International Conference on Rewriting Techniques and Applications 2010 (Edinburgh), pp. 243-258
http://rewriting.loria.fr/rta/

POLYNOMIAL INTERPRETATIONS OVER THE REALS DO NOT SUBSUME POLYNOMIAL INTERPRETATIONS OVER THE INTEGERS

FRIEDRICH NEURAUTER AND AART MIDDELDORP

Institute of Computer Science
University of Innsbruck, Austria
E-mail address: {friedrich.neurauter,aart.middeldorp}@uibk.ac.at

ABSTRACT. Polynomial interpretations are a useful technique for proving termination of term rewrite systems. They come in various flavors: polynomial interpretations with real, rational and integer coefficients. In 2006, Lucas proved that there are rewrite systems that can be shown polynomially terminating by polynomial interpretations with real (algebraic) coefficients, but cannot be shown polynomially terminating using polynomials with rational coefficients only. He also proved a similar theorem with respect to the use of rational coefficients versus integer coefficients. In this paper we show that polynomial interpretations with real or rational coefficients do not subsume polynomial interpretations with integer coefficients, contrary to what is commonly believed. We further show that polynomial interpretations with real coefficients subsume polynomial interpretations with rational coefficients.

1. Introduction

Polynomial interpretations are a simple yet useful technique for proving termination of term rewrite systems (TRSs, for short). While originally conceived in the late seventies by Lankford [Lan79] as a means for establishing direct termination proofs, polynomial interpretations are nowadays often used in the context of the dependency pair (DP) framework [Art00, Gie05, Hir05]. In the classical approach of Lankford, one considers polynomials with integer coefficients inducing polynomial algebras over the well-founded domain of the natural numbers. To be precise, every n-ary function symbol f is interpreted by a polynomial P_f in n indeterminates with integer coefficients, which induces a mapping or *interpretation* from terms to integer numbers in the obvious way. In order to conclude termination of a given TRS, three conditions have to be satisfied. First, every polynomial must be *well-defined*, i.e., it must induce a well-defined polynomial function $f_{\mathbb{N}}\colon \mathbb{N}^n \to \mathbb{N}$ over the natural numbers. In addition, the interpretation functions $f_{\mathbb{N}}$ are required to be *strictly monotone* in all arguments. Finally, one has to show *compatibility* of the interpretation with the given TRS. More precisely, for every rewrite rule $l \to r$ the polynomial P_l associated with the left-hand side must be greater than P_r, the corresponding polynomial of the right-hand side, i.e., $P_l > P_r$ for all values of the indeterminates.

1998 ACM Subject Classification: F.4.2 Grammars and Other Rewriting Systems, F.4.1 Mathematical Logic: Computational logic.

Key words and phrases: term rewriting, termination, polynomial interpretations.

Proceedings of the 21st International Conference on Rewriting Techniques and Applications, Edinburgh, July, 2010
Editors: Christopher Lynch
LIPIcs - Leibniz International Proceedings in Informatics. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, Germany
Digital Object Identifier: 10.4230/LIPIcs.RTA.2010.243

Already back in the seventies, an alternative approach using polynomials with real coefficients instead of integers was proposed by Dershowitz [Der79]. However, as the real numbers $\mathbb{R}$ equipped with the standard order $>_{\mathbb{R}}$ are not well-founded, a subterm property is explicitly required to ensure well-foundedness. And it was not until 2005 that this limitation was overcome, when Lucas [Luc05] presented a framework for proving polynomial termination over the real numbers, where well-foundedness is basically achieved by replacing $>_{\mathbb{R}}$ with a new ordering $>_{\mathbb{R},\delta}$ requiring comparisons between terms to not be below a given positive real number δ. Moreover, this framework also facilitates polynomial interpretations over the rational numbers.

Thus, one can distinguish three variants of polynomial interpretations, polynomial interpretations with real, rational and integer coefficients, and the obvious question is: what is their relationship with regard to termination proving power? For Knuth-Bendix orders it is known [Kor03, Lep01] that extending the range of the underlying weight function from natural numbers to non-negative reals does not result in an increase in termination proving power. In 2006, a partial answer to this question was given by Lucas [Luc06], who managed to show that there are rewrite systems that can be shown polynomially terminating by polynomial interpretations with rational coefficients, but cannot be shown polynomially terminating using polynomials with integer coefficients only. Likewise, he proved that there are systems that can be handled by polynomial interpretations with real (algebraic) coefficients, but cannot be handled by polynomial interpretations with rational coefficients. Based on these results and the fact that we have the strict inclusions $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, there is the common yet unproven belief in the term rewriting community that polynomial interpretations with real coefficients properly subsume polynomial interpretations with rational coefficients, which in turn properly subsume polynomial interpretations with integer coefficients.[1] However, in this paper we show that it is not true by (constructively) proving that polynomial interpretations with real or rational coefficients do not properly subsume polynomial interpretations with integer coefficients. Besides, we also prove that polynomial interpretations with real coefficients subsume polynomial interpretations with rational coefficients.

