method to prove confluence of term rewriting systems that contain non-terminating rewrite rules such as commutativity and associativity. Usually, confluence of term rewriting systems containing such rules is proved by treating them as equational term rewriting systems and considering E-critical pairs and/or termination modulo E. In contrast, our method is based solely on usual critical pairs and usual termination. We first present confluence criteria for term rewriting systems whose rewrite rules can be partitioned into terminating part and possibly non-terminating part. We then give a reduction-preserving completion procedure so that the applicability of the criteria is enhanced. In contrast to the well-known Knuth-Bendix completion procedure which preserves the equivalence relation of the system, our completion procedure preserves the reduction relation of the system, by which confluence of the original system is inferred from that of the completed system.

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1 Introduction

Confluence is one of the most important properties of term rewriting systems (TRSs for short) and hence many efforts have been spent on developing techniques to prove this property [3, 15]. One of the classes of TRSs for which many known confluence proving methods are not effective is the class of TRSs containing associativity and commutativity rules (AC-rules). Such TRSs are non-terminating by the existence of AC-rules (more precisely, commutativity rules are self-looping and associativity rules are looping under the presence of commutativity rules) and hence the Knuth-Bendix criterion does not apply. Furthermore, confluence criteria regardless of termination based on critical pairs often do not apply either.

A well-known approach to deal with TRSs containing AC-rules is to deal them as equational term rewriting systems [6, 7, 13]. In this approach, non-terminating rules such as AC-rules are treated exceptionally as an equational subsystem $\mathcal{E}$. Then the confluence of equational term rewriting system $\langle \mathcal{R}, \mathcal{E} \rangle$ is obtained if $\mathcal{R}$ is terminating modulo $\mathcal{E}$ [6, 7, 13]

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and either $\mathcal{E}$-critical pairs of $\mathcal{R}$ satisfy certain conditions [7, 13] or $\mathcal{R}$ is left-linear and $\mathcal{E}/\mathcal{R}$-critical pairs satisfy a certain condition [6]. This approach, however, only works if $\mathcal{R}$ is terminating modulo $\mathcal{E}$. Furthermore, the computation of $\mathcal{E}$-critical pairs requires a finite and complete $\mathcal{E}$-unification algorithm which depends on $\mathcal{E}$.

In this paper, we give a method to prove confluence of TRSs that contain non-terminating rewrite rules such as AC-rules. In contrast to the traditional approach described above, our method is based solely on usual critical pairs and usual termination. Thus the implementation of the method requires little special ingredients and the method is easily integrated into confluence provers to combine with other confluence proving methods. We first present confluence criteria for TRSs whose rewrite rules can be partitioned into terminating part and possibly non-terminating part (Section 3). We then give a reduction-preserving completion procedure so that the applicability of the criteria is enhanced (Section 4). In contrast to the well-known Knuth-Bendix completion procedure which preserves the equivalence relation of the system, our completion procedure preserves the reduction relation of the system, by which confluence of the original system is inferred from that of the completed system. Finally we report on our implementation and results of experiments (Section 5).

2 Preliminaries

This section fixes some notions and notations used in this paper. We refer to [3] for omitted definitions.

Let $\to$ be a relation on a set A. The *reflexive closure* (the *symmetric closure*, the *transitive closure*, the *reflexive and transitive closure*, the *equivalence closure*) of $\to$ is denoted by $\xrightarrow{=}$ ($\leftrightarrow$, $\xrightarrow{+}$, $\xrightarrow{*}$, $\xleftrightarrow{*}$, respectively). The union $\to_i \cup \to_j$ of indexed relations $\to_i$ and $\to_j$ is written as $\to_{i \cup j}$. A symmetric relation is written as $\vdash\!\dashv$. A relation $\to$ is *well-founded* if there exists no infinite descending chain $a_0 \to a_1 \to \cdots$. The composition of relations R, S is written as $R \circ S$. A relation $\to$ on a set A is *confluent* if $\xleftarrow{*} \circ \xrightarrow{*} \subseteq \xrightarrow{*} \circ \xleftarrow{*}$ holds.

Let $\mathcal{F}$ be a set of arity-fixed function symbols and $\mathcal{V}$ be the set of variables. The set of terms over $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathrm{T}(\mathcal{F}, \mathcal{V})$. The sets of function symbols and variables occurring in a term t are denoted by $\mathcal{F}(t)$ and $\mathcal{V}(t)$, respectively. A *linear* term is a term in which any variable occur at most once. *Positions* are finite sequences of positive integers. The *empty sequence* is denoted by ϵ. The set of positions in a term t is denoted by $\mathrm{Pos}(t)$. The *concatenation* of positions p, q is denoted by $p.q$. We use $\leq$ for prefix ordering on positions, i.e. $p \leq q$ iff $\exists o.\ p.o = q$. For p, q such that $p \leq q$, the position o satisfying $p.o = q$ is denoted by $p \backslash q$. Positions $p_1, \ldots, p_n$ are *parallel* if $p_i \not\leq p_j$ for any $i \neq j$. We write $p \parallel q$ if two positions p, q are parallel. If p is a position in a term t, then the symbol in t at the position p is written as $t(p)$, the subterm of t at the position p is written as t/p, and the term obtained by replacing the subterm t/p by a term s is written as $t[s]_p$. For $X \subseteq \mathcal{F} \cup \mathcal{V}$, we put $\mathrm{Pos}_X(t) = \{p \in \mathrm{Pos}(t) \mid t(p) \in X\}$. For parallel positions $p_1, \ldots, p_n$ in a term t, the term obtained by replacing each subterm t/p_i by a term s_i is written as $t[s_1, \ldots, s_n]_{p_1,\ldots,p_n}$. A map σ from $\mathcal{V}$ to $\mathrm{T}(\mathcal{F}, \mathcal{V})$ is a *substitution* if the domain $\mathrm{dom}(\sigma)$ of σ is finite where $\mathrm{dom}(\sigma) = \{x \in \mathcal{V} \mid \sigma(x) \neq x\}$. As usual, we identify each substitution with its homomorphic extension. For a substitution σ and a term t, $\sigma(t)$ is also written as $t\sigma$. For a set $\mathcal{E}$ of equations, we write $\mathcal{E}^{-1} = \{r \approx l \mid l \approx r \in \mathcal{E}\}$. A set $\mathcal{E} = \{s_1 \approx t_1, \ldots, s_n \approx t_n\}$ of equations is *unifiable* if there exists a substitution σ such that $s_i\sigma = t_i\sigma$ for all i; the substitution σ is a *unifier* of $\mathcal{E}$. A relation R on $\mathrm{T}(\mathcal{F}, \mathcal{V})$ is *stable* if for any terms $s, t \in \mathrm{T}(\mathcal{F}, \mathcal{V})$, $s\,R\,t$ implies $s\theta\,R\,t\theta$ for any substitution θ; it is *monotone* if $s\,R\,t$ implies $f(\ldots, s, \ldots)\,R\,f(\ldots, t, \ldots)$ for any $f \in \mathcal{F}$. A relation R on $\mathrm{T}(\mathcal{F}, \mathcal{V})$ is a *rewrite relation* if it is stable and monotone.

An equation $l \approx r$ is a *rewrite rule* if it satisfies the conditions (1) $l \notin \mathcal{V}$ and (2) $\mathcal{V}(l) \subseteq \mathcal{V}(r)$. A rewrite rule $l \approx r$ is written as $l \to r$. Rewrite rules are identified modulo renaming of variables. A rewrite rule $l \to r$ is *linear* (*left-linear*) if l, r is linear (l is linear, respectively); it is *bidirectional* if $r \approx l$ is a rewrite rule. A *term rewriting system* (*TRS* for short) is a finite set of rewrite rules. A TRS is left-linear (linear, bidirectional) if so are all its rewrite rules. If a TRS $\mathcal{R}$ is bidirectional then $\mathcal{R}^{-1} = \{r \to l \mid l \to r \in \mathcal{R}\}$ is a TRS. Let $\mathcal{R}$ be a TRS. If there exists a rewrite rule $l \to r \in \mathcal{R}$ and a position p in a term s and substitution θ such that $s/p = l\theta$ and $t = s[r\theta]_p$, we write $s \to_{p,\mathcal{R}} t$. If not necessary, $s \to_{p,\mathcal{R}} t$ is written as $s \to_{\mathcal{R}} t$ or $s \to t$. We call $s \to_{\mathcal{R}} t$ a *rewrite step*; $\to_{\mathcal{R}}$ is a rewrite relation and called *the* rewrite relation of $\mathcal{R}$. A term s is *normal* if $s \to_{\mathcal{R}} t$ for no term t. The set of normal terms is denoted by $\mathrm{NF}(\mathcal{R})$. A *normal form* (or *$\mathcal{R}$-normal form*) of a term s is a term $t \in \mathrm{NF}(\mathcal{R})$ such that $s \xrightarrow{*}_{\mathcal{R}} t$. A TRS $\mathcal{R}$ is *terminating* if $\to_{\mathcal{R}}$ is well-founded; $\mathcal{R}$ is confluent if $\to_{\mathcal{R}}$ is confluent. The *parallel extension* $\dashv\!\!\!\!\!\vdash\!\!\!\!\rightarrow_{\mathcal{R}}$ of the rewrite relation $\to_{\mathcal{R}}$ and the parallel extension $\leftarrow\!\!\!\!\dashv\!\!\!\!\vdash\!\!\!\!\rightarrow_{\mathcal{R}}$ of the symmetric closure $\leftrightarrow_{\mathcal{R}}$ of the rewrite relation $\to_{\mathcal{R}}$ are defined like this: $s \dashv\!\!\!\!\!\vdash\!\!\!\!\rightarrow_{\{p_1,\ldots,p_n\},\mathcal{R}} t$ ($s \leftarrow\!\!\!\!\dashv\!\!\!\!\vdash\!\!\!\!\rightarrow_{\{p_1,\ldots,p_n\},\mathcal{R}} t$) iff $p_1, \ldots, p_n$ are parallel positions in the term s and there exist rewrite rules $l_1 \to r_1, \ldots, l_n \to r_n \in \mathcal{R}$ (equations $l_1 \approx r_1, \ldots, l_n \approx r_n \in \mathcal{R} \cup \mathcal{R}^{-1}$, respectively) and substitution $\theta_1, \ldots, \theta_n$ such that $s/p_i = l_i\theta_i$ for each i and $t = s[r_1\theta_1, \ldots, r_n\theta_n]_{p_1,\ldots,p_n}$. If not necessary, $s \dashv\!\!\!\!\!\vdash\!\!\!\!\rightarrow_{\{p_1,\ldots,p_n\},\mathcal{R}} t$ ($s \leftarrow\!\!\!\!\dashv\!\!\!\!\vdash\!\!\!\!\rightarrow_{\{p_1,\ldots,p_n\},\mathcal{R}} t$) is written as $s \dashv\!\!\!\!\!\vdash\!\!\!\!\rightarrow_{\mathcal{R}} t$ or $s \dashv\!\!\!\!\!\vdash\!\!\!\!\rightarrow t$ ($s \leftarrow\!\!\!\!\dashv\!\!\!\!\vdash\!\!\!\!\rightarrow_{\mathcal{R}} t$ or $s \leftarrow\!\!\!\!\dashv\!\!\!\!\vdash\!\!\!\!\rightarrow t$, respectively). We call $s \dashv\!\!\!\!\!\vdash\!\!\!\!\rightarrow_{\mathcal{R}} t$ a *parallel rewrite step*. We note that $\dashv\!\!\!\!\!\vdash\!\!\!\!\rightarrow_{\mathcal{R}}$ is a reflexive rewrite relation and $\leftarrow\!\!\!\!\dashv\!\!\!\!\vdash\!\!\!\!\rightarrow_{\mathcal{R}}$ is a reflexive symmetric rewrite relation. Note that $\leftarrow\!\!\!\!\dashv\!\!\!\!\vdash\!\!\!\!\rightarrow_{\mathcal{R}}$ differs from the symmetric closure of $\dashv\!\!\!\!\!\vdash\!\!\!\!\rightarrow_{\mathcal{R}}$ in general and coincides with $\dashv\!\!\!\!\!\vdash\!\!\!\!\rightarrow_{\mathcal{R} \cup \mathcal{R}^{-1}}$ if $\mathcal{R}$ is bidirectional.

