Bounds on the maximum multiplicity of some common geometric graphs*

Adrian Dumitrescu[1], André Schulz[2], Adam Sheffer[3], and Csaba D. Tóth[4]

1 **Department of Computer Science, University of Wisconsin–Milwaukee, USA**
dumitres@uwm.edu
2 **Institut für Mathematsche Logik und Grundlagenforschung, Universität Münster, Germany**
andre.schulz@uni-muenster.de
3 **School of Computer Science, Tel Aviv University, Israel**
sheffera@tau.ac.il
4 **Department of Mathematics and Statistics, University of Calgary, Canada**
cdtoth@ucalgary.ca

Abstract

We obtain new lower and upper bounds for the maximum multiplicity of some weighted, and respectively non-weighted, common geometric graphs drawn on n points in the plane in general position (with no three points collinear): perfect matchings, spanning trees, spanning cycles (tours), and triangulations.

(i) We present a new lower bound construction for the maximum number of triangulations a set of n points in general position can have. In particular, we show that a *generalized double chain* formed by two *almost convex* chains admits $\Omega(8.65^n)$ different triangulations. This improves the bound $\Omega(8.48^n)$ achieved by the previous best construction, the *double zig-zag chain* studied by Aichholzer *et al.*

(ii) We present a new lower bound of $\Omega(11.97^n)$ for the number of *non-crossing* spanning trees of the *double chain* composed of two *convex* chains. The previous bound, $\Omega(10.42^n)$, stood unchanged for more than 10 years.

(iii) Using a recent upper bound of 30^n for the number of triangulations, due to Sharir and Sheffer, we show that n points in the plane in general position admit at most $O(68.664^n)$ non-crossing spanning cycles.

(iv) We derive exponential lower bounds for the number of maximum and minimum *weighted* geometric graphs (matchings, spanning trees, and tours). It was known that the number of longest non-crossing spanning trees of a point set can be exponentially large, and here we show that this can be also realized with points in convex position. For points in convex position we obtain tight bounds for the number of longest and shortest tours. We give a combinatorial characterization of the longest tours, which leads to an $O(n \log n)$ time algorithm for computing them.

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1 Introduction

Let P be a set of n points in the plane in *general position*, i.e., no three points lie on a common line. A *geometric graph* $G = (P, E)$ is a graph drawn in the plane so that the vertex set consists of the points in P and the edges are drawn as straight line segments between points in P. All graphs we consider in this paper are geometric graphs. We call a graph *non-crossing* if edges intersect only at common endpoints.

It is a fundamental question to determine the maximum number of non-crossing geometric graphs on n points in the plane. We follow common conventions (see e.g., [19]) and denote by $\mathtt{pg}(P)$ the number of non-crossing (plane) graphs on P, and by $\mathtt{pg}(n) = \max_{|P|=n} \mathtt{pg}(P)$ the *maximum number* of non-crossing graphs an n-element point set can admit. Analogously, we introduce shorthand notation for the maximum number of triangulations, perfect matchings, spanning trees, and spanning cycles (i.e., Hamiltonian cycles); see Table 1.

Abbr.	Graph class	Lower bound	Upper bound
$\mathtt{pg}$(n)	graphs	$\Omega(41.18^n)$ [1, 13]	$O(207.85^n)$ [14, 21]
$\mathtt{cf}$(n)	cycle-free graphs	$\mathbf{\Omega(12.23^n)}$ [new, Thm. 2]	$O(164.49^n)$ [14, 21]
$\mathtt{pm}$(n)	perfect matchings	$\Omega^*(3^n)$ [13]	$O(10.07^n)$ [19]
$\mathtt{st}$(n)	spanning trees	$\mathbf{\Omega(11.97^n)}$ [new, Thm. 2]	$O(146.37^n)$ [14, 21]
$\mathtt{sc}$(n)	spanning cycles	$\Omega(4.64^n)$ [13]	$\mathbf{O(68.664^n)}$ [new, Thm. 3]
$\mathtt{tr}$(n)	triangulations	$\mathbf{\Omega(8.65^n)}$ [new, Thm. 1]	$O(30^n)$ [21]

Table 1 Classes of non-crossing geometric graphs, current best upper and lower bounds.

In the past 30 years numerous researchers have tried to estimate these quantities. In a pivotal result, Ajtai *et al.* [2] showed that $\mathtt{pg}(n) = O(c^n)$ for an absolute, but very large constant $c > 0$. The constant c has been improved several times since then, the best bound today is $c < 207.85$, which follows form the combination of the result of Sharir and Sheffer [21] with the result of Hoffmann *et al.* [14]. Interestingly, this upper bound, as well as the currently best upper bounds for $\mathtt{st}(n)$, $\mathtt{sc}(n)$, and $\mathtt{cf}(n)$, are derived from upper bounds on $\mathtt{tr}(n)$. This underlines the importance of the bound for $\mathtt{tr}(n)$ in this setting. For example, the best known upper bound for $\mathtt{st}(n)$ is the combination of $\mathtt{tr}(n) \le 30^n$ [21] with the ratio $\mathtt{sc}(n)/\mathtt{tr}(n) = O^*(4.879^n)$ [14]; see also previous work [17, 18, 19, 20]. To our knowledge, the only upper bound derived via a different approach is for the number of perfect matchings by Sharir and Welzl [19], $\mathtt{pm}(n) = O(10.07^n)$.