The remainder of this paper is organized as follows. In Section 2, we introduce some preliminary definitions and terminology concerning polynomials and polynomial interpretations. In Section 3, we show that polynomial interpretations with real coefficients subsume polynomial interpretations with rational coefficients. Section 4 is dedicated to our main result showing that polynomial interpretations with real or rational coefficients do not properly subsume polynomial interpretations with integer coefficients. We conclude in Section 5.

2. Preliminaries

As usual, we denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ the sets of natural, integer, rational and real numbers, respectively. An *irrational* number is a real number, which is not in $\mathbb{Q}$. Given some $N \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ and $m \in N$, $>_N$ denotes the standard order of the respective domain and $N_m := \{x \in N \mid x \geq m\}$. A sequence of real numbers $(x_n)_{n \in \mathbb{N}}$ *converges* to the *limit* x if for every real number $\varepsilon > 0$ there exists a natural number N such that the absolute distance $|x_n - x|$ is less than ε for all $n > N$; we denote this by $\lim_{n \to \infty} x_n = x$. As

[1] E.g., [Luc07] states that "polynomial interpretations over the reals are strictly better for proving polynomial termination of rewriting than those which only use integer coefficients".

convergence in $\mathbb{R}^k$ is equivalent to componentwise convergence, we use the same notation also for limits of converging sequences of vectors of real numbers $(\vec{x}_n \in \mathbb{R}^k)_{n\in\mathbb{N}}$. A real function $f\colon \mathbb{R}^k \to \mathbb{R}$ is *continuous* in $\mathbb{R}^k$ if for every converging sequence $(\vec{x}_n \in \mathbb{R}^k)_{n\in\mathbb{N}}$ it holds that $\lim_{n\to\infty} f(\vec{x}_n) = f(\lim_{n\to\infty} \vec{x}_n)$. Finally, as $\mathbb{Q}$ is dense in $\mathbb{R}$, every real number is a rational number or the limit of a converging sequence of rational numbers.

Polynomials

For any ring R (e.g., $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$), we denote the associated *polynomial ring* in n *indeterminates* $x_1, \ldots, x_n$ by $R[x_1, \ldots, x_n]$, the elements of which are finite sums of *monomials* of the form $c \cdot x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$, where the *coefficient* c is an element of R and the *exponents* $i_1, \ldots, i_n$ are natural numbers. An element $P \in R[x_1, \ldots, x_n]$ is called an *(n-variate) polynomial with coefficients in R*. For example, the polynomial $2x^2 - x + 1$ is an element of $\mathbb{Z}[x]$, the ring of all univariate polynomials with integer coefficients. The *degree* of a monomial $c \cdot x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$ is just the sum of its exponents.

In the special case $n = 1$, a polynomial $P \in R[x]$ can be written as follows: $P(x) = \sum_{k=0}^{d} a_k x^k$ $(d \geq 0)$. For the largest k such that $a_k \neq 0$, we call $a_k x^k$ the *leading monomial* of P, a_k its *leading coefficient* and k its *degree*, which is denoted by $\deg(P) = k$. A polynomial $P \in R[x]$ is said to be *linear* if $\deg(P) = 1$, and *quadratic* if $\deg(P) = 2$.

Polynomial Interpretations

We assume familiarity with the basics of term rewriting and polynomial interpretations (e.g. [Baa98, Ter03]). The key concept for establishing (direct) termination of TRSs via polynomial interpretations is the notion of well-founded monotone algebras as they induce reduction orders on terms.

Definition 2.1. Let $\mathcal{F}$ be a signature, i.e., a set of function symbols equipped with fixed arities. A *(well-founded) monotone $\mathcal{F}$-algebra* $(\mathcal{A}, >_A)$ is a non-empty algebra $\mathcal{A} = (A, \{f_A\}_{f\in\mathcal{F}})$ together with a (well-founded) order $>_A$ on the *carrier* A of $\mathcal{A}$ such that every algebra operation f_A is strictly monotone in all arguments, i.e., if $f \in \mathcal{F}$ has arity $n \geq 1$ then $f_A(a_1, \ldots, a_i, \ldots, a_n) >_A f_A(a_1, \ldots, b, \ldots, a_n)$ for all $a_1, \ldots, a_n, b \in A$ and $i \in \{1, \ldots, n\}$ with $a_i >_A b$. Moreover, every function symbol $f \in \mathcal{F}$ is said to be *interpreted* by its associated *interpretation function* f_A.