Let s, t be terms whose variables are disjoint. The term s *overlaps* on t (at a position p) when there exists a non-variable subterm $u = t/p$ of t such that u and s are unifiable. Let $l_1 \to r_1$ and $l_2 \to r_2$ be rewrite rules w.l.o.g. whose variables are disjoint. Suppose that l_1 overlaps on l_2 at a position p and σ is the most general unifier of l_1 and l_2/p. Then the term $l_2[l_1]_p\sigma$ yields a *critical pair* $\langle l_2[r_1]_p\sigma, r_2\sigma \rangle$ obtained by the overlap of $l_1 \to r_1$ on $l_2 \to r_2$ at the position p. In the case of self-overlap (i.e. when $l_1 \to r_1$ and $l_2 \to r_2$ are identical modulo renaming), we do not consider the case $p = \epsilon$. We call the critical pair *outer* if $p = \epsilon$ and *inner* if $p > \epsilon$. The set of outer (inner) critical pairs obtained by the overlaps of a rewrite rule from $\mathcal{R}_1$ on a rewrite rule from $\mathcal{R}_2$ is denoted by $\mathrm{CP}_{out}(\mathcal{R}_1, \mathcal{R}_2)$ ($\mathrm{CP}_{in}(\mathcal{R}_1, \mathcal{R}_2)$, respectively). We put $\mathrm{CP}(\mathcal{R}_1, \mathcal{R}_2) = \mathrm{CP}_{out}(\mathcal{R}_1, \mathcal{R}_2) \cup \mathrm{CP}_{in}(\mathcal{R}_1, \mathcal{R}_2)$. Critical pairs are often regarded as equations.

3 Confluence criteria

In this section, we give new confluence criteria for term rewriting systems. We first present an abstract confluence criterion that will be used as the basis of our confluence criteria.

▶ **Lemma 3.1.** *Let $\vdash\!\dashv_0, \to_1$ be relations on a set A such that $\vdash\!\dashv_0$ is symmetric and $\to_1$ is well-founded. Let $\to_{0\cup1} = \vdash\!\dashv_0 \cup \to_1$. Suppose (i) $\leftarrow_1 \circ \to_1 \subseteq \xrightarrow{*}_1 \circ \overset{=}{\vdash\!\dashv}_0 \circ \xleftarrow{*}_1$ and (ii) $\vdash\!\dashv_0 \circ \to_1 \subseteq \xrightarrow{*}_1 \circ \overset{=}{\vdash\!\dashv}_0 \circ \xleftarrow{*}_1$. Then $\xleftrightarrow{*}_{0\cup1} \subseteq \xrightarrow{*}_1 \circ \overset{*}{\vdash\!\dashv}_0 \circ \xleftarrow{*}_1$.*

Proof. Let the weight of a rewrite step $a \leftrightarrow_{0\cup1} b$ be given by the multiset $w(a \leftrightarrow_{0\cup1} b)$ defined like this: $w(a \vdash\!\dashv_0 b) = \{a, b\}$, $w(a \to_1 b) = \{a\}$ and $w(a \leftarrow_1 b) = \{b\}$. For each rewrite sequence $a_0 \leftrightarrow_{0\cup1} a_1 \leftrightarrow_{0\cup1} \cdots \leftrightarrow_{0\cup1} a_n$ let its weight be the multiset consisting of the weights of the rewrite steps $a_i \leftrightarrow_{0\cup1} a_{i+1}$, i.e. $\{w(a_0 \leftrightarrow_{0\cup1} a_1), w(a_1 \leftrightarrow_{0\cup1} a_2), \ldots, w(a_{n-1} \leftrightarrow_{0\cup1} a_n)\}$. Let $\gg$ be the multiset extension of the well-founded order $\xrightarrow{+}_1$ and $\gg_{\mathrm{mul}}$ the multiset extension of $\gg$. We show by noetherian induction on the weight of the rewrite sequence w.r.t. $\gg_{\mathrm{mul}}$ that for any rewrite sequence $a_0 \stackrel{*}{\leftrightarrow}_{0\cup 1} a_n$ there exists a rewrite sequence $a_0 \stackrel{*}{\to}_1 \circ \stackrel{*}{\vdash\!\dashv}_0 \circ \stackrel{*}{\leftarrow}_1 a_n$.

1. Suppose there exists k such that $a_{k-1} \leftarrow_1 a_k \to_1 a_{k+1}$. Then by assumption (i), there exist $b_0, \ldots, b_m$ such that $a_{k-1} = b_0 \stackrel{*}{\to}_1 b_l \stackrel{=}{\vdash\!\dashv}_0 b_{l+1} \stackrel{*}{\leftarrow}_1 b_m = a_{k+1}$. Thus we have a rewrite sequence $a_0 \stackrel{*}{\leftrightarrow}_{0\cup 1} a_{k-1} = b_0 \stackrel{*}{\to}_1 b_l \stackrel{=}{\vdash\!\dashv}_0 b_{l+1} \stackrel{*}{\leftarrow}_1 b_m = a_{k+1} \stackrel{*}{\leftrightarrow}_{0\cup 1} a_n$. We now show this new rewrite sequence has less weight than the original rewrite sequence $a_0 \stackrel{*}{\leftrightarrow}_{0\cup 1} a_n$. We here only show the case of $l \neq 0$, $l+1 \neq m$ and $b_l \vdash\!\dashv_0 b_{l+1}$. Then the weight decreases as $\{\ldots, \{a_k\}, \{a_k\}, \ldots\} \gg_{\mathrm{mul}} \{\ldots, \{b_0\}, \ldots, \{b_{l-1}\}, \{b_l, b_{l+1}\}, \{b_{l+2}\}, \ldots, \{b_m\}, \ldots\}$. For other cases, one can easily check that the weight of the rewrite sequence decreases in a similar way. Thus, it follows that there exists a rewrite sequence $a_0 \stackrel{*}{\to}_1 \circ \stackrel{*}{\vdash\!\dashv}_0 \circ \stackrel{*}{\leftarrow}_1 a_n$ by the induction hypothesis.
2. Suppose that there exists k such that $a_{k-1} \vdash\!\dashv_0 a_k \to_1 a_{k+1}$. Then by assumption (ii), there exist $b_0, \ldots, b_m$ such that $a_{k-1} = b_0 \stackrel{*}{\to}_1 b_l \stackrel{=}{\vdash\!\dashv}_0 b_{l+1} \stackrel{*}{\leftarrow}_1 b_m = a_{k+1}$. Thus we have a rewrite sequence $a_0 \stackrel{*}{\leftrightarrow}_{0\cup 1} a_{k-1} = b_0 \stackrel{*}{\to}_1 b_l \stackrel{=}{\vdash\!\dashv}_0 b_{l+1} \stackrel{*}{\leftarrow}_1 b_m = a_{k+1} \stackrel{*}{\leftrightarrow}_{0\cup 1} a_n$. In a way similar to the first case, one can easily check that this new rewrite sequence has less weight than the original rewrite sequence $a_0 \stackrel{*}{\leftrightarrow}_{0\cup 1} a_n$. We here only show the case of $l \neq 0$, $l+1 \neq m$ and $b_l \vdash\!\dashv_0 b_{l+1}$. Then the weight decreases as $\{\ldots, \{b_0, a_k\}, \{a_k\}, \ldots\} \gg_{\mathrm{mul}} \{\ldots, \{b_0\}, \ldots, \{b_{l-1}\}, \{b_l, b_{l+1}\}, \{b_{l+2}\}, \ldots, \{b_m\}, \ldots\}$. Thus, it follows that there exists a rewrite sequence $a_0 \stackrel{*}{\to}_1 \circ \stackrel{*}{\vdash\!\dashv}_0 \circ \stackrel{*}{\leftarrow}_1 a_n$ by the induction hypothesis.
3. Suppose that there exists k such that $a_{k-1} \leftarrow_1 a_k \vdash\!\dashv_0 a_{k+1}$. Then one can show that there exists a rewrite sequence $a_0 \stackrel{*}{\to}_1 \circ \stackrel{*}{\vdash\!\dashv}_0 \circ \stackrel{*}{\leftarrow}_1 a_n$ in the same way as the case (2).
4. It remains to show the case that (α) there exists no k such that $a_{k-1} \leftarrow_1 a_k \to_1 a_{k+1}$, ($\beta$) there exists no k such that $a_{k-1} \vdash\!\dashv_0 a_k \to_1 a_{k+1}$ and (γ) there exists no k such that $a_{k-1} \leftarrow_1 a_k \vdash\!\dashv_0 a_{k+1}$. We show by induction on the length of $a_0 \stackrel{*}{\leftrightarrow}_{0\cup 1} a_n$ that this rewrite sequence has the form $a_0 \stackrel{*}{\to}_1 \circ \stackrel{*}{\vdash\!\dashv}_0 \circ \stackrel{*}{\leftarrow}_1 a_n$. The case $n = 0$ is trivial. Suppose $a_0 \leftrightarrow_{0\cup 1} a_1 \stackrel{*}{\leftrightarrow}_{0\cup 1} a_n$. By induction hypothesis we have $a_1 \stackrel{*}{\to}_1 a_l \stackrel{*}{\vdash\!\dashv}_0 a_m \stackrel{*}{\leftarrow}_1 a_n$. We distinguish three cases:
   a. Case of $a_0 \vdash\!\dashv_0 a_1$. By (β), it follows that we have $a_0 \vdash\!\dashv_0 a_1 = a_l \stackrel{*}{\vdash\!\dashv}_0 a_m \stackrel{*}{\leftarrow}_1 a_n$. Hence the conclusion follows.
   b. Case of $a_0 \to_1 a_1$. Since we have $a_0 \to_1 a_1 \stackrel{*}{\to}_1 a_l \stackrel{*}{\vdash\!\dashv}_0 a_m \stackrel{*}{\leftarrow}_1 a_n$, the conclusion follows.
   c. Case of $a_0 \leftarrow_1 a_1$. Then by (α), it follows that we have $a_0 \leftarrow_1 a_1 = a_l \stackrel{*}{\vdash\!\dashv}_0 a_m \stackrel{*}{\leftarrow}_1 a_n$. Furthermore, by ($\gamma$), it follows that $a_0 \leftarrow_1 a_1 = a_l = a_m \stackrel{*}{\leftarrow}_1 a_n$. Hence the conclusion follows.

◀

▶ Remark. Let $\sim$ be an equivalence relation on a set A. Then a relation $\to_1$ on A is said to be *confluent modulo* $\sim$ (CR$\sim$) if $\stackrel{*}{\leftarrow}_1 \circ \sim \circ \stackrel{*}{\to}_1 \subseteq \stackrel{*}{\to}_1 \circ \sim \circ \stackrel{*}{\leftarrow}_1$ holds; *locally confluent modulo* $\sim$ (LCR$\sim$) if (i′) $\leftarrow_1 \circ \to_1 \subseteq \stackrel{*}{\to}_1 \circ \sim \circ \stackrel{*}{\leftarrow}_1$ and (ii′) $\sim \circ \to_1 \subseteq \stackrel{*}{\to}_1 \circ \sim \circ \stackrel{*}{\leftarrow}_1$ hold [6]. It is shown in [6] that CR$\sim$ and LCR$\sim$ coincide provided that $\to_1$ is well-founded. Suppose $\to_1$ is well-founded and $\sim = \stackrel{*}{\vdash\!\dashv}_0$. Then the property CR$\sim$ is equivalent to the conclusion of the lemma, i.e. $\stackrel{*}{\leftrightarrow}_{0\cup 1} \subseteq \stackrel{*}{\to}_1 \circ \stackrel{*}{\vdash\!\dashv}_0 \circ \stackrel{*}{\leftarrow}_1$; hence so are (i′) and (ii′). In our lemma, in contrast to (i′) and (ii′), the condition part of (ii) are localized (i.e. we only assume $\vdash\!\dashv_0 \circ \to_1$ rather than $\stackrel{*}{\vdash\!\dashv}_0 \circ \to_1$) in price of requesting joinability sequences to have zero or one $\vdash\!\dashv_0$-step in the conclusion part of (i) and (ii) (i.e. we need to guarantee $\stackrel{*}{\to}_1 \circ \stackrel{=}{\vdash\!\dashv}_0 \circ \stackrel{*}{\leftarrow}_1$ rather than $\stackrel{*}{\to}_1 \circ \stackrel{*}{\vdash\!\dashv}_0 \circ \stackrel{*}{\leftarrow}_1$). A different localization given in [6] is that if $\to_1 \circ \sim$ is well-founded then (i′) $\leftarrow_1 \circ \to_1 \subseteq \stackrel{*}{\to}_1 \circ \sim \circ \stackrel{*}{\leftarrow}_1$ and (iii′) $\vdash\!\dashv_0 \circ \to_1 \subseteq \stackrel{*}{\to}_1 \circ \sim \circ \stackrel{*}{\leftarrow}_1$ imply CR$\sim$. Contrast to our lemma, this localization allows an arbitrary number of $\vdash\!\dashv_0$-steps in the conclusion part of (i′) and (iii′) in price of requesting (not only $\to_1$ but) $\to_1 \circ \sim$ is well-founded. In [8] (see also

[9]), another localization is obtained: if $\to_1$ is well-founded then (i′) $\leftarrow_1 \circ \to_1 \subseteq \xrightarrow{*}_1 \circ \sim \circ \xleftarrow{*}_1$ and (iv′) $\vdash\!\dashv_0 \circ \to_1 \subseteq \xrightarrow{+}_1 \circ \sim$ imply CR$\sim$ (and that $\to_1 \circ \sim$ is well-founded). Contrast to our lemma, this localization allows an arbitrary number of $\vdash\!\dashv_0$-steps in the conclusion part of (i′) and (iv′) in price of restricting the form of joinability sequences in the conclusion part of (iv′).