So far, we recalled various upper bounds on the maximum number of geometric graphs in certain classes. In this paper we mostly conduct our offensive from the other direction, on improving the corresponding *lower bounds*. Lower bounds for unweighted non-crossing graph classes were obtained in [1, 7, 13]. García, Noy, and Tejel [13] were the first to recognize the power of the *double chain* configuration in establishing good lower bounds for the maximum number of matchings, triangulations, spanning cycles and trees. It was widely believed for some time that the double chain gives asymptotically the highest number of triangulations, namely $\Theta^*(8^n)$. This was until 2006, when Aichholzer *et al.* [1] showed that another configuration, the so-called *double zig-zag chain*, admits $\Theta^*(\sqrt{72}^n) = \Omega(8.48^n)$ triangulations[1]. In this paper we further exploit the power of *almost convex* polygons and establish a new lower bound $\mathtt{tr}(n) = \Omega(8.65^n)$. For matchings, spanning cycles, and plane graphs, the double chain still holds the current record.

[1] We use the Θ^*, O^*, Ω^* notation for the asymptotic growth of functions ignoring polynomial factors.

Less studied are multiplicities of *weighted* geometric graphs. The weight of a geometric graph is the sum of its (Euclidean) edge lengths. This leads to the question how many graphs of a certain type (e.g., matchings, spanning trees, or tours) with *minimum* or *maximum* weight can be realized on an n-element point set. The notation is analogous; see Table 2. Dumitrescu [8] showed that the longest and shortest matchings can have exponential multiplicity, $2^{\Omega(n)}$, for a point set in general position. Furthermore, the longest and shortest spanning trees can also have multiplicity of $2^{\Omega(n)}$. Both bounds count explicitly geometric graphs with crossings; however these minima are automatically non-crossing. The question for the maximum multiplicity for non-crossing geometric graphs remained open for most of the geometric graph classes. Since we do not have any upper bounds that are better than those for the corresponding unweighted classes, the "upper bound" column is missing from Table 2.

Abbr.	Graph class	Lower bound
$\mathtt{pm}_{\min}(n)$	shortest perfect matchings	$\Omega(2^{n/4})$ [8]
$\mathtt{pm}_{\max}(n)$	longest perfect matchings	$\mathbf{\Omega(2^{n/4})}$ [new, Theorem 4]
$\mathtt{st}_{\min}(n)$	shortest spanning trees	$\Omega(2^{n/2})$ [8]
$\mathtt{st}_{\max}(n)$	longest spanning trees	$\mathbf{\Omega(2^{n})}$ [new, Theorem 7]
$\mathtt{sc}_{\min}(n)$	shortest spanning cycles	$\mathbf{\Omega(2^{n/3})}$ [new, Theorem 8]
$\mathtt{sc}_{\max}(n)$	longest spanning cycles	$\mathbf{\Omega(2^{n/3})}$ [new, Theorem 5]

Table 2 Classes of *weighted* non-crossing geometric graphs: exponential lower bounds.

Our results. Due to space constraints, some of the proofs are omitted from this extended abstract (all proofs are available in the full version of this paper [10]).

(I) A new lower bound, $\Omega(8.65^n)$, for the maximum number of triangulations a set of n points can have. We first re-derive the bound given by Aichholzer *et al.* [1] with a simpler analysis, which allows us to extend it to more complex point sets. Our estimate might be the best possible for the type of construction we consider.

(II) A new lower bound, $\Omega(11.97^n)$, for the maximum number of non-crossing spanning trees a set of n points can have. This is obtained by refining the analysis of the number of such trees on the "double chain" point configuration. The previous bound was $\Omega(10.42^n)$. A slight modification of the construction improves also the lower bound for cycle-free non-crossing graphs. In particular, we improve the old bound of $\Omega(11.62^n)$ to $\Omega(12.23^n)$,

(III) A new upper bound, $O(68.664^n)$, for the number of non-crossing spanning cycles on n points in the plane. This improves the latest upper bound of 70.21^n obtained by a combination of the results of Buchin *et al.* [4] and a recent upper bound of 30^n on the number of triangulations by Sharir and Sheffer [21].

(IV) Bounds on the maximum multiplicity of various weighted geometric graphs (weighted by Euclidean length). We show that the maximum number of longest non-crossing perfect matchings, spanning trees, spanning cycles, as well as shortest tours are all exponential in n. We also derive tight bounds, as well as a combinatorial characterization of longest tours over points in convex position. This yields an $O(n \log n)$ algorithm to compute a longest tour for such sets.

1.1 Preliminaries

Asymptotics of multinomial coefficients. Denote by $H(q) = -q\log q - (1-q)\log(1-q)$ the *binary entropy function*, where log stands for the logarithm in base 2 (by convention,

$0 \log 0 = 0$). For a constant $0 \le \alpha \le 1$, the following estimate can be easily derived from Stirling's formula for the factorial:

$$\binom{n}{\alpha n} = \Theta(n^{-1/2} 2^{H(\alpha)n}), \tag{1}$$

We also need the following bound on the sum of binomial coefficients; see [3] for a proof and [9, 11] for an application. If $0 < \alpha \le \frac{1}{2}$ is a constant,

$$\sum_{k=0}^{k \le \alpha n} \binom{n}{k} \le 2^{H(\alpha)n}. \tag{2}$$

Define similarly the *generalized entropy function* of k parameters $\alpha_1, \ldots, \alpha_k$, satisfying

$$\sum_{i=1}^{k} \alpha_i = 1, \quad \alpha_1, \ldots, \alpha_k \ge 0, \text{ as } \quad H_k(\alpha_1, \ldots, \alpha_k) = -\sum_{i=1}^{k} \alpha_i \log \alpha_i. \tag{3}$$

Clearly, $H(q) = H_2(q, 1-q)$. Recall, the multinomial coefficient

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \ldots n_k!},$$

where $\sum_{i=1}^{k} n_i = n$, counts the number of distinct ways to permute a multiset of n elements, k of which are distinct, with n_i, $i = 1, \ldots, k$, being the multiplicities of each of the k distinct elements.