Given some monotone algebra $(\mathcal{A}, >_A)$, we define the relations $\succeq_{\mathcal{A}}$ and $\succ_{\mathcal{A}}$ on terms as follows: $s \succeq_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) \geq_A [\alpha]_{\mathcal{A}}(t)$ and $s \succ_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) >_A [\alpha]_{\mathcal{A}}(t)$, for all assignments α of elements of A to the variables in s and t ($[\alpha]_{\mathcal{A}}(\cdot)$ denotes the usual evaluation function associated with the algebra $\mathcal{A}$). Now if $(\mathcal{A}, >_A)$ is a well-founded monotone algebra, then $\succ_{\mathcal{A}}$ is a reduction order that can be used to prove termination of TRSs via the following theorem.

Theorem 2.2. *A TRS is terminating if and only if it is compatible with a well-founded monotone algebra.*

Here, a TRS $\mathcal{R}$ is *compatible* with a well-founded monotone algebra $(\mathcal{A}, >_A)$ if $l \succ_{\mathcal{A}} r$ for every rewrite rule $l \to r \in \mathcal{R}$.

Definition 2.3. A *polynomial interpretation over* $\mathbb{N}$ for a signature $\mathcal{F}$ consists of a polynomial $f_{\mathbb{N}} \in \mathbb{Z}[x_1, \ldots, x_n]$ for every n-ary function symbol $f \in \mathcal{F}$, such that for all $f \in \mathcal{F}$ the following two properties are satisfied:

(1) *well-definedness*: $f_{\mathbb{N}}(x_1, \ldots, x_n) \in \mathbb{N}$ for all $x_1, \ldots, x_n \in \mathbb{N}$,
(2) *strict monotonicity* of $f_{\mathbb{N}}$ in all arguments with respect to $>_{\mathbb{N}}$, the standard order on $\mathbb{N}$.

Now $(\mathbb{N}, \{f_{\mathbb{N}}\}_{f \in \mathcal{F}}, >_{\mathbb{N}})$ constitutes a well-founded monotone algebra, and we say that a polynomial interpretation over $\mathbb{N}$ is *compatible* with a TRS $\mathcal{R}$ if the well-founded monotone algebra $(\mathbb{N}, \{f_{\mathbb{N}}\}_{f \in \mathcal{F}}, >_{\mathbb{N}})$ is compatible with $\mathcal{R}$. Finally, a TRS is *polynomially terminating over* $\mathbb{N}$ if it admits a compatible polynomial interpretation over $\mathbb{N}$.

Remark 2.4. In principle, one could take any set $\mathbb{N}_m$ (or even $\mathbb{Z}_m$) instead of $\mathbb{N}$ as the carrier for polynomial interpretations. However, it is well-known [Ter03, Con05] that all these sets are order-isomorphic to $\mathbb{N}$ and hence do not change the class of polynomially terminating TRSs. In other words, a TRS $\mathcal{R}$ is polynomially terminating over $\mathbb{N}$ if and only if it is polynomially terminating over $\mathbb{N}_m$. Thus, we can restrict to $\mathbb{N}$ as carrier without loss of generality.

Now if one wants to extend the notion of polynomial interpretations to the rational or real numbers, the main problem one is confronted with is the non-well-foundedness of these domains with respect to the standard orders $>_{\mathbb{Q}}$ and $>_{\mathbb{R}}$. In [Hof01, Luc05], this problem is overcome by replacing these orders with new non-total orders $>_{\mathbb{R},\delta}$ and $>_{\mathbb{Q},\delta}$, the first of which is defined as follows: given some fixed positive real number δ,

$$x >_{\mathbb{R},\delta} y \quad :\Longleftrightarrow \quad x - y \geq_{\mathbb{R}} \delta \quad \text{for all } x, y \in \mathbb{R}.$$

Analogously, one defines $>_{\mathbb{Q},\delta}$ on $\mathbb{Q}$. Thus, $>_{\mathbb{R},\delta}$ ($>_{\mathbb{Q},\delta}$) is well-founded on subsets of $\mathbb{R}$ ($\mathbb{Q}$) that are bounded from below. Therefore, any set $\mathbb{R}_m$ ($\mathbb{Q}_m$) could be used as carrier for polynomial interpretations over $\mathbb{R}$ ($\mathbb{Q}$). However, without loss of generality we may restrict to $\mathbb{R}_0$ ($\mathbb{Q}_0$) because the main argument of Remark 2.4 also applies to polynomials over $\mathbb{R}$ ($\mathbb{Q}$), as is already mentioned in [Luc05].