▶ **Theorem 3.2** (abstract confluence criterion). *Let $\vdash\!\dashv_0, \to_1$ be relations on a set A such that $\vdash\!\dashv_0$ is symmetric and $\to_1$ is well-founded. Let $\to_{0\cup 1} = \vdash\!\dashv_0 \cup \to_1$. Suppose (i) $\leftarrow_1 \circ \to_1 \subseteq \xrightarrow{*}_1 \circ \overset{=}{\vdash\!\dashv}_0 \circ \xleftarrow{*}_1$ and (ii) $\vdash\!\dashv_0 \circ \to_1 \subseteq \xrightarrow{*}_1 \circ \overset{=}{\vdash\!\dashv}_0 \circ \xleftarrow{*}_1$. Then $\to_{0\cup 1}$ is confluent.*

Proof. We prove $\xleftrightarrow{*}_{0\cup 1} \subseteq \xrightarrow{*}_{0\cup 1} \circ \xleftarrow{*}_{0\cup 1}$. Suppose $a \xleftrightarrow{*}_{0\cup 1} b$. Then $a \xrightarrow{*}_1 \circ \overset{*}{\vdash\!\dashv}_0 \circ \xleftarrow{*}_1 b$ by Lemma 3.1. Hence $a \xrightarrow{*}_{0\cup 1} \circ \xleftarrow{*}_{0\cup 1} b$. ◀

For the rest of this section, we develop some confluence criteria for TRSs based on this abstract confluence criterion.

▶ **Lemma 3.3.** *Let $\mathcal{S}$ be a TRS and $\vdash\!\dashv$ be a symmetric rewrite relation. Suppose that* CP$(\mathcal{S}, \mathcal{S}) \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \overset{=}{\vdash\!\dashv} \circ \xleftarrow{*}_{\mathcal{S}}$*. Then $\leftarrow_{\mathcal{S}} \circ \to_{\mathcal{S}} \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \overset{=}{\vdash\!\dashv} \circ \xleftarrow{*}_{\mathcal{S}}$.*

Proof. Suppose $t_0 \leftarrow_{p,\mathcal{S}} s \to_{q,\mathcal{S}} t_1$. We distinguish the cases by relative positions of p and q. The case of $p \parallel q$ is straightforward. Suppose $q \leq p$. Let $s/q = l\sigma$ and $l \to r \in \mathcal{S}$. Then either (1) $q \backslash p \in \mathrm{Pos}_{\mathcal{F}}(l)$ or (2) there exists $q_x \in \mathrm{Pos}_{\mathcal{V}}(l)$ such that $l/q_x = x \in \mathcal{V}$ and $q.q_x \leq p$.

1. Then $t_0 = s[u\rho]_q$ and $t_1 = s[v\rho]_q$ for some $\langle u, v\rangle \in$ CP$(\mathcal{S}, \mathcal{S})$ and substitution ρ. Thus by assumption $u \xrightarrow{*}_{\mathcal{S}} u' \overset{=}{\vdash\!\dashv} v' \xleftarrow{*}_{\mathcal{S}} v$ for some u', v'. Then, since $\overset{=}{\vdash\!\dashv}$ and $\to_{\mathcal{S}}$ are rewrite relations, we have $t_0 = s[u\rho]_q \xrightarrow{*}_{\mathcal{S}} s[u'\rho]_q \overset{=}{\vdash\!\dashv} s[v'\rho]_q \xleftarrow{*}_{\mathcal{S}} s[v\rho]_q = t_1$.
2. Then $t_1 = s[r\sigma]_q$ and $s = s[l\sigma]_q \to_{p,\mathcal{S}} t_0 \xrightarrow{*}_{\mathcal{S}} s[l\sigma']_q$ for some substitution σ' such that $\sigma(x) \to_{(q.q_x)\backslash p,\mathcal{S}} \sigma'(x)$ and $\sigma'(y) = \sigma(y)$ for any $y \neq x$. Thus $t_0 \xrightarrow{*}_{\mathcal{S}} s[l\sigma']_q \to_{\mathcal{S}} s[r\sigma']_q \xleftarrow{*}_{\mathcal{S}} s[r\sigma]_q = t_1$. The claim follows since $\overset{=}{\vdash\!\dashv}$ is reflexive.

The case of $p \leq q$ follows similarly to the case of $q \leq p$, using the symmetry of $\overset{=}{\vdash\!\dashv}$. ◀

▶ **Lemma 3.4.** *Let $\mathcal{P}, \mathcal{S}$ be TRSs. Suppose that* CP$(\mathcal{S}, \mathcal{S}) \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \mathrel{\leftarrow\!\!\parallel\!\!\rightarrow}_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$*. Then $\leftarrow_{\mathcal{S}} \circ \to_{\mathcal{S}} \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \mathrel{\leftarrow\!\!\parallel\!\!\rightarrow}_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$.*

Proof. Take $\vdash\!\dashv \,:=\, \mathrel{\leftarrow\!\!\parallel\!\!\rightarrow}_{\mathcal{P}}$ (hence $\overset{=}{\vdash\!\dashv} = \mathrel{\leftarrow\!\!\parallel\!\!\rightarrow}_{\mathcal{P}}$) in Lemma 3.3. ◀

▶ **Lemma 3.5.** *Let $\mathcal{P}, \mathcal{S}$ be TRSs such that $\mathcal{S}$ is left-linear and $\mathcal{P}$ is bidirectional. Suppose (i)* CP$_{in}(\mathcal{P} \cup \mathcal{P}^{-1}, \mathcal{S}) = \emptyset$ *and (ii)* CP$(\mathcal{S}, \mathcal{P} \cup \mathcal{P}^{-1}) \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \mathrel{\leftarrow\!\!\parallel\!\!\rightarrow}_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$*. Then $\mathrel{\leftarrow\!\!\parallel\!\!\rightarrow}_{\mathcal{P}} \circ \to_{\mathcal{S}} \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \mathrel{\leftarrow\!\!\parallel\!\!\rightarrow}_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$.*

Proof. Suppose $t_0 \mathrel{\leftarrow\!\!\parallel}_{U,\mathcal{P}\cup\mathcal{P}^{-1}} s \to_{q,\mathcal{S}} t_1$. Let $U = \{p_1, \ldots, p_n\}$ where $p_1, \ldots, p_n$ are positions from left to right, $s/p_i = l_i\sigma_i$ for $l_i \to r_i \in \mathcal{P} \cup \mathcal{P}^{-1}$ and substitutions σ_i $(1 \leq i \leq n)$ and $s/q = l'\rho$ for $l' \to r' \in \mathcal{S}$ and a substitution ρ. We distinguish two cases: (1) the case that $\exists p \in U.\ p \leq q$ and (2) the case that $\forall p \in U.\ p \not\leq q$.

1. Suppose $p_i \in U$ and $p_i \leq q$. Then either (a) $p_i \backslash q \in \mathrm{Pos}_{\mathcal{F}}(l_i)$ or (b) there exists $p_x \in \mathrm{Pos}_{\mathcal{V}}(l_i)$ such that $l_i/p_x = x \in \mathcal{V}$ and $p_i.p_x \leq q$.
   a. Then $t_0/p_i = v\rho$ and $t_1/p_i = u\rho$ for some $\langle u, v\rangle \in$ CP$(\mathcal{S}, \mathcal{P} \cup \mathcal{P}^{-1})$ and substitution ρ. Then, from our assumption (ii), we have $u \xrightarrow{*}_{\mathcal{S}} u' \mathrel{\leftarrow\!\!\parallel\!\!\rightarrow}_{\mathcal{P}} v' \xleftarrow{*}_{\mathcal{S}} v$ for some u', v'. Thus $t_0/p_i = v\rho \xrightarrow{*}_{\mathcal{S}} v'\rho \mathrel{\leftarrow\!\!\parallel\!\!\rightarrow}_{\mathcal{P}} u'\rho \xleftarrow{*}_{\mathcal{S}} u\rho = t_1/p_i$. Hence we have

$$\begin{array}{lll} t_0 & = & s[r_1\sigma_1, \ldots, t_0/p_i, \ldots, r_n\sigma_n]_{p_1,\ldots,p_i,\ldots,p_n} \\ & \xrightarrow{*}_{\mathcal{S}} & s[r_1\sigma_1, \ldots, v'\rho, \ldots, r_n\sigma_n]_{p_1,\ldots,p_i,\ldots,p_n} \\ & \mathrel{\leftarrow\!\!\parallel\!\!\rightarrow}_{\mathcal{P}} & s[l_1\sigma_1, \ldots, u'\rho, \ldots, l_n\sigma_n]_{p_1,\ldots,p_i,\ldots,p_n} \\ & \xleftarrow{*}_{\mathcal{S}} & s[l_1\sigma_1, \ldots, t_1/p_i, \ldots, l_n\sigma_n]_{p_1,\ldots,p_i,\ldots,p_n} = t_1. \end{array}$$

b. Then $t_0/p_i = r_i\sigma_i$ and $t_1/p_i \xrightarrow{*}_{\mathcal{S}} l_i\sigma'_i$ for some substitution σ'_i such that $\sigma_i(x) \to_{(p_i.p_x)\backslash q,\mathcal{S}} \sigma'_i(x)$ and $\sigma'_i(y) = \sigma_i(y)$ for any $y \neq x$. Thus we have

$$\begin{array}{lcl} t_0 & = & C[r_1\sigma_1,\dots,r_i\sigma_i,\dots,r_n\sigma_n]_{p_1,\dots,p_i,\dots,p_n} \\ & \xrightarrow{*}_{\mathcal{S}} & C[r_1\sigma_1,\dots,r_i\sigma'_i,\dots,r_n\sigma_n]_{p_1,\dots,p_i,\dots,p_n} \\ & \Leftrightarrow\!\!\!\!\!\!\|\;\;_{\mathcal{P}} & C[l_1\sigma_1,\dots,l_i\sigma'_i,\dots,l_n\sigma_n]_{p_1,\dots,p_i,\dots,p_n} \\ & \xleftarrow{*}_{\mathcal{S}} & C[l_1\sigma_1,\dots,t_1/p_i,\dots,l_n\sigma_n]_{p_1,\dots,p_i,\dots,p_n} = t_1. \end{array}$$