Assuming that $n_i = \alpha_i n$, $i = 1, \ldots, k$, for constants $\alpha_1, \ldots, \alpha_k$, satisfying (3), again by using Stirling's formula for the factorial, one gets an expression analogous to (1):

$$\binom{n}{n_1, n_2, \ldots, n_k} = \Theta(n^{-(k-1)/2}) \cdot \left(\prod_{i=1}^{k} \alpha_i^{-\alpha_i} \right)^n = \Theta(n^{-(k-1)/2}) \cdot 2^{H_k(\alpha_1, \ldots, \alpha_k)n}. \tag{4}$$

Notations and conventions. For a polygonal chain P, let $|P|$ denote the number of vertices. If $1 < c_1 < c_2$ are two constants, we frequently write $\Omega^*(c_2^n) = \Omega(c_1^n)$. We also write $f(n) \sim g(n)$ whenever $f(n) = \Theta(g(n))$.

2 Lower bound on the maximum number of triangulations

Following the notation from [15], we denote by $P(n, k^r)$ the class of *almost convex polygons* with n vertices, formed by concatenating r flat reflex chains, each having k interior vertices. For example, $P(n, 0^r)$ is the class of convex polygons with $n = r$ vertices. Note that, for $r \ge 3$ and $k \ge 0$, every polygon in $P(n, k^r)$ has $n = r(k+1)$ vertices, r of which are convex. See Fig. 1 for a small example. To further simplify notation, we denote by $P(n, k^r)$ any polygon in this class; note they are all equivalent in the sense that they have the same visibility graph.

In establishing our new bound on the maximum number of triangulations, we go through the following steps: We first describe the double zig-zag chain from [1] in our framework, and re-derive the $\Theta^*(\sqrt{72}^n)$ bound of [1] for the number of its triangulations. Our simpler analysis extends to some variations of the double zig-zag chains, and leads to a new lower bound of $\texttt{tr}(n) = \Omega(8.65^n)$.

Two x-monotone polygonal chains L and U are said to be *mutually visible* if every pair of points $p \in L$ and $q \in U$, are *visible* from each other. Let us call $D(n, k^r)$ the *generalized double*

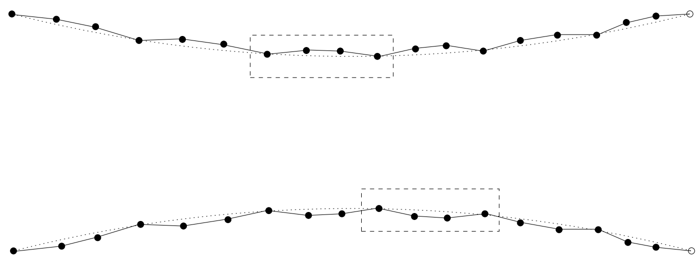

Figure 1 Two (flat) mutually visible copies of $P(18, 2^6)$ that form $D(36, 2^{12})$. Two consecutive hull vertices of $P(18, 2^6)$ with a reflex chain of two vertices in between are indicated in both the upper and the lower chain.

chain of n points made up of the set of points in two mutually visible copies of $P(n/2, k^r)$, each with $n/2 = r(k+1)$ vertices, with opposite concavities as in Fig. 1[2]. Generalized double chains are a family of point configurations, containing, among others, the double chain and double zig-zag chain configurations. In particular, $D(n, 1^r)$ is the *double zig-zag chain* used by Aichholzer *et al.* [1].

▶ **Theorem 1.** *The point set $D(n, 3^r)$ with $n = 8r$ points admits $\Omega(8.65^n)$ triangulations.*

Proof. We start by estimating the number of triangulations of $P(n, k^r)$. Denote this number by $t(n, k^r) = \texttt{tr}(P(n, k^r))$. Recall that $P(n, k^r)$ has $n = r(k+1)$ vertices. According to [15, Theorem 3],

$$t(n, k^r) \sim \left(\frac{1 + k/2}{2^k}\right)^r \cdot t(n) \sim \left(\frac{k+2}{2^{k+1}}\right)^r \cdot 4^{r(k+1)} = \left((k+2)^{\frac{1}{k+1}} \cdot 2\right)^n.$$

In particular,

- for $k = 1$, $t(n, 1^r) \sim (2\sqrt{3})^n = \sqrt{12}^n$. This estimate was used for counting triangulations in the construction $D(n, 1^r)$ with $\Omega(8.48^n)$ triangulations from [1].
- for $k = 2$, $t(n, 2^r) \sim (2^{5/3})^n$.
- for $k = 3$, $t(n, 3^r) \sim (5^{1/4} \cdot 2)^n$.
- for $k = 4$, $t(n, 4^r) \sim (6^{1/5} \cdot 2)^n$.