Definition 2.5. A *polynomial interpretation over* $\mathbb{R}$ for a signature $\mathcal{F}$ consists of a polynomial $f_{\mathbb{R}} \in \mathbb{R}[x_1, \ldots, x_n]$ for every n-ary function symbol $f \in \mathcal{F}$ and some positive real number $\delta >_{\mathbb{R}} 0$, such that for all $f \in \mathcal{F}$:

(a) *well-definedness*: $f_{\mathbb{R}}(x_1, \ldots, x_n) \in \mathbb{R}_0$ for all $x_1, \ldots, x_n \in \mathbb{R}_0$
(b) *strict monotonicity* of $f_{\mathbb{R}}$ in all arguments with respect to $>_{\mathbb{R}_0,\delta}$, the restriction of $>_{\mathbb{R},\delta}$ to $\mathbb{R}_0$.

Analogously, one defines polynomial interpretations over $\mathbb{Q}$ by the obvious adaptation of the definition above. Again, $(\mathbb{R}_0, \{f_{\mathbb{R}}\}_{f \in \mathcal{F}}, >_{\mathbb{R}_0,\delta})$ and $(\mathbb{Q}_0, \{f_{\mathbb{Q}}\}_{f \in \mathcal{F}}, >_{\mathbb{Q}_0,\delta})$ constitute well-founded monotone algebras, and we say that a TRS is *polynomially terminating over* $\mathbb{R}$ ($\mathbb{Q}$) if it is compatible with such an algebra.

We conclude this section with a more useful characterization of monotonicity with respect to the orders $>_{\mathbb{R}_0,\delta}$ and $>_{\mathbb{Q}_0,\delta}$ than the one obtained by specializing Definition 2.1. To this end, we note that a function $f\colon \mathbb{R}_0^n \to \mathbb{R}_0$ is strictly monotone in its i-th argument with respect to $>_{\mathbb{R}_0,\delta}$ if and only if $f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n) \geq_{\mathbb{R}} \delta$ for all $x_1, \ldots, x_n, h \in \mathbb{R}_0$ with $h \geq_{\mathbb{R}} \delta$. From this and from the analogous characterization of $>_{\mathbb{Q}_0,\delta}$-monotonicity, it is easy to derive the following lemmata.

Lemma 2.6. *A linear polynomial* $f_{\mathbb{R}}(x_1,\ldots,x_n) = \sum_{i=1}^{n} a_i x_i + a_0$ *in* $\mathbb{R}[x_1,\ldots,x_n]$ *is strictly monotone in all arguments with respect to* $>_{\mathbb{R}_0,\delta}$ *if and only if* $a_i \geq_{\mathbb{R}} 1$ *for all* $i \in \{1,\ldots,n\}$.

Lemma 2.7. *A linear polynomial* $f_{\mathbb{Q}}(x_1,\ldots,x_n) = \sum_{i=1}^{n} a_i x_i + a_0$ *in* $\mathbb{Q}[x_1,\ldots,x_n]$ *is strictly monotone in all arguments with respect to* $>_{\mathbb{Q}_0,\delta}$ *if and only if* $a_i \geq_{\mathbb{Q}} 1$ *for all* $i \in \{1,\ldots,n\}$.

In the remainder of this paper we will sometimes use the term "polynomial interpretations with integer coefficients" as a synonym for polynomial interpretations over $\mathbb{N}$. Likewise, the term "polynomial interpretations with real (rational) coefficients" refers to polynomial interpretations over $\mathbb{R}$ ($\mathbb{Q}$).

3. Polynomial Termination over the Reals and Rationals

In this section we show that polynomial termination over $\mathbb{Q}$ implies polynomial termination over $\mathbb{R}$. The proof is based upon the fact that polynomials induce continuous functions, whose behavior at irrational points is completely defined by the values they take at rational points.