2. Suppose $\forall p \in U.\ p \not\leq q$. Let $U' = \{p_i \in U \mid q < p_i\} = \{p_l,\dots,p_k\}$, $q_i = q\backslash p_i$ for $l \leq i \leq k$, and thus $l'\rho = l'\rho[l_l\sigma_l,\dots,l_k\sigma_k]_{q_l,\dots,q_k}$. By our assumption (i), for each $p_i \in U'$ there exists $q_x \in \mathrm{Pos}_{\mathcal{V}}(l')$ such that $l'/q_x = x \in \mathcal{V}$ and $q.q_x \leq p_i$. Thus, $s/q = l'\rho = l'\rho[l_l\sigma_l,\dots,l_k\sigma_k]_{q_l,\dots,q_k} \to_{\mathcal{S}} r'\rho = r'\rho[l_{j_1}\sigma_{j_1},\dots,l_{j_m}\sigma_{j_m}]_{o_1,\dots,o_m} = t_1/q$ for some positions $o_1,\cdots,o_m$ and $j_1,\dots,j_m \in \{l,\dots,k\}$. Furthermore, by the left-linearity of $\mathcal{S}$, we have $l'\rho[r_l\sigma_l,\dots,r_k\sigma_k]_{q_l,\dots,q_k} \to_{\mathcal{S}} r'\rho[r_{j_1}\sigma_{j_1},\dots,r_{j_m}\sigma_{j_m}]_{o_1,\dots,o_m}$. Thus,

$$\begin{array}{lcl} t_0 & = & s[r_1\sigma_1,\dots,l'\rho[r_l\sigma_l,\dots,r_k\sigma_k]_{q_l,\dots,q_k},\dots,r_n\sigma_n]_{p_1,\dots,q,\dots,p_n} \\ & \to_{\mathcal{S}} & s[r_1\sigma_1,\dots,r'\rho[r_{j_1}\sigma_{j_1},\dots,r_{j_m}\sigma_{j_m}]_{o_1,\dots,o_m},\dots,r_n\sigma_n]_{p_1,\dots,q,\dots,p_n} \\ & \Leftrightarrow\!\!\!\!\!\!\|\;\;_{\mathcal{P}} & s[l_1\sigma_1,\dots,r'\rho[l_{j_1}\sigma_{j_1},\dots,l_{j_m}\sigma_{j_m}]_{o_1,\dots,o_m},\dots,l_n\sigma_n]_{p_1,\dots,q,\dots,p_n} = t_1. \end{array}$$

◀

▶ **Definition 3.6** (reversible relation). A relation $\to$ is said to be *reversible* if $\to \subseteq \xleftarrow{*}$. A TRS $\mathcal{R}$ is reversible if $\to_{\mathcal{R}}$ is reversible.

Note that, by the definition of rewrite rules, reversible TRSs are bidirectional.

▶ **Theorem 3.7** (confluence criterion). *Let $\mathcal{P},\mathcal{S}$ be TRSs such that $\mathcal{S}$ is left-linear and terminating and $\mathcal{P}$ is reversible. Suppose (i)* $\mathrm{CP}(\mathcal{S},\mathcal{S}) \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \Leftrightarrow\!\!\!\!\!\!\|\;\;_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$*, (ii)* $\mathrm{CP}_{in}(\mathcal{P} \cup \mathcal{P}^{-1},\mathcal{S}) = \emptyset$ *(iii)* $\mathrm{CP}(\mathcal{S},\mathcal{P} \cup \mathcal{P}^{-1}) \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \Leftrightarrow\!\!\!\!\!\!\|\;\;_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$*. Then $\mathcal{S} \cup \mathcal{P}$ is confluent.*

Proof. By our assumption (i) and Lemma 3.4, we have (a) $\leftarrow_{\mathcal{S}} \circ \to_{\mathcal{S}} \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \Leftrightarrow\!\!\!\!\!\!\|\;\;_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$. From our assumptions (ii) and (iii), it follows that (b) $\Leftrightarrow\!\!\!\!\!\!\|\;\;_{\mathcal{P}} \circ \to_{\mathcal{S}} \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \Leftrightarrow\!\!\!\!\!\!\|\;\;_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$ by Lemma 3.5. Take $\vdash\!\dashv_0 := \Leftrightarrow\!\!\!\!\!\!\|\;\;_{\mathcal{P}}$ and $\to_1 := \to_{\mathcal{S}}$. Then, by the termination of $\mathcal{S}$, $\to_1$ is well-founded. Hence by Theorem 3.2, $\Leftrightarrow\!\!\!\!\!\!\|\;\;_{\mathcal{P}} \cup \to_{\mathcal{S}}$ is confluent. Furthermore, since $\to_{\mathcal{P}}$ is reversible, $\to_{\mathcal{P}} \subseteq \Leftrightarrow\!\!\!\!\!\!\|\;\;_{\mathcal{P}} \subseteq \xrightarrow{*}_{\mathcal{P}}$. Hence $\to_{\mathcal{P}\cup\mathcal{S}}$ is confluent. ◀

We are now going to slightly weaken the condition (ii) $\mathrm{CP}_{in}(\mathcal{P}\cup\mathcal{P}^{-1},\mathcal{S}) = \emptyset$ of the theorem using the notion of parallel critical pairs [5, 14]. Let $s_1,\dots,s_n,t$ be terms whose variables are disjoint. The terms $s_1,\dots,s_n$ *parallel-overlap* on t (at parallel positions $p_1,\dots,p_n$) if $t/p_i \notin \mathcal{V}$ for any $1 \leq i \leq n$ and $\{s_1 \approx t/p_1,\dots,s_n \approx t/p_n\}$ is unifiable. Let $l_1 \to r_1,\dots,l_n \to r_n$ and $l' \to r'$ be rewrite rules w.l.o.g. whose variables are mutually disjoint. Suppose that $l_1,\dots,l_n$ parallel-overlap on l' at parallel positions $p_1,\dots,p_n$ and σ is the most general unifier of $\{l_1 \approx l'/p_1,\dots,l_n \approx l'/p_n\}$. Then the term $l'[l_1,\dots,l_n]_{p_1,\dots,p_n}\sigma$ yields a *parallel critical pair* $\langle l'[r_1,\dots,r_n]_{p_1,\dots,p_n}\sigma, r'\sigma\rangle$ obtained by the parallel-overlap of $l_1 \to r_1,\dots,l_n \to r_n$ on $l' \to r'$ at positions $p_1,\dots,p_n$. In the case of self-overlap (i.e. when $n = 1$ and $l_1 \to r_1$ and $l' \to r'$ are identical modulo renaming), we do not consider the case $p_1 = \epsilon$. We write $\langle l'[r_1,\dots,r_n]_{p_1,\dots,p_n}\sigma, r'\sigma\rangle_X$ if $X = \bigcup_{1\leq i\leq n} \mathcal{V}(l'\sigma/p_i)$. We call the parallel critical pair *outer* if $n = 1$ and $p_1 = \epsilon$, and *inner* if $p_i > \epsilon$ for all i. The set of outer (inner) parallel critical pairs obtained by the parallel-overlaps of rewrite rules from $\mathcal{R}_1$ on a rewrite rule from $\mathcal{R}_2$ is denoted by $\mathrm{PCP}_{out}(\mathcal{R}_1,\mathcal{R}_2)$ ($\mathrm{PCP}_{in}(\mathcal{R}_1,\mathcal{R}_2)$, respectively). (Note, however, that $\mathrm{PCP}_{out}(\mathcal{R}_1,\mathcal{R}_2) = \mathrm{CP}_{out}(\mathcal{R}_1,\mathcal{R}_2)$.) We put $\mathrm{PCP}(\mathcal{R}_1,\mathcal{R}_2) = \mathrm{PCP}_{out}(\mathcal{R}_1,\mathcal{R}_2) \cup \mathrm{PCP}_{in}(\mathcal{R}_1,\mathcal{R}_2)$.

▶ **Lemma 3.8.** *Let $\mathcal{P}, \mathcal{S}$ be TRSs such that $\mathcal{S}$ is left-linear and $\mathcal{P}$ is bidirectional. Suppose that (i) for all $\langle u,v\rangle_X \in \mathrm{PCP}_{in}(\mathcal{P}\cup\mathcal{P}^{-1},\mathcal{S})$, $u \xrightarrow{*}_{\mathcal{S}} u' \mathrel{\#\!\!\!\!\leftrightarrow}_{V,\mathcal{P}} v' \xleftarrow{*}_{\mathcal{S}} v$ for some u',v' and V satisfying $\bigcup_{o\in V}\mathcal{V}(v'/o) \subseteq X$, and (ii) $\mathrm{CP}(\mathcal{S},\mathcal{P}\cup\mathcal{P}^{-1}) \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \mathrel{\#\!\!\!\!\leftrightarrow}_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$. Then $\mathrel{\#\!\!\!\!\leftrightarrow}_{\mathcal{P}} \circ \rightarrow_{\mathcal{S}} \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \mathrel{\#\!\!\!\!\leftrightarrow}_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$.*

Proof. Suppose $t_0 \mathrel{\#\!\!\!\!\leftarrow}_{U,\mathcal{P}\cup\mathcal{P}^{-1}} s \rightarrow_{q,\mathcal{S}} t_1$. Let $U = \{p_1,\ldots,p_n\}$ where $p_1,\ldots,p_n$ are positions from left to right, $s/p_i = l_i\sigma_i$ for $l_i \rightarrow r_i \in \mathcal{P}\cup\mathcal{P}^{-1}$ and substitutions σ_i ($1 \leq i \leq n$) and $s/q = l'\rho$ for $l' \rightarrow r' \in \mathcal{S}$ and substitution ρ. The same proof as in Lemma 3.5 applies other than the case of $\forall p \in U.\, p \not\leq q$. Let $\{p_k,\ldots,p_m\} = \{p_i \in U \mid q \leq p_i\}$. For each p_i ($k \leq i \leq m$) either $p_i\backslash q \in \mathrm{Pos}_{\mathcal{F}}(l')$ or there exists $q_x \in \mathrm{Pos}_{\mathcal{V}}(l')$ such that $q.q_x \leq p_i$. W.l.o.g. let $\{p_k,\ldots,p_l\} = \{p_i \mid p_i\backslash q \in \mathrm{Pos}_{\mathcal{F}}(l')\}$ and $\{p_{l+1},\ldots,p_m\} = \{p_i \mid \exists q_x \in \mathrm{Pos}_{\mathcal{V}}(l').\, q.q_x \leq p_i\}$. Then there exists a parallel critical pair $\langle u,v\rangle_X$ obtained from overlaps of $l_k \rightarrow r_k,\ldots,l_l \rightarrow r_l$ on $l' \rightarrow r'$ at $p_k\backslash q,\ldots,p_l\backslash q$. Then, by our assumption $u \xrightarrow{*}_{\mathcal{S}} u' \mathrel{\#\!\!\!\!\leftrightarrow}_{V,\mathcal{P}} v' \xleftarrow{*}_{\mathcal{S}} v$ for some u',v' satisfying $\bigcup_{o\in V}\mathcal{V}(v'/o) \subseteq X$. Let $Y = \mathcal{V}(l'\sigma) \setminus X$. Then, since l' is linear (and $\mathcal{V}(l'), \mathcal{V}(l_l),\ldots,\mathcal{V}(l_m)$ are mutually disjoint), we have $\{l'(q_x) \mid q_x \in \mathrm{Pos}_{\mathcal{V}}(l'), \exists i\,(q.q_x \leq p_i)\} \subseteq Y$. Furthermore, $t_0/q = u\theta'$ and $t_1/q = v\theta$ for some substitution θ,θ' such that $\theta(y) \xrightarrow{*}_{\mathcal{S}} \theta'(y)$ for $y \in Y$ and $\theta(z) = \theta'(z)$ for $z \notin Y$. Hence, by the left-linearity of $\mathcal{S}$, we have $u\theta' \xrightarrow{*}_{\mathcal{S}} u'\theta'$. Now we claim that any position $o_1 \in \mathrm{Pos}_Y(v')$ and $o_2 \in V$ are parallel. Since $Y \subseteq \mathcal{V}$, it suffices to show $o_2 \not\leq o_1$. If $o_2 \leq o_1$ then $v'/o_1 \in \mathcal{V}(v'/o_2)$ holds, and hence $\mathcal{V}(v'/o_2) \cap Y \neq \emptyset$. Then, by $\mathcal{V}(v'/o_2) \subseteq X$, $X \cap Y \neq \emptyset$ holds. This is a contradiction. Hence any position $o_1 \in \mathrm{Pos}_Y(u') \cup \mathrm{Pos}_Y(v')$ and $o_2 \in V$ are parallel. Now, we have