The following estimate is used in all our triangulation bounds. Consider two mutually visible polygonal chains, L and U, with m vertices each (L is the lower chain and U is the upper chain). As in the proof of [13, Theorem 4.1], the region between the two chains consists of $2m - 2$ triangles, such that exactly $m - 1$ triangles have an edge along L and the remaining $m - 1$ triangles have an edge adjacent to U. It follows that the number of distinct triangulations of this middle region is

$$\binom{2m-2}{m-1} = \Theta(m^{-1/2} \cdot 4^m). \tag{5}$$

The old $\Omega(8.48^n)$ lower bound in a new perspective. We estimate from below the number of triangulations of $D(n, 1^r)$ as follows. Recall that $|L| = |U| = n/2 = 2r$. Include all edges of L and U in any of the triangulations we construct. Now construct different triangulations as follows. Independently select a subset of $\alpha_1 r$ short edges of $\mathrm{conv}(U)$ and

[2] For convenience, an extra vertex is added to each chain to complete the last group in the figure.

similarly, a subset of $\alpha_1 r$ short edges of conv(L). Here $\alpha_1 \in (0,1)$ is a constant to be determined later. According to (1), this can be done in

$$\binom{r}{\alpha_1 r} = \Theta(r^{-1/2} \cdot 2^{H(\alpha_1)r})$$

ways in each of the two chains. Include these edges in the triangulation. Observe that after adding these short edges the middle region between the (initial) chains L and U is sandwiched between two mutually visible shorter chains, say $L' \subset L$ and $U' \subset U$, where

$$|L'| = |U'| = 2r - \alpha_1 r = (2 - \alpha_1)r. \tag{6}$$

Triangulate this middle region in all possible ways, as outlined in the paragraph above (5). Let N denote the total number of triangulations of $D(n, 1^r)$ obtained in this way. By the above estimate, we have $t(n, 1^r) \sim (2\sqrt{3})^n$. Combining this with (5) and (6),

$$N = \Omega^*\left(\left[(2\sqrt{3})^{2r} 2^{H(\alpha_1)r}\right]^2 4^{(2-\alpha_1)r}\right) = \Omega^*\left(\left[2^{2r} 3^r 2^{(2-\alpha_1)r} 2^{H(\alpha_1)r}\right]^2\right) =$$
$$= \Omega^*\left(\left[2^2 \cdot 3 \cdot 2^{(2-\alpha_1)} 2^{H(\alpha_1)}\right]^{2r}\right) = \Omega^*\left(\left[2^{4-\alpha_1+H(\alpha_1)} \cdot 3\right]^{n/2}\right) = \Omega^*\left(a^n\right),$$

where

$$a = \left[2^{4-\alpha_1+H(\alpha_1)} \cdot 3\right]^{(1/2)}.$$

By setting $\alpha_1 = 1/3$, as in [1], this yields $a = 6\sqrt{2} = 8.485\ldots$, and $N = \Omega^*(8.485^n) = \Omega(8.48^n)$.

Applying a similar analysis for a generalized double chain with reflex chains of length 3 implies Theorem 1. The details are in the full paper [10]. ◀

3 Lower bound on the maximum number of non-crossing spanning trees and forests

In this section we derive a new lower bound for the number of non-crossing spanning trees on the double-chain $D(n, 0^r)$, hence also for the maximum number of non-crossing spanning trees an n-element planar point set can have. The previous best bound, $\Omega(10.42^n)$, is due to Dumitrescu [8]. By refining the analysis of [8] we obtain a new bound $\Omega(11.97^n)$.

▶ **Theorem 2.** *For the double chain $D(n, 0^r)$, we have*

$$\Omega(11.97^n) < \mathtt{st}(D(n, 0^r)) < O(24.68^n), \text{ and}$$
$$\Omega(12.23^n) < \mathtt{cf}(D(n, 0^r)) < O(24.68^n).$$

These bounds imply that $\mathtt{st}(n) = \Omega(11.97^n)$ *and* $\mathtt{cf}(n) = \Omega(12.23^n)$.

Instead of spanning trees, we count (spanning) forests formed by two trees, similarly to [8]. One of the trees will be associated with the lower chain L and is called *lower tree*, the other tree will be associated with the upper chain U and is called *upper tree*. Since the two trees can be connected in at most $O(n^2)$ ways, it is enough the bound the number of two trees. Fig. 2 shows an example. We count only special kinds of forests: no edge of the lower tree connects two vertices of the upper chain, and similarly, no edge of the upper tree connects two vertices of the lower chain. We call the connected components of the edges between U and L *bridges*. For the class of forests we consider, bridges are subtrees of the

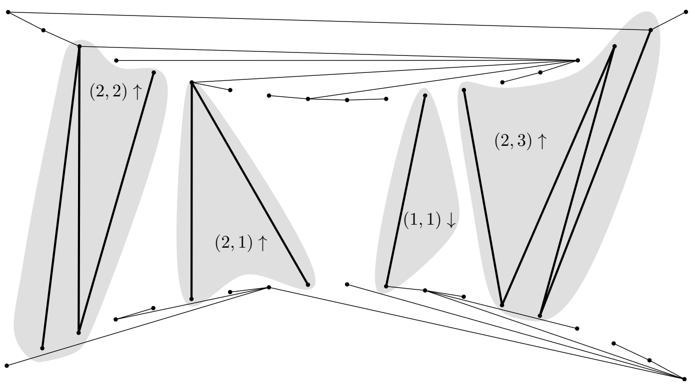

Figure 2 A double chain with lower and upper tree and four bridges.

lower or the upper tree. A bridge is called an (i, j)-*bridge* if it has i vertices in L and j vertices in U. Every bridge is part of either the upper or the lower tree. We say that in the first case the bridge is oriented *upwards* and in the latter case it is oriented *downwards.* Since edges cannot cross, the bridges have a natural left-to-right order. Fig. 2 shows four bridges, the first bridge is an upward oriented $(2, 2)$-bridge. We consider only bridges (i, j), with $1 \leq i, j \leq z$, for some fixed positive integer z. For $z = 1$, our analysis coincides with the one in [8], and we rederive the lower bound of $\Omega(10.42^n)$ found there. Successive improvements will be achieved by considering $z = 2, 3, 4$.

Let $m = n/2$ be the number of points on one chain. The distribution of bridges is specified by a set of parameters α_{ij}, to be determined later, where the number of (i, j)-bridges is $\alpha_{ij}m$. To simplify further expressions we introduce the following wildcard-notation:

$$\alpha_{i*} = \sum_{k=1}^{z} \alpha_{ik}, \quad \alpha_{*j} = \sum_{k=1}^{z} \alpha_{kj}, \quad \text{and} \quad \alpha_{**} = \sum_{k=1}^{z} \alpha_{*k} = \sum_{k=1}^{z} \alpha_{k*}.$$

A vertex is called a *bridge vertex*, if it is part of some bridge, and it is a *tree vertex* otherwise. We denote by $\alpha_L m$ the number of bridge vertices along L, and by $\alpha_U m$ the number of bridge vertices along U, we have

$$\alpha_L = \sum_{k=1}^{z} k\alpha_{k*}, \quad \text{and} \quad \alpha_U = \sum_{k=1}^{z} k\alpha_{*k}.$$

To count the forests we proceed as follows. We first count the distributions of the vertices that belong to bridges on the lower (N_L) and upper chain (N_U). We then count the different ways how bridges can be realized (N_{bridges}) and how the bridges can be connected to the two trees (N_{links}). Finally, we estimate the number of the trees within the two chains (N_{trees}). All these numbers are parameterized by the variables α_{ij}.

Consider the feasible locations of bridge vertices at the lower chain. We have $\binom{m}{\alpha_L m}$ choices to select the bridge vertices in L. Every bridge vertex belongs to some (i, j)-bridge. The vertices of the bridges cannot interleave, thus we can describe the configuration of bridges by a sequence of (i, j) tuples that denotes the appearance of the $\alpha_{**}m$ bridges from left to right on L. There are $\binom{\alpha_{**}m}{\alpha_{11}m, \alpha_{12}m, \ldots, \alpha_{zz}m}$ such sequences. This give us a total of

$$N_L := \binom{m}{\alpha_L m}\binom{\alpha_{**}m}{\alpha_{11}m, \alpha_{12}m, \ldots, \alpha_{zz}m} = \Theta^*\left(2^{H(\alpha_L)m + \alpha_{**}H_{(z^2)}(\alpha_{11}/\alpha_{**}, \ldots, \alpha_{zz}/\alpha_{**})m}\right)$$

such “configurations” of bridge vertices along L.

We now determine how many options we have to place the bridge vertices on U. Since we have already specified the sequence of the (i,j)-bridges at the lower chain, all we can do is to select the bridge vertices in U. This gives

$$N_U := \binom{m}{\alpha_U m} = \Theta^*\left(2^{H(\alpha_U)m}\right)$$

possibilities for the configuration on U.

We now study in how many ways the bridges can be added to the two trees. Since all bridges are subtrees, we can link one of the bridge vertices with the lower or upper tree. From this perspective the whole bridge acts like a super-node in one of the trees. The orientation of the bridges determine which tree they are glued to: upwards bridges to the upper tree, downwards bridges to the lower tree. For every pair (i,j) we orient half of the (i,j)-bridges upwards and half of them downwards. To glue the bridges to the trees we have to specify a vertex that will be linked to one of the trees. Depending on the orientation of the (i,j)-bridge, we have i candidates for a downwards oriented bridge and j candidates for an upward oriented bridge. In total we have

$$N_{\text{links}} := \prod_{i,j} \binom{\alpha_{i,j}m}{\alpha_{i,j}m/2} \left(i^{\alpha_{ij}/2} j^{\alpha_{ij}/2}\right)^m = \prod_{i,j} \Theta^*\left(2^{\alpha_{i,j}m}\right)(ij)^{\frac{\alpha_{ij}m}{2}} = \Theta^*\left(2^{\alpha_{**}m}\right)\prod_{i,j}(ij)^{\frac{\alpha_{ij}m}{2}}$$

ways to link the bridges with the trees.