$$\begin{array}{lll}
t_0 & = & s[r_1\sigma_1,\ldots,u\theta',\ldots,r_n\sigma_n]_{p_1,\ldots,q,\ldots,p_n} \\
 & \xrightarrow{*}_{\mathcal{S}} & s[r_1\sigma_1,\ldots,u'\theta',\ldots,r_n\sigma_n]_{p_1,\ldots,q,\ldots,p_n} \\
 & \mathrel{\#\!\!\!\!\leftrightarrow}_{U',\mathcal{P}} & s[l_1\sigma_1,\ldots,v'\theta,\ldots,l_n\sigma_n]_{p_1,\ldots,q,\ldots,p_n} \\
 & \xleftarrow{*}_{\mathcal{S}} & s[l_1\sigma_1,\ldots,v\theta,\ldots,l_n\sigma_n]_{p_1,\ldots,q,\ldots,p_n} = t_1
\end{array}$$

where $U' = \{p_1,\ldots,p_{k-1}\} \cup \{p_{m+1},\ldots,p_n\} \cup \{q.o \mid o \in V\} \cup W$ where W is the set of descendants of $\{q_{l+1},\ldots,q_m\}$ in s along the rewrite steps $s = s[l'\theta]_q \rightarrow_{q,\mathcal{S}} t_1 = s[v\theta]_q \xrightarrow{*}_{\mathcal{S}} s[v'\theta]_q = s[l_1\sigma_1,\ldots,v'\theta,\ldots,l_n\sigma_n]_{p_1,\ldots,q,\ldots,p_n}$. Clearly, $U' \setminus W$ and $U' \setminus \{q.o \mid o \in V\}$ are sets of parallel positions. Thus it remains to show that positions from W are parallel to the positions from $\{q.o \mid o \in V\}$. By the fact $\{l'(q_x) \mid q_x \in \mathrm{Pos}_{\mathcal{V}}(l'), \exists i\,(q.q_x \leq p_i)\} \subseteq Y$, for any $o_1 \in W$ there exists $o_y \in \mathrm{Pos}_Y(v')$ such that $q.o_y \leq o_1$. Since any position $o_y \in \mathrm{Pos}_Y(v')$ and $o_2 \in V$ are parallel, any $o_1 \in W$ and any $q.o_2$ ($o_2 \in V$) are parallel. ◀

▶ **Theorem 3.9** (confluence criterion using parallel critical pairs)**.** *Let $\mathcal{P}, \mathcal{S}$ be TRSs such that $\mathcal{S}$ is left-linear and terminating and $\mathcal{P}$ is reversible. Suppose (i) $\mathrm{CP}(\mathcal{S},\mathcal{S}) \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \mathrel{\#\!\!\!\!\leftrightarrow}_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$, (ii) for all $\langle u,v\rangle_X \in \mathrm{PCP}_{in}(\mathcal{P}\cup\mathcal{P}^{-1},\mathcal{S})$, $u \xrightarrow{*}_{\mathcal{S}} u' \mathrel{\#\!\!\!\!\leftrightarrow}_{V,\mathcal{P}} v' \xleftarrow{*}_{\mathcal{S}} v$ for some u',v' and V satisfying $\bigcup_{q\in V}\mathcal{V}(v'/q) \subseteq X$ and (iii) $\mathrm{CP}(\mathcal{S},\mathcal{P}\cup\mathcal{P}^{-1}) \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \mathrel{\#\!\!\!\!\leftrightarrow}_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$. Then $\mathcal{S}\cup\mathcal{P}$ is confluent.*

Proof. Similar to proof of the Theorem 3.7, using Lemmas 3.4, 3.8. ◀

Since, by the definition of parallel critical pairs, $\mathrm{CP}_{in}(\mathcal{P}\cup\mathcal{P}^{-1},\mathcal{S}) \subseteq \mathrm{PCP}_{in}(\mathcal{P}\cup\mathcal{P}^{-1},\mathcal{S})$ holds. Thus the condition (ii) of Theorem 3.7 is a particular case of condition (ii) of Theorem 3.9. Hence Theorem 3.7 is subsumed by Theorem 3.9.

▶ **Example 3.10.** Let

$$\mathcal{R} = \left\{ \begin{array}{llllllll}
(a) & +(0,y) & \rightarrow & y & (b) & +(\mathsf{s}(x),y) & \rightarrow & \mathsf{s}(+(x,y)) \\
(c) & +(x,0) & \rightarrow & x & (d) & +(x,\mathsf{s}(y)) & \rightarrow & \mathsf{s}(+(x,y)) \\
(e) & +(x,y) & \rightarrow & +(y,x) & (f) & +(+(x,y),z) & \rightarrow & +(x,+(y,z))
\end{array} \right\}.$$

Put $\mathcal{S} = \{(a), (b), (c), (d)\}$ and $\mathcal{P} = \{(e), (f)\}$. Then $\mathcal{S}$ is linear and terminating. We have $+(x, +(y, z)) \to_{\mathcal{P}} +(+(y, z), x) \to_{\mathcal{P}} +(+(z, y), x) \to_{\mathcal{P}} +(z, +(y, x)) \to_{\mathcal{P}} +(z, +(x, y)) \to_{\mathcal{P}} +(+(x, y), z)$. Thus $\mathcal{P}$ is reversible. We have $\mathrm{CP}(\mathcal{S}, \mathcal{S}) =$

$$\left\{\begin{array}{llll}
\langle 0, 0\rangle & \in \xleftarrow{*}_{\mathcal{S}} & \langle \mathsf{s}(y), \mathsf{s}(+(0, y))\rangle & \in \leftarrow_{\mathcal{S}} \\
\langle \mathsf{s}(+(x, 0)), \mathsf{s}(x)\rangle & \in \to_{\mathcal{S}} & \langle \mathsf{s}(x), \mathsf{s}(+(x, 0))\rangle & \in \leftarrow_{\mathcal{S}} \\
\langle \mathsf{s}(+(0, y)), \mathsf{s}(y)\rangle & \in \to_{\mathcal{S}} & \langle \mathsf{s}(+(x, \mathsf{s}(y))), \mathsf{s}(+(\mathsf{s}(x), y))\rangle & \in \to_{\mathcal{S}} \circ \leftarrow_{\mathcal{S}} \\
\langle \mathsf{s}(+(\mathsf{s}(x), y)), \mathsf{s}(+(x, \mathsf{s}(y)))\rangle & \in \to_{\mathcal{S}} \circ \leftarrow_{\mathcal{S}} & &
\end{array}\right\},$$

$\mathrm{CP}_{in}(\mathcal{P} \cup \mathcal{P}^{-1}, \mathcal{S}) = \emptyset$ and $\mathrm{CP}(\mathcal{S}, \mathcal{P} \cup \mathcal{P}^{-1}) =$

$$\left\{\begin{array}{llll}
\langle y, +(y, 0)\rangle & \in \leftarrow_{\mathcal{S}} & \langle +(y, z), +(0, +(y, z))\rangle & \in \leftarrow_{\mathcal{S}} \\
\langle +(y, z), +(+(0, y), z)\rangle & \in \leftarrow_{\mathcal{S}} & \langle +(x, z), +(+(x, 0), z)\rangle & \in \leftarrow_{\mathcal{S}} \\
\langle \mathsf{s}(+(x, y)), +(y, \mathsf{s}(x))\rangle & \in \leftrightarrow_{\mathcal{P}} \circ \leftarrow_{\mathcal{S}} & & \\
\langle +(\mathsf{s}(+(x, y)), z), +(\mathsf{s}(x), +(y, z))\rangle & \in \to_{\mathcal{S}} \circ \leftrightarrow_{\mathcal{P}} \circ \leftarrow_{\mathcal{S}} & & \\
\langle \mathsf{s}(+(x, +(y, z))), +(+(\mathsf{s}(x), y), z)\rangle & \in \leftrightarrow_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}} & & \\
\langle +(x, \mathsf{s}(+(y, z))), +(+(x, \mathsf{s}(y)), z)\rangle & \in \to_{\mathcal{S}} \circ \leftrightarrow_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}} & & \\
\langle x, +(0, x)\rangle & \in \leftarrow_{\mathcal{S}} & \langle +(x, y), +(x, +(y, 0))\rangle & \in \leftarrow_{\mathcal{S}} \\
\langle +(y, z), +(y, +(0, z))\rangle & \in \leftarrow_{\mathcal{S}} & \langle +(x, y), +(+(x, y), 0)\rangle & \in \leftarrow_{\mathcal{S}} \\
\langle \mathsf{s}(+(x, y)), +(\mathsf{s}(y), x)\rangle & \in \leftrightarrow_{\mathcal{P}} \circ \leftarrow_{\mathcal{S}} & & \\
\langle \mathsf{s}(+(+(x, y), z)), +(x, +(y, \mathsf{s}(z)))\rangle & \in \leftrightarrow_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}} & & \\
\langle +(\mathsf{s}(+(x, y)), z), +(x, +(\mathsf{s}(y), z))\rangle & \in \to_{\mathcal{S}} \circ \leftrightarrow_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}} & & \\
\langle +(x, \mathsf{s}(+(y, z))), +(+(x, y), \mathsf{s}(z))\rangle & \in \to_{\mathcal{S}} \circ \leftrightarrow_{\mathcal{P}} \circ \leftarrow_{\mathcal{S}} & &
\end{array}\right\}.$$

Thus one can apply Theorem 3.9 (or Theorem 3.7) to obtain the confluence of $\mathcal{R} = \mathcal{S} \cup \mathcal{P}$.

For the case the terminating TRS $\mathcal{S}$ is linear, one can obtain another confluence criterion from the abstract confluence criterion using $\vdash\!\dashv_0 := \leftrightarrow_{\mathcal{P}}$ instead of $\vdash\!\dashv_0 := \mathrel{\parallel\!\!\!\!\!\leftrightarrow}_{\mathcal{P}}$.

▶ **Lemma 3.11.** *Let $\mathcal{P}, \mathcal{S}$ be TRSs. Suppose* $\mathrm{CP}(\mathcal{S}, \mathcal{S}) \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \stackrel{=}{\leftrightarrow}_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$. *Then* $\leftarrow_{\mathcal{S}} \circ \to_{\mathcal{S}} \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \stackrel{=}{\leftrightarrow}_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$.

Proof. Take $\vdash\!\dashv := \leftrightarrow_{\mathcal{P}}$ in Lemma 3.3. ◀

▶ **Lemma 3.12.** *Let $\mathcal{P}, \mathcal{S}$ be TRSs such that $\mathcal{S}$ is linear and $\mathcal{P}$ is bidirectional. Suppose* $\mathrm{CP}(\mathcal{S}, \mathcal{P} \cup \mathcal{P}^{-1}) \cup \mathrm{CP}(\mathcal{P} \cup \mathcal{P}^{-1}, \mathcal{S}) \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \stackrel{=}{\leftrightarrow}_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$. *Then* $\stackrel{=}{\leftrightarrow}_{\mathcal{P}} \circ \to_{\mathcal{S}} \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \stackrel{=}{\leftrightarrow}_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$.

Proof. In a similar way to the proof of Lemma 3.5. ◀

▶ **Theorem 3.13** (confluence criterion for linear $\mathcal{S}$)**.** *Let $\mathcal{P}, \mathcal{S}$ be TRSs such that $\mathcal{S}$ is linear and terminating and $\mathcal{P}$ is reversible. Suppose (i)* $\mathrm{CP}(\mathcal{S}, \mathcal{S}) \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \stackrel{=}{\leftrightarrow}_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$, *(ii)* $\mathrm{CP}(\mathcal{P} \cup \mathcal{P}^{-1}, \mathcal{S}) \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \stackrel{=}{\leftrightarrow}_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$ *and (iii)* $\mathrm{CP}(\mathcal{S}, \mathcal{P} \cup \mathcal{P}^{-1}) \subseteq \xrightarrow{*}_{\mathcal{S}} \circ \stackrel{=}{\leftrightarrow}_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$. *Then $\mathcal{S} \cup \mathcal{P}$ is confluent.*

Proof. Similar to proof of the Theorem 3.7, using Lemmas 3.11, 3.12. ◀

The next examples show that Theorem 3.13 and Theorems 3.7/3.9 are incomparable (Figure 1).