Until now we have specified which vertices belong to which type of bridges, the orientation of the bridges, and the vertex where the bridge will be linked to its tree. It remains to count the number of ways to actually "draw" the bridges. Let us consider an (i,j)-bridge. All edges have to go from L to U and the bridge has to be a tree. The number of such trees equals the number of triangulations of a polygon with point set $\{(k,0) \mid 0 \le k \le i\} \cup \{(k,1) \mid 0 \le k \le j\}$. By deleting the edges along the horizontal lines $y = 0$ and $y = 1$, we define a bijection between these triangulations and the combinatorial types of (i,j)-bridges. The number of triangulations is now easy to express similarly to Equation (5): We have $i+j-2$ triangles, and each triangle is adjacent to a horizontal edge along either $y = 0$ or $y = 1$, where exactly $i-1$ triangles are adjacent to line $y = 0$. In total we have $B_{ij} := \binom{i+j-2}{i-1}$ different triangulations and therefore we can express the number of different bridges by

$$N_{\text{bridges}} = \left(\prod_{ij} B_{ij}^{\alpha_{ij}}\right)^m.$$

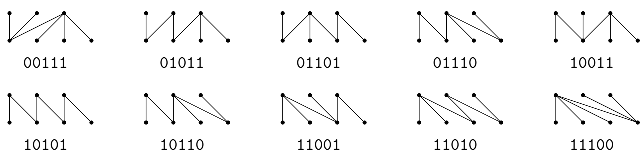

Figure 3 All $B_{34} = 10$ combinatorial types of $(3,4)$-bridges. If an edge differs form its predecessor at the top we write a `0`, otherwise a `1`. We obtain a bijection between the bridges and sequences with three `1`s and two `0`s.

Observe that the upper and the lower trees are trees on a convex point set. By considering the bridges as super-nodes, we treat the lower chain as a convex chain of n_L vertices. Similarly, we think of the upper chain as a convex chain with n_U vertices. We have

$$n_U = \left(1 - \sum_{k=1}^{n} \frac{2k-1}{2}\alpha_{k*}\right)m, \qquad \text{and} \qquad n_L = \left(1 - \sum_{k=1}^{n} \frac{2k-1}{2}\alpha_{*k}\right)m.$$

(Notice that the bridges take away all of its vertices, except one, depending on the orientation). Since the number of non-crossing spanning trees on an n-element convex point set equals $\Theta^*((27/4)^n)$ [12], the number of spanning trees within the two chains is given by

$$N_{\text{trees}} = O^*\Big(\Big(27/4\Big)^{n_L+n_U}\Big).$$

To finish our analysis we have to find the optimal parameters α_{ij} such that

$$\mathtt{st}(D(n,0^r)) = \Omega^*\left(N_L \cdot N_U \cdot N_{\text{bridges}} \cdot N_{\text{links}} \cdot N_{\text{trees}}\right) \tag{7}$$

is maximized. The details are presented in the full paper [10].

4 Upper bound for the number of non-crossing spanning cycles

Newborn and Moser [16] asked what is the maximum number of non-crossing spanning cycles for n points in the plane, and they proved $\Omega((10^{1/3})^n) \le \mathtt{sc}(n) \le O(6^n \lfloor \frac{n}{2} \rfloor !)$. The first exponential upper bound $\mathtt{sc}(n) \le 10^{13n}$ was obtained by Ajtai *et al.* [2], and has been followed by a series of improved bounds (e.g., see [4, 7, 19], a more comprehensive history can be found in [6]). Currently, the best known lower bound $4.462^n \le \mathtt{sc}(n)$ is by García *et al.* [13]. The previous best upper bound $O(70.21^n)$ is obtained by combining the upper bound $30^{n/4}$ of Buchin *et al.* [4] for the number of spanning cycles in a triangulation with a new upper bound of $\mathtt{tr}(n) \le 30^n$ by Sharir and Sheffer [21].

The bound by Buchin *et al.* [4] cannot be improved much further, since they also present triangulations with $\Omega(2.0845^n)$ spanning cycles. However, the bound for $\mathtt{sc}(n)$ still seems rather weak since it potentially counts some spanning cycles many times. To overcome this inefficiency, we use the notion of *pseudo-simultaneously flippable edges* (*ps-flippable edges* for short), introduced in [14]. A set F of edges in a triangulation is ps-flippable if after deleting all edges in F, the bounded faces are convex. One can obtain a lower bound for the support of a spanning cycle C in terms of the number of ps-flippable edges that are *not* in C.

▶ **Theorem 3.** *We have* $\mathtt{sc}(n) = O\left(68.664^n\right)$.

The proof is available in the full paper [10].

5 Weighted geometric graphs

Longest perfect matchings. Let n be even, and consider perfect matchings on a set of n points in the plane. It is easy to construct n-element point sets (no three of which are collinear) with an exponential number of *longest* matchings: [8] gives constructions with $\Omega(2^{n/4})$ such matchings. Moreover, the same lower bound can be achieved with yet another restriction, convex position, imposed on the point set; see [8]. Here, we present constructions with an exponential number of *maximum* (longest) non-crossing matchings.

▶ **Theorem 4.** *For every even n, there exist n-element point sets with at least $2^{\lfloor n/4 \rfloor}$ longest non-crossing perfect matchings. Consequently,* $\mathtt{pm}_{\max}(n) = \Omega(2^{n/4})$.

Proof. (sketch) Assume first that n is a multiple of 4. Let $S_4 = \{a, b, c, d\}$ be a 4-element point set such that segment ab is vertical, cd lies on the orthogonal bisector of ab (hence, $|ac| = |bc|$ and $|ad| = |bd|$), $|ab| = |cd| = \frac{1}{n}$ and $\min\{|ac|, |ad|\} = |ac| = |bc| = 2n$. Then S_4 has two maximum matchings, $\{ac, bd\}$ and $\{ad, bc\}$, each of which has length at least $4n$. Let the n-element point set P be the union of $n/4$ translated copies of S_4 lying in disjoint horizontal strips such that the copies of a are almost collinear, all the copies of points a and b lie in a disk of unit diameter, and all the copies of points c and d lie in a disk of unit diameter; see Fig. 4.

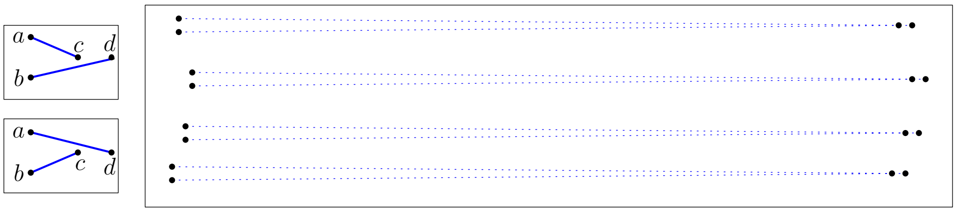

Figure 4 Left: Two possible maximum matchings for the point set $S_4 = \{a, b, c, d\}$. Right: A set of $n = 16$ points that admit 2^4 maximum non-crossing perfect matchings.

If we combine the maximum matchings of all copies of S_4, then we obtain $2^{n/4}$ non-crossing perfect matchings of P. All these matchings have the same length, which is at least $\frac{n}{4} \cdot 4n = n^2$. In the full paper [10], we show that this is the *maximum* possible length of a non-crossing perfect matching of P. ◀

Longest non-crossing tours. By Theorem 8, the maximum number of shortest non-crossing spanning cycles on n points is exponential in n. We show here that the maximum number of longest non-crossing spanning cycles is also exponential in n.

▶ **Theorem 5.** *Let $\mathtt{sc}_{\max}(n)$ denote the maximum number of longest non-crossing spanning cycles that an n-element point set can have. Then we have $\mathtt{sc}_{\max}(n) = \Omega(2^{n/3})$.*

Proof. (sketch) For every $k \in \mathbb{N}$, we construct a set Q of $4k + 1$ points that admits $2^k = \Omega(2^{n/4})$ longest non-crossing tours. We start by constructing an auxiliary set P of $2k$ points. The auxiliary point set P may contain collinear triples, however our final set Q does not. Recall that two segments cross if and only if their relative interiors intersect. We construct $P = \{c_i, x_i : i = 1, 2, \ldots, k\}$ with the following properties: (i) for every x_i, the farthest point in P is c_i; (ii) the perfect matching $M = \{c_i x_i : i = 1, 2, \ldots, k\}$ is non-crossing; and (iii) the convex hull of P is $\mathrm{conv}(P) = (x_1, c_1, c_2, \ldots, c_k)$. Note that property (i) implies that M is the maximum matching of P.

For $k \in \mathbb{N}$, let $\alpha = \frac{\pi}{3k}$. We construct $P = \{c_i, x_i : i = 1, \ldots, k\}$ iteratively. During the iterative process, we maintain the properties that $|x_i c_i| > \max_{j<i} |x_i c_j|$ and $|x_{i+1} c_i| > \max_{j<i} |x_{i+1} c_j|$. Initially, let $c_1 = (0, 0)$, $x_1 = (2, 0)$, and $x_2 = (2 - \frac{1}{k}, 0)$. Let $\vec{\ell}_1$ be a ray emitted by x_1 and incident to c_1. Refer to Fig. 5. If c_i, x_i and x_{i+1} are already defined, we construct points c_{i+1} and x_{i+2} (in the last iteration, only c_{i+1}) as follows. Let $\vec{\ell}_{i+1}$ be a ray emitted by x_{i+1} such that $\angle(\vec{\ell}_{i+1}, \vec{\ell}_i) = \alpha$. Compute the intersections of ray $\vec{\ell}_{i+1}$ with the circle centered at x_i of radius $|x_i c_i|$ and the circle centered at x_{i+1} of radius $|x_{i+1} c_i|$. Let $c_{i+1} \in \vec{\ell}_{i+1}$ be the midpoint of the segment between these two intersection points. This choice guarantees that $|x_{i+1} c_{i+1}| > |x_{i+1} c_j|$ and $|x_j c_{i+1}| < |x_j c_j|$ for all $j \leq i$. Now let $x_{i+2} \in c_{i+1} x_{i+1}$ be a point at distance at most $\frac{1}{k}$ from x_{i+1} such that we have $|x_{i+2} c_{i+1}| > |x_{i+2} c_j|$ for all $j \leq i$. This completes the description of P.

Note that $|x_i x_{i+1}| \leq \frac{1}{k}$, and so the points $x_1, \ldots, x_k$ lie in a disk of diameter 1. Hence, for every point x_i, the farthest point in P is in $\{c_j : j = 1, \ldots, k\}$. By the above construction, the farthest point from x_i in $\{c_j : j = 1, \ldots, k\}$ is c_i. This proves that P has property (i). It is easy to verify that P has properties (ii) and (iii), as well.