▶ **Example 3.14.** Let $\mathcal{R}$ be the one given in Example 3.10. Consider a TRS $\mathcal{R}_1 = \mathcal{R} \cup \{\mathsf{dbl}(x) \to +(x, x)\}$. One can easily confirm that the confluence of $\mathcal{R}_1$ is shown in the same way as $\mathcal{R}$ using Theorem 3.7. Since $\mathcal{R}_1$ is not linear, however, Theorem 3.13 does not apply. Consider a TRS $\mathcal{R}_2 = \mathcal{R} \cup \{\mathsf{s}(x) \to \mathsf{s}(\mathsf{s}(x)), \mathsf{s}(\mathsf{s}(x)) \to \mathsf{s}(x)\}$. By putting $\mathcal{S}_2 = \mathcal{S}$ and $\mathcal{P}_2 = \mathcal{P} \cup \{\mathsf{s}(x) \to \mathsf{s}(\mathsf{s}(x)), \mathsf{s}(\mathsf{s}(x)) \to \mathsf{s}(x)\}$, one can show the confluence of $\mathcal{R}_2$ using Theorem 3.13. On the other hand, $\langle +(\mathsf{s}(\mathsf{s}(x)), y), \mathsf{s}(+(x, y))\rangle_{\{x\}} \in \mathrm{PCP}_{in}(\mathcal{P} \cup \mathcal{P}^{-1}, \mathcal{S})$ and thus the condition of Theorem 3.9 is not satisfied. Theorem 3.9 does not apply to other partitions of $\mathcal{R}_2$ either. Thus one can not show the confluence of $\mathcal{R}_2$ using Theorem 3.9.

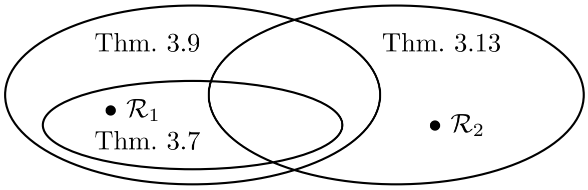

Figure 1 Relation of three confluence criterion

4 Reduction-preserving completion

There are cases where our confluence criteria can be applicable indirectly. The idea is to construct a TRS suitable for applying our theorems by exchanging or adding rewrite rules without changing the equivalence of the reduction so that the confluence of the transformed TRS implies that of the original TRS. Using the reversibility of $\mathcal{P}$, there are several flexibilities on such transformations. The notion of reduction equivalence and following properties of reduction equivalence are well-known in literature and the latter are easily proved.

▶ **Definition 4.1** (reduction equivalence). Two relation $\rightarrow_0$ and $\rightarrow_1$ are said to be *reduction equivalent* if $\xrightarrow{*}_0 = \xrightarrow{*}_1$. Two TRSs $\mathcal{R}$ and $\mathcal{R}'$ are reduction equivalent if so are $\rightarrow_{\mathcal{R}}$ and $\rightarrow_{\mathcal{R}'}$.

▶ **Proposition 4.2** (properties of reduction equivalence). *(i) If $\rightarrow_{\mathcal{R}} \subseteq \xrightarrow{*}_{\mathcal{R}'}$ and $\rightarrow_{\mathcal{R}'} \subseteq \xrightarrow{*}_{\mathcal{R}}$ then $\mathcal{R}$ and $\mathcal{R}'$ are reduction equivalent. (ii) If $\mathcal{R}$ and $\mathcal{R}'$ are reduction equivalent then the confluence of $\mathcal{R}$ and $\mathcal{R}'$ coincide.*

We now demonstrate how the confluence criteria in the previous section can be applied indirectly using the notion of reduction equivalence.

▶ **Example 4.3** (confluence by reduction equivalence). Let (a)–(f) be rewrite rules given in Example 3.10. We show the confluence of $\mathcal{R} = \{(a),(b),(e),(f)\}$. Theorems 3.9 and 3.13 can not be applied directly to prove this—for example, if we put $\mathcal{S} = \{(a),(b)\}$ and $\mathcal{P} = \{(e),(f)\}$, then we have $\langle y, +(y, \mathsf{0})\rangle \in \mathrm{CP}(\mathcal{S}, \mathcal{P} \cup \mathcal{P}^{-1})$ which is not joinable by $\xrightarrow{*}_{\mathcal{S}} \circ \nleftrightarrow_{\mathcal{P}} \circ \xleftarrow{*}_{\mathcal{S}}$. Let $\mathcal{R}' = \mathcal{R} \cup \{(c),(d)\}$. Then since we have $+(x, \mathsf{0}) \rightarrow_{\mathcal{P}} +(\mathsf{0}, x) \rightarrow_{\mathcal{S}} x$ and $+(x, \mathsf{s}(y)) \rightarrow_{\mathcal{P}} +(\mathsf{s}(y), x) \rightarrow_{\mathcal{S}} \mathsf{s}(+(y,x) \rightarrow_{\mathcal{P}} \mathsf{s}(+(x,y))$, the inclusions $\rightarrow_{\mathcal{R}} \subseteq \rightarrow_{\mathcal{R}'} \subseteq \xrightarrow{*}_{\mathcal{R}}$ hold. Hence $\mathcal{R}$ and $\mathcal{R}'$ are reduction equivalent by Proposition 4.2 (i). As we have shown in Example 3.10, $\mathcal{R}'$ is confluent. Thus by Proposition 4.2 (ii), $\mathcal{R}$ is confluent either.

In this example, two additional rewrite rules (c) and (d) are given by hand. But in automated confluence proving procedures, one needs to find such new rewrite rules automatically. We next present a completion-like procedure to automate such additions (or more generally transformations) of rewrite rules. We first present an abstract version of the procedure in the form of inference rules and prove its soundness w.r.t. the confluence proof.

▶ **Definition 4.4** (abstract reduction-preserving completion procedure). Inference rules of an *abstract reduction-preserving completion procedure* are listed in Figure 2. The inference rules act on a pair of TRSs. One step derivation using any of inference rules (from upper to lower) is denoted by $\rightsquigarrow$. We also write $\rightsquigarrow^p$ ($\rightsquigarrow^r, \rightsquigarrow^a$) for an inference step by the rule *Partition* (*Replacement*, *Addition*, respectively).

$$\textit{Partition}\quad \frac{\langle \mathcal{S}, \mathcal{P}\rangle}{\langle \mathcal{S}', \mathcal{P}'\rangle}\ \mathcal{S}\cup\mathcal{P} = \mathcal{S}'\cup\mathcal{P}',\ \mathcal{P}'\text{: reversible}$$

$$\textit{Replacement}\quad \frac{\langle \mathcal{S}\cup\{l\to r\}, \mathcal{P}\rangle}{\langle \mathcal{S}\cup\{l\to r'\}, \mathcal{P}\rangle}\ r \overset{*}{\leftrightarrow}_{\mathcal{P}} r'$$

$$\textit{Addition}\quad \frac{\langle \mathcal{S}, \mathcal{P}\rangle}{\langle \mathcal{S}\cup\{l\to r\}, \mathcal{P}\rangle}\ l \overset{*}{\leftrightarrow}_{\mathcal{P}} \circ \overset{*}{\to}_{\mathcal{S}} r$$

Figure 2 Inference rules of reduction-preserving completion

▶ **Theorem 4.5** (soundness of the abstract reduction-preserving completion procedure)**.** *Let $\langle \mathcal{R}, \emptyset\rangle = \langle \mathcal{S}_0, \mathcal{P}_0\rangle \overset{*}{\rightsquigarrow} \langle \mathcal{S}_n, \mathcal{P}_n\rangle$ be a derivation of abstract reduction-preserving completion procedure. Suppose that $\mathcal{S}_n, \mathcal{P}_n$ satisfy the conditions of Theorem 3.9 or Theorem 3.13. Then $\mathcal{R}$ is confluent.*

Proof. We show, for any inference step $\langle \mathcal{S}_i, \mathcal{P}_i\rangle \rightsquigarrow \langle \mathcal{S}_{i+1}, \mathcal{P}_{i+1}\rangle$, that $\mathcal{S}_i \cup \mathcal{P}_i$ and $\mathcal{S}_{i+1}\cup\mathcal{P}_{i+1}$ are reduction equivalent and that $\mathcal{P}_{i+1}$ is reversible whenever so is $\mathcal{P}_i$.

- Case $\langle \mathcal{S}_i, \mathcal{P}_i\rangle \rightsquigarrow \langle \mathcal{S}_{i+1}, \mathcal{P}_{i+1}\rangle$ by *Partition*. Then since $\mathcal{S}_i \cup \mathcal{P}_i = \mathcal{S}_{i+1}\cup\mathcal{P}_{i+1}$ and $\mathcal{P}_{i+1}$ is reversible by the side condition, the claim follows immediately.
- Case $\langle \mathcal{S}_i, \mathcal{P}_i\rangle \rightsquigarrow \langle \mathcal{S}_{i+1}, \mathcal{P}_{i+1}\rangle$ by *Replacement*. Then $\mathcal{S}_i = \mathcal{S}_i' \cup \{l \to r\}$, $r \overset{*}{\leftrightarrow}_{\mathcal{P}_i} r'$ and $\mathcal{S}_{i+1} = \mathcal{S}_i' \cup \{l \to r'\}$ for some $\mathcal{S}_i'$, l, r, r' and $\mathcal{P}_{i+1} = \mathcal{P}_i$. By the reversiblity of $\mathcal{P}_i$, we have $l \to_{\mathcal{S}_i} r \overset{*}{\to}_{\mathcal{P}_i} r'$ hence $\to_{\mathcal{S}_{i+1}\cup\mathcal{P}_{i+1}} \subseteq \overset{*}{\to}_{\mathcal{S}_i\cup\mathcal{P}_i}$. By the reversiblity of $\mathcal{P}_i$, we also have $l \to_{\mathcal{S}_{i+1}} r' \overset{*}{\to}_{\mathcal{P}_i} r$, hence $\to_{\mathcal{S}_i\cup\mathcal{P}_i} \subseteq \overset{*}{\to}_{\mathcal{S}_{i+1}\cup\mathcal{P}_{i+1}}$. Thus by Proposition 4.2 (i), $\mathcal{S}_i \cup \mathcal{P}_i$ and $\mathcal{S}_{i+1}\cup\mathcal{P}_{i+1}$ are reduction equivalent. Hence, by $\mathcal{P}_{i+1} = \mathcal{P}_i$, the claim follows.
- Case $\langle \mathcal{S}_i, \mathcal{P}_i\rangle \rightsquigarrow \langle \mathcal{S}_{i+1}, \mathcal{P}_{i+1}\rangle$ by *Addition*. Then $l \overset{*}{\leftrightarrow}_{\mathcal{P}_i} \circ \overset{*}{\to}_{\mathcal{S}_i} r$ and $\mathcal{S}_{i+1} = \mathcal{S}_i \cup \{l \to r\}$ for some l, r and and $\mathcal{P}_{i+1} = \mathcal{P}_i$. Since $\mathcal{S}_i\cup\mathcal{P}_i \subseteq \mathcal{S}_{i+1}\cup\mathcal{P}_{i+1}$, we have $\to_{\mathcal{S}_i\cup\mathcal{P}_i} \subseteq \overset{*}{\to}_{\mathcal{S}_{i+1}\cup\mathcal{P}_{i+1}}$. By the reversiblity of $\mathcal{P}_i$, $l \overset{*}{\to}_{\mathcal{P}_i} \circ \overset{*}{\to}_{\mathcal{S}_i} r'$. Hence $\to_{\mathcal{S}_{i+1}\cup\mathcal{P}_{i+1}} \subseteq \overset{*}{\to}_{\mathcal{S}_i\cup\mathcal{P}_i}$. Thus by Proposition 4.2 (i), $\mathcal{S}_i\cup\mathcal{P}_i$ and $\mathcal{S}_{i+1}\cup\mathcal{P}_{i+1}$ are reduction equivalent. Hence, by $\mathcal{P}_{i+1} = \mathcal{P}_i$, the claim follows.

Thus by induction on n, it follows that $\mathcal{R}$ and $\mathcal{S}_n \cup \mathcal{P}_n$ are reduction equivalent. By Theorem 3.9 or 3.13, $\mathcal{S}_n \cup \mathcal{P}_n$ is confluent, and hence $\mathcal{R}$ is confluent by Proposition 4.2 (ii). ◀

▶ **Example 4.6** (derivations in abstract reduction-preserving completion procedure)**.** The confluence proof of Example 4.3 is derived by the abstract reduction-preserving completion procedure. Let rewrite rules (a)–(f) be those given in Example 3.10. Give $\mathcal{R} = \{(a), (b), (e), (f)\}$ as the input to the procedure. Let us consider the following derivation.

$$\begin{array}{llll}
\langle \mathcal{S}_0, \mathcal{P}_0\rangle = \langle\{(a),(b),(e),(f)\},\emptyset\rangle & \rightsquigarrow^p & \langle\{(a),(b)\},\{(e),(f)\}\rangle & = \langle \mathcal{S}_1, \mathcal{P}_1\rangle \\
& \rightsquigarrow^a & \langle\{(a),(b),(c)\},\{(e),(f)\}\rangle & = \langle \mathcal{S}_2, \mathcal{P}_2\rangle \\
& \rightsquigarrow^a & \langle\{(a),(b),(c),(d')\},\{(e),(f)\}\rangle & = \langle \mathcal{S}_3, \mathcal{P}_3\rangle \\
& \rightsquigarrow^r & \langle\{(a),(b),(c),(d)\},\{(e),(f)\}\rangle & = \langle \mathcal{S}_4, \mathcal{P}_4\rangle
\end{array}$$

where $(d') : +(x, \mathsf{s}(y)) \to \mathsf{s}(+(y, x))$. Then $\mathcal{S}_4 = \{(a), (b), (c), (d)\}$ and $\mathcal{P}_4 = \{(e), (f)\}$ satisfy the conditions of Theorem 3.7. Thus, by Theorem 4.5, $\mathcal{R}$ is confluent.

Next we present a concrete reduction-preserving completion procedure that can be used as the basis of an automated completion procedure. The procedure presented below is designed so as to apply Theorem 3.7, but it is straightforward to modify the procedure suitable for Theorem 3.9 and/or Theorem 3.13.

▶ **Definition 4.7** (concrete reduction-preserving completion procedure).

Input: a TRS $\mathcal{R}$
Output: `Success` or `Failure` (or may diverge)

Step 1. Put $\mathcal{R}_0 := \mathcal{R}$ and $i := 0$. Proceed to Step 2.
Step 2. Take a partition $\mathcal{S}_i \cup \mathcal{P}_i = \mathcal{R}_i$ such that $\mathcal{S}_i$ is left-linear and terminating, $\mathcal{P}_i$ is reversible and $\mathrm{CP}_{in}(\mathcal{P}_i \cup {\mathcal{P}_i}^{-1}, \mathcal{S}_i) = \emptyset$. Proceed to Step 3. If there is no such a partition then return `Failure`.
Step 3. Set $\mathcal{U} := \emptyset$. For each $\langle u, v\rangle \in \mathrm{CP}(\mathcal{S}_i, \mathcal{P}_i \cup {\mathcal{P}_i}^{-1})$, take $\mathcal{S}_i$-normal forms $\hat{u}, \hat{v}$ of u, v, respectively and check whether $\hat{u} \mathrel{+\!\!\!\leftrightarrow\!\!\!+}_{\mathcal{P}_i} \hat{v}$. If not $\hat{u} \mathrel{+\!\!\!\leftrightarrow\!\!\!+}_{\mathcal{P}_i} \hat{v}$ then put $\mathcal{U} := \mathcal{U} \cup \{v \to \hat{u}\}$. Finally if $\mathcal{U} = \emptyset$ then proceed to Step 4. Otherwise take some non-empty $\mathcal{U}' \subseteq \mathcal{U}$ and put $\mathcal{R}_{i+1} := \mathcal{R}_i \cup \mathcal{U}'$, $i := i + 1$ and go to Step 2.
Step 4. Set $\mathcal{U} := \emptyset$. For each $\langle u, v\rangle \in \mathrm{CP}(\mathcal{S}_i, \mathcal{S}_i)$, take $\mathcal{S}_i$-normal forms $\hat{u}, \hat{v}$ of u, v, respectively and check whether $\hat{u} \mathrel{+\!\!\!\leftrightarrow\!\!\!+}_{\mathcal{P}_i} \hat{v}$. If not $\hat{u} \mathrel{+\!\!\!\leftrightarrow\!\!\!+}_{\mathcal{P}_i} \hat{v}$ then put $\mathcal{U} := \mathcal{U} \cup \{\hat{u} \approx \hat{v}\}$. Finally if $\mathcal{U} = \emptyset$ then return `Success`. Otherwise take some set $\mathcal{U}' \subseteq (\mathcal{U} \cup \mathcal{U}^{-1}) \cap \stackrel{*}{\leftrightarrow}_{\mathcal{P}_i}$ of rewrite rules and put $\mathcal{R}_{i+1} := \mathcal{R}_i \cup \mathcal{U}'$, $i := i + 1$ and go to Step 2.

During the step 2, one may perform the following additional steps.

Step 2a. If there exist $l \to r \in \mathcal{S}_i$ and r' such that $r \leftrightarrow_{\mathcal{P}_i} r'$ and $\mathrm{CP}_{in}(\mathcal{P}_i \cup {\mathcal{P}_i}^{-1}, \{l \to r\}) \neq \emptyset$, then put $\mathcal{R}_{i+1} := (\mathcal{R}_i \setminus \{l \to r\}) \cup \{l \to r'\}$, $i := i + 1$.
Step 2b. Let $\langle u, v\rangle \in \mathrm{CP}_{in}(\mathcal{P}_i \cup {\mathcal{P}_i}^{-1}, \mathcal{S}_i)$ and let $\hat{v}$ be $\mathcal{S}_i$-normal form of v. Then put $\mathcal{R}_{i+1} := \mathcal{R}_i \cup \{u \to \hat{v}\}$ and $i := i + 1$.

Before moving from the step 3 to the step 2, one may perform the following additional step.

Step 3a. Set $\mathcal{S}_i := \mathcal{S}_{i-1}, \mathcal{P}_i := \mathcal{P}_{i-1}$. If there exist $l \to r \in \mathcal{S}_i$ and r' such that $r \leftrightarrow_{\mathcal{P}_i} r'$ and there exists $\langle u, v\rangle \in \mathrm{CP}(\{l \to r\}, \mathcal{P}_i \cup {\mathcal{P}_i}^{-1})$ such that $\hat{u} \mathrel{+\!\!\!\leftrightarrow\!\!\!+}_{\mathcal{P}_i} \hat{v}$ does not hold where $\hat{u}, \hat{v}$ are $\mathcal{S}_i$-normal forms of u, v, respectively, then put $\mathcal{R}_{i+1} := (\mathcal{R}_i \setminus \{l \to r\}) \cup \{l \to r'\}$, $i := i + 1$.

Before moving from the step 4 to the step 2, one may perform the following additional step.

Step 4a. Set $\mathcal{S}_i := \mathcal{S}_{i-1}, \mathcal{P}_i := \mathcal{P}_{i-1}$. If there exist $l \to r \in \mathcal{S}_i$ and r' such that $r \leftrightarrow_{\mathcal{P}_i} r'$ and there exists $\langle u, v\rangle \in \mathrm{CP}(\{l \to r\}, \mathcal{S}_i) \cup \mathrm{CP}(\mathcal{S}_i, \{l \to r\})$ such that $\hat{u} \mathrel{+\!\!\!\leftrightarrow\!\!\!+}_{\mathcal{P}_i} \hat{v}$ does not hold where $\hat{u}, \hat{v}$ are $\mathcal{S}_i$-normal forms of u, v, respectively, then put $\mathcal{R}_{i+1} := (\mathcal{R}_i \setminus \{l \to r\}) \cup \{l \to r'\}$, $i := i + 1$.

▶ **Corollary 4.8** (soundness of the concrete reduction-preserving completion procedure). *If the procedure of Definition 4.7 succeeds for the input $\mathcal{R}$, then $\mathcal{R}$ is confluent.*

Proof. It suffices to show if the procedure succeeds then there exists a successful derivation of the abstract reduction-preserving completion procedure. Step 1 corresponds to the empty derivation. Step 2 corresponds to an inference step by *Partition*. For any $\langle u, v\rangle \in \mathrm{CP}(\mathcal{S}_i, \mathcal{P}_i \cup {\mathcal{P}_i}^{-1})$, we have $u \leftarrow_{\mathcal{S}_i} \circ \leftrightarrow_{\mathcal{P}_i} v$, and hence $v \leftrightarrow_{\mathcal{P}_i} \circ \stackrel{*}{\to}_{\mathcal{S}_i} \hat{u}$. Thus, Step 3 is simulated by multiple inference steps by *Addition*. Similarly, Steps 4 and 2b are simulated by multiple inference steps by *Addition*. Steps 2a, 3a, 4a are simulated by inference steps by *Replace*. ◀

▶ **Example 4.9.** Let

$$\mathcal{R} = \left\{ \begin{array}{llllllll} (a) & +(0,y) & \to & y & (b) & +(x,\mathsf{s}(y)) & \to & \mathsf{s}(+(x,y)) \\ (c) & +(x,y) & \to & +(y,x) & (d) & +(+(x,y),z) & \to & +(x,+(y,z)) \end{array} \right\}$$

1. (Step 1) We put $\mathcal{R}_0 := \{(a),(b),(c),(d)\}$.
2. (Step 2) We take $\mathcal{S}_0 = \{(a),(b)\}$ and $\mathcal{P}_0 = \{(c),(d)\}$. Then $\mathcal{S}_0$ is left-linear and terminating, $\mathcal{P}_0$ is reversible and $\mathrm{CP}_{in}(\mathcal{P}_0 \cup {\mathcal{P}_0}^{-1}, \mathcal{S}_0) = \emptyset$.
3. (Step 3) We have $\mathrm{CP}(\mathcal{S}_0, \mathcal{P}_0 \cup {\mathcal{P}_0}^{-1}) =$

$$\left\{ \begin{array}{llll} (1) & \langle +(y,z), +(0,+(y,z)) \rangle & (5) & \langle \mathsf{s}(+(+(x,z),y)), +(x,+(z,\mathsf{s}(y))) \rangle \\ (2) & \langle +(y,z), +(+(0,y),z) \rangle & (6) & \langle +(\mathsf{s}(+(x,y)),z), +(x,+(\mathsf{s}(y),z)) \rangle \\ (3) & \langle +(x,y), +(+(x,0),y) \rangle & (7) & \langle +(z,\mathsf{s}(+(x,y))), +(+(z,x),\mathsf{s}(y)) \rangle \\ (4) & \langle y, +(y,0) \rangle & (8) & \langle \mathsf{s}(+(x,y)), +(\mathsf{s}(y),x) \rangle \end{array} \right\}.$$

   Then for $\langle u,v \rangle \in \{(3),(4),(6),(8)\}$, $\mathcal{S}_0$-normal forms of u,v are not joinable by a $\twoheadleftarrow\!\!\!\twoheadrightarrow_{\mathcal{P}_0}$-step. Put $\mathcal{R}_1 := \mathcal{R}_0 \cup \mathcal{U}' =$

$$\mathcal{R}_0 \cup \{ \ (e) \quad +(y,0) \to y \qquad (f) \quad +(\mathsf{s}(y),x) \to \mathsf{s}(+(x,y)) \ \}.$$

   and go to the step 2.
4. (Step 2) We take $\mathcal{S}_1 = \{(a),(b),(e),(f)\}$ and $\mathcal{P}_1 = \{(c),(d)\}$. Then $\mathcal{S}_1$ is left-linear and terminating, $\mathcal{P}_1$ is reversible, and $\mathrm{CP}_{in}(\mathcal{P}_1 \cup {\mathcal{P}_1}^{-1}, \mathcal{S}_1) = \emptyset$.
5. (Step 3) There are four elements including (9) $\langle +(\mathsf{s}(+(x,y)),z), +(x,+(\mathsf{s}(y),z)) \rangle$ in $\mathrm{CP}(\mathcal{S}_1, \mathcal{P}_1 \cup {\mathcal{P}_1}^{-1})$ whose $\mathcal{S}_1$-normal forms are not joinable by a $\twoheadleftarrow\!\!\!\twoheadrightarrow_{\mathcal{P}_1}$-step. Here we put $\mathcal{U}' := \emptyset$, $\mathcal{R}_2 := \mathcal{R}_1$, $i := 2$ and proceed to Step 3a.
6. (Step 3a) Since $(9) \in \mathrm{CP}(\{(f)\}, \mathcal{P}_1 \cup {\mathcal{P}_1}^{-1})$ and $\mathsf{s}(+(x,y)) \to_{\mathcal{P}_2} \mathsf{s}(+(y,x))$. Hence put $\mathcal{R}_3 := (\mathcal{R}_2 \setminus \{(f)\}) \cup \{ \ (g) \quad +(\mathsf{s}(y),x) \to \mathsf{s}(+(y,x)) \ \}$ and $i := 3$ and go to Step 2.
7. (Step 2) We take $\mathcal{S}_3 = \{(a),(b),(e),(g)\}$ and $\mathcal{P}_3 = \{(c),(d)\}$. Then $\mathcal{S}_3$ is left-linear and terminating, $\mathcal{P}_3$ is reversible and $\mathrm{CP}_{in}(\mathcal{P}_3 \cup {\mathcal{P}_3}^{-1}, \mathcal{S}_3) = \emptyset$. Thus proceed to Step 3.
8. (Step 3) For any $\langle u,v \rangle \in \mathrm{CP}(\mathcal{S}_3, \mathcal{P}_3 \cup {\mathcal{P}_3}^{-1})$, $\mathcal{S}_3$-normal forms of u,v are joinable by a $\twoheadleftarrow\!\!\!\twoheadrightarrow_{\mathcal{P}_3}$-step (Example 3.10). Hence proceed to Step 4.
9. (Step 4) For any $\langle u,v \rangle \in \mathrm{CP}(\mathcal{S}_3, \mathcal{S}_3)$, $\mathcal{S}_3$-normal forms of u,v are joinable by a $\twoheadleftarrow\!\!\!\twoheadrightarrow_{\mathcal{P}_3}$-step (Example 3.10). Thus `Success` is returned.