We now construct the point set Q based on P. Let $\delta > 0$ be a sufficiently small constant. For every segment $c_i x_i$ we construct a skinny deltoid $\Delta_i = (a_i, b_i, c_i, d_i)$, see

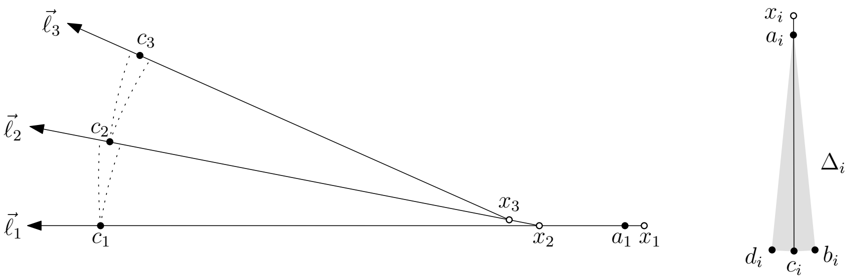

Figure 5 Left: The auxiliary point set P for $k = 3$. Right: A long and skinny deltoid $\Delta_i = (a_i, b_i, c_i, d_i)$.

Fig. 5, such that $a_i \in c_i x_i$ is at distance δ from x_i, we have $|b_i c_i| = |c_i d_i| = \delta$, and $|a_i b_i| = |a_i c_i| = |a_i d_i| = |c_i x_i| - \delta$. Since the segments $c_i x_i$ are pairwise non-crossing and $\delta > 0$ is small, the deltoids Δ_i are pairwise interior disjoint. Let Q be the set of vertices of all deltoids Δ_i, $i = 1, \ldots, k$, and the point x_1. Since $\mathrm{conv}(P) = (c_1, c_2, \ldots, c_k, x_1)$, we have $\mathrm{conv}(Q) = (b_1, c_1, d_1, b_2, c_2, d_2, \ldots, b_k, c_k, d_k, x_1)$, and the points $\{a_i : i = 1, \ldots, k\}$ lie in the interior of $\mathrm{conv}(Q)$. If $\delta > 0$ is sufficiently small, then the farthest points from a_i in Q are b_i, c_i, and d_i, for every $i = 1, 2, \ldots, k$.

Every non-crossing tour of Q visits the convex hull vertices in the cyclic order determined by $\mathrm{conv}(Q)$. We obtain a non-crossing tour by replacing some edges of $\mathrm{conv}(Q)$ with non-crossing paths visiting the points lying in the interior of $\mathrm{conv}(Q)$. If we replace either edge $b_i c_i$ or $c_i d_i$ with the path (b_i, a_i, c_i) or (c_i, a_i, d_i), respectively, for every $i = 1, 2, \ldots, k$, then we obtain a tour. Let $\mathcal{H}$ be the set of 2^k tours obtained in this way. These tours are non-crossing, since for every i, we exchange an edge of Δ_i with a path lying in Δ_i, and the deltoids Δ_i are interior disjoint. The tours in $\mathcal{H}$ have the same length, $L = |\mathrm{conv}(Q)| - k\delta + 2\sum_{i=1}^{k} |a_i c_i|$, since $|a_i b_i| = |a_i d_i| = |a_i c_i|$. In the full paper [10], we show that this length is maximal over all non-crossing tours (cycles).

To obtain the asserted bound, we use a skinny hexagon (instead of deltoid Δ_i) with five equidistant vertices on a circle centered at a_i. We now have four possible ways to insert each a_i into the tour, which implies $\mathtt{sc}_{\max}(n) = \Omega(4^{n/6}) = \Omega(2^{n/3})$. ◀

Typically for the longest matching, spanning tree or spanning cycle, one expects to see many crossings. Somewhat surprisingly, we show that this is not always the case.

▶ **Corollary 6.** *For every even $n \geq 2$, there exists an n-element point set (in general position) whose longest perfect matching is non-crossing.*

Longest spanning trees and shortest spanning cycles. We state without proof our results on the maximum multiplicity $\mathtt{st}_{\max}(n)$ of the longest crossing-free spanning tree on points, and the maximum multiplicity $\mathtt{sc}_{\min}(n)$ of the shortest non-crossing Hamiltonian cycle on n points.

▶ **Theorem 7.** *The vertex set of a regular convex n-gon admits $\Omega(2^n)$ longest non-crossing spanning trees. Consequently, $\mathtt{st}_{\max}(n) = \Omega(2^n)$.*

▶ **Theorem 8.** *Let $\mathtt{sc}_{\min}(n)$ denote the maximum number of shortest tours that an n-element point set can have.*

(i) *If S is a set of $n \geq 3$ points in convex position, then $\mathtt{sc}_{\min}(S) = 1$.*

(ii) *For points in general position, we have $\mathtt{sc}_{\min}(n) \geq 2^{\lfloor n/3 \rfloor}$.*

Recall that a geometric graph $G = (V, E)$ is called a *(geometric) thrackle*, if any two edges in E either cross or share a common endpoint.

▶ **Theorem 9.** *Let* $\mathtt{tc}_{\max}(n)$ *denote the maximum number of longest tours that an* n*-element point set in convex position can have. For* n *odd we have* $\mathtt{tc}_{\max}(n) = 1$ *and the (unique) longest tour is a thrackle. For* n *even we have* $\mathtt{tc}_{\max}(n) = n/2$*.*

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