5 Implementation and experiments

All results of this paper have been implemented. The program is written in SML/NJ[1] and is built upon confluence prover ACP[2] [1, 2, 21].

In Figure 3, we present a pseudo-code of main function of our implementation of reduction-preserving completion procedure enough for describing some heuristics employed in the implementation. A short description of functions involved in our pseudo-code and heuristics employed follows.

- (`checkConfluence` $\mathcal{R}$) is the main function of the procedure. It simulates multiple runs in the breadth-first strategy.

  Let $\mathcal{D} = \{l(\epsilon) \mid l \to r \in \mathcal{R}\}$ and $\mathcal{C} = \mathcal{F} \setminus \mathcal{C}$.

[1] `http://www.smlnj.org/`
[2] `http://www.nue.riec.tohoku.ac.jp/tools/acp/`

```
fun check (S,P,i) = if i = 0 then (apply Theorem 3.9)
                    else (apply Theorem 3.13)
fun checkConfluence R =
    let fun step [] = Failure
        |   step ((S,P,i)::rest) = case check (S,P,i) of
                NONE ⇒ step rest
              | SOME ([],[]) ⇒ Succeess
              | SOME nj ⇒ step (rest @
                             (mapAppend decompose (trans (S,P) nj)))
    in step (decompose R) end
```

Figure 3 Pseudo-code of the main function

- (`decompose` $\mathcal{R}$) decomposes $\mathcal{R}$ into $\mathcal{S} \cup \mathcal{P}$ and duplicates $\mathcal{S} \cup \mathcal{P}$. Hence a list of triples $(\mathcal{S}, \mathcal{P}, i)$ where $\mathcal{S} \cup \mathcal{P} = \mathcal{R}$ and $i \in \{0, 1\}$ are returned. Here, however, not all partitions but only one partition of $\mathcal{R}$ are returned based on a heuristic, namely that $\mathcal{P}$ is the set of the rules $l \to r$ satisfying either (1) $r \to l \in \mathcal{R}$ or (1′) $\mathcal{F}(l) = \mathcal{F}(r) \subseteq \mathcal{D}$ and (2′) $l(\epsilon), r(\epsilon) \in \mathcal{D}$ implies $l(\epsilon) = r(\epsilon)$.
- (`check` $(\mathcal{S}, \mathcal{P}, i)$) checks whether conditions of Theorem 3.9 (or Theorem 3.13) are satisfied. If $\mathcal{S}$ is not left-linear or it fails to prove termination of $\mathcal{S}$ or reversibility of $\mathcal{P}$, then `NONE` is returned. Reversibility is tested by checking $r \overset{\leq k}{\to} l$ for some constant k (in our implementation, we set $k = 10$). If all conditions other than the critical pairs conditions are satisfied then non-joinable critical pairs and rewrite rules generating such critical pairs are returned in the form `SOME` (U_1, U_2). For example, in the case of $i = 0$, from $\mathrm{CP}(\mathcal{S}, \mathcal{S})$ the list $U_1 = \bigcup_{l \to r, l' \to r' \in \mathcal{S}} \{\langle l \to r, l' \to r', u, v\rangle \mid \langle u, v\rangle \in \mathrm{CP}(\{l \to r\}, \{l' \to r'\}) \setminus \overset{*}{\to}_{\mathcal{S}} \circ \Leftrightarrow\!\!\!\!\!\!\not\;\;_{\mathcal{P}} \circ \overset{*}{\leftarrow}_{\mathcal{S}}\}$ is returned. Similarly U_2 is obtained from $\mathrm{PCP}_{in}(\mathcal{P} \cup \mathcal{P}^{-1}, \mathcal{S}) \cup \mathrm{CP}(\mathcal{S}, \mathcal{P} \cup \mathcal{P}^{-1})$. If both of these lists are empty then the conditions of Theorem 3.9 (or Theorem 3.13) are satisfied and thus the procedure succeeds (`Success` is returned).
- (`trans` $(\mathcal{S}, \mathcal{P})$ (U_1, U_2)) returns a collection of transformed TRSs obtained by addition and replacement of rewrite rules constructed from non-joinable critical pairs and rewrite rules generating such critical pairs as described in the Definition 4.7. Here, the addition of rewrite rules are restricted based on the following heuristic: $l \to r$ is added if (1) $l \in \mathrm{NF}(\mathcal{S})$, (2) $l(\epsilon) = r(\epsilon) \in \mathcal{D}$ implies $(\mathcal{F}(l) \cup \mathcal{F}(r)) \cap \mathcal{C} = \emptyset$ and (3) $l(\epsilon) \neq r(\epsilon)$ and $l(\epsilon), r(\epsilon) \in \mathcal{D}$ imply $\mathcal{F}(r) \cap \mathcal{C} = \emptyset$.

Table 1 shows the summary of our experiments. We have tested various combinations of our results: (1)–(4) are proofs by confluence criterion of Theorem 3.7, of Theorem 3.9, of Theorem 3.13 and by the combination of those of Theorem 3.9 and Theorem 3.13. (5)–(7) are proofs by the reduction-preserving completion without the *Replacement* rule, i.e. without the Steps $2a, 3a, 4a$ of the concrete reduction-preserving completion (Definition 4.7). (8)–(10) are proofs by the reduction-preserving completion with the *Replacement* rule. For the experiments, we used a collection of 81 TRSs involving non-terminating rules such as commutativity and associativity rules which have been developed in the course of experiments. All experiments have been performed on a FreeBSD platform of a PC equipped with 1.2GHz CPU and 1GB memory. We set the timout 60 sec. Total time is indicated in millisecond.

Table 1 Summary of experiments

	success	failure	diverge	timeout	time(msec.)
(1) main (Theorem 3.7)	19	62	0	0	1308
(2) PCP (Theorem 3.9)	28	53	0	0	1318
(3) linear (Theorem 3.13)	27	54	0	0	901
(4) PCP&linear	29	52	0	0	1725
(5) completion (PCP)	50	31	0	0	2258
(6) completion (linear)	46	35	0	0	1451
(7) completion (PCP&linear)	51	30	0	0	2995
(8) completion (repl., PCP)	64	17	(3)	0	3773
(9) completion (repl., linear)	59	22	0	0	2146
(10) completion (repl., PCP&linear)	66	15	(2)	0	4885
ACP [1, 2, 21]	12	67	—	2	164943

The maximal steps of the completion procedure is limited to 20 steps; the columns below the title "diverge" show the numbers of examples which exceed this limit, where these numbers are included in those of "failure."

The applicability of our incomparable confluent criteria (Theorem 3.9 and Theorem 3.13) does not have much differences. The applicability of Theorem 3.7, which is subsumed by Theorem 3.9, is limited compared to these two criteria. There is a clear advantage of using the completion procedure. The introducton of the *Replacement* inference rule also makes clear difference. The increase of total time by the introduction of completion procedure based on a confluence criterion are within 3 times of total time required in proving confluence only by checking that confluence criterion. This is partly due to our heuristics and the limitation on the number of limit of completion steps. The number of successful examples, however, does not change in the case we increase that limit to 100 steps. The collection of examples and all details of the experiments are available on the webpage `http://www.nue.riec.tohoku.ac.jp/tools/acp/experiments/rta11/all.html`.

We have also tested the confluence prover ACP on our collection. ACP is an automated confluence prover in which divide–and–conquer approach based on the persistent, layer-preserving, commutative decompositions is employed and involving many confluence criteria [4, 6, 10, 11, 14, 16, 17, 12, 19] as well as the decreasing diagram techniques [18, 20]. As shown in the table, most of our examples are not coped with the confluence prover ACP.

We have also tested on the 71 examples containing associativity and commutativity rules selected from the termination problem database 8.0[3] which have been developed to test termination modulo AC or C. ACP succeeded at 30 examples among which 27 examples are proved as non-confluent and 3 examples are proved as confluent. By our methods, 7 examples have been proved as confluent. We have also tested on a collection of 106 examples from [2, 1]. By enhancing ACP by our methods, confluence proving succeeded at 3 more examples.

6 Conclusion

We have presented a method for proving confluence of TRSs which can be applied even if the TRSs contain non-terminating rules such as commutativity and associativity. We have given

[3] `http://www.termination-portal.org`

confluence criteria for TRSs that can be partitioned into terminating part and reversible part which may be non-terminating. Then we have given a reduction-preserving completion procedure so that the criteria can be applied indirectly. In contrast to the well-known method for proving confluence of equational TRSs [7], our method is based solely on usual critical pairs and usual termination and hence easily integrated into confluence provers based on other confluence proving methods for TRSs. We have implemented the proposed techniques and reported experimental results. It turns out that our method is effective for TRSs for which most of standards methods for proving confluence of TRSs are not effective.

The following examples show that our method and the methods of [6, 7] are incomparable.

▶ **Example 6.1.** Let

$$\mathcal{R} = \left\{ \begin{array}{ll} +(x, 0) & \to x \\ +(x, \mathsf{s}(y)) & \to \mathsf{s}(+(x, y)) \\ *(x, 0) & \to 0 \\ *(x, \mathsf{s}(y)) & \to +(*(x, y), x) \\ *(x, +(y, z)) & \to +(*(x, y), *(x, z)) \end{array} \right\} \text{ and } \mathcal{E} = \left\{ \begin{array}{ll} +(x, y) & \leftrightarrow +(y, x) \\ +(+(x, y), z) & \leftrightarrow +(x, +(y, z)) \\ *(x, y) & \leftrightarrow *(y, x) \\ *(*(x, y), z) & \leftrightarrow *(x, *(y, z)) \end{array} \right\}.$$

It can be shown by the method of [7] that $\mathcal{R}$ is confluent modulo $\mathcal{E}$ and hence $\mathcal{R} \cup \mathcal{E}$ is confluent. Our method, however, failed to prove this example. Let

$$\mathcal{R}' = \left\{ *(+(x, y), z) \to +(*(x, z), *(y, z)) \right\} \text{ and } \mathcal{E}' = \left\{ \begin{array}{ll} +(x, y) & \leftrightarrow +(y, x) \\ +(+(x, y), z) & \leftrightarrow +(x, +(y, z)) \end{array} \right\}.$$

It can be shown by the method of [6] that $\mathcal{R}'$ is confluent modulo $\mathcal{E}'$ and hence $\mathcal{R}' \cup \mathcal{E}'$ is confluent. Our method, however, failed to prove this example. Let

$$\mathcal{R}'' = \left\{ \begin{array}{l} \mathsf{f}(0, 0) \to \mathsf{f}(0, 1) \\ \mathsf{f}(1, 0) \to \mathsf{f}(0, 0) \end{array} \right\} \text{ and } \mathcal{E}'' = \left\{ \mathsf{f}(x, y) \leftrightarrow \mathsf{f}(y, x) \right\}.$$

It can be shown by our method that $\mathcal{R}'' \cup \mathcal{E}''$ is confluent. Because $\mathcal{R}''$ is not terminating modulo $\mathcal{E}''$, the methods of [6, 7] fail to prove this example. We also note that the method of [8] also fails to prove this example by the same reason.

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