Lemma 3.1. *Let* $f\colon \mathbb{R}^k \to \mathbb{R}$ *be continuous in* $\mathbb{R}^k$. *If* $f(x_1,\ldots,x_k) \geq 0$ *for all* $x_1,\ldots,x_k \in \mathbb{Q}_0$ *then* $f(x_1,\ldots,x_k) \geq 0$ *for all* $x_1,\ldots,x_k \in \mathbb{R}_0$.

Proof. Let $\vec{x} := (x_1,\ldots,x_k) \in \mathbb{R}_0^k$ and let $(\vec{x}_n)_{n\in\mathbb{N}}$ be a sequence of vectors of non-negative rational numbers $\vec{x}_n \in \mathbb{Q}_0^k$ whose limit is $\vec{x}$. Such a sequence exists because $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$. Then

$$f(\vec{x}) = f(\lim_{n\to\infty} \vec{x}_n) = \lim_{n\to\infty} f(\vec{x}_n)$$

by continuity of f. Thus $f(\vec{x})$ is the limit of $(f(\vec{x}_n))_{n\in\mathbb{N}}$, which is a sequence of non-negative real numbers by assumption. Hence, $f(\vec{x})$ is non-negative, too. ∎

Theorem 3.2. *If a TRS is polynomially terminating over* $\mathbb{Q}$, *then it is also polynomially terminating over* $\mathbb{R}$.

Proof. Let $\mathcal{R}$ be a TRS over the signature $\mathcal{F}$ that is polynomially terminating over $\mathbb{Q}$. So there exists some polynomial interpretation $\mathcal{I}$ over $\mathbb{Q}$ consisting of a positive rational number δ and a polynomial $f_{\mathbb{Q}} \in \mathbb{Q}[x_1,\ldots,x_n]$ for every n-ary function symbol $f \in \mathcal{F}$ such that:

(a) for all n-ary $f \in \mathcal{F}$, $f_{\mathbb{Q}}(x_1,\ldots,x_n) \geq 0$ for all $x_1,\ldots,x_n \in \mathbb{Q}_0$,
(b) for all $f \in \mathcal{F}$, $f_{\mathbb{Q}}$ is strictly monotone with respect to $>_{\mathbb{Q}_0,\delta}$ in all arguments,
(c) for every rewrite rule $l \to r \in \mathcal{R}$, $P_l >_{\mathbb{Q}_0,\delta} P_r$ for all $x_1,\ldots,x_m \in \mathbb{Q}_0$.

Here P_l (P_r) denotes the polynomial associated with l (r) and the variables $x_1,\ldots,x_m$ are those occurring in $l \to r$. Next we note that all three conditions are quantified polynomial inequalities of the shape "$P(x_1,\ldots,x_k) \geq 0$ for all $x_1,\ldots,x_k \in \mathbb{Q}_0$" for some polynomial P with rational coefficients. This is easy to see for the first and third condition. As to the second condition, the function $f_{\mathbb{Q}}$ is strictly monotone in its i-th argument with respect to $>_{\mathbb{Q}_0,\delta}$ if and only if $f_{\mathbb{Q}}(x_1,\ldots,x_i+h,\ldots,x_n) - f_{\mathbb{Q}}(x_1,\ldots,x_i,\ldots,x_n) \geq \delta$ for all $x_1,\ldots,x_n,h \in \mathbb{Q}_0$ with $h \geq \delta$, which is equivalent to

$$f_{\mathbb{Q}}(x_1,\ldots,x_i+\delta+h,\ldots,x_n) - f_{\mathbb{Q}}(x_1,\ldots,x_i,\ldots,x_n) - \delta \geq 0$$

for all $x_1,\ldots,x_n,h \in \mathbb{Q}_0$. From Lemma 3.1 and the fact that polynomials induce continuous functions we infer that all these polynomial inequalities do not only hold in $\mathbb{Q}_0$ but also in $\mathbb{R}_0$. Hence, the polynomial interpretation $\mathcal{I}$ proves termination over $\mathbb{R}$. ∎

We conclude this section with the following remark that emphasizes the essence of the proof of Theorem 3.2.

Remark 3.3. Not only does the result established in this section show that polynomial termination over $\mathbb{Q}$ implies polynomial termination over $\mathbb{R}$, but it even reveals that the same interpretation applies.

4. Polynomial Termination over the Reals and Integers

As far as the relationship of polynomial interpretations with real, rational and integer coefficients with regard to termination proving power is concerned, the only results published to date are due to Lucas [Luc06], who managed to prove the following two theorems.

Theorem 4.1 (Lucas, 2006). *There are TRSs that are polynomially terminating over* $\mathbb{Q}$ *but not over* $\mathbb{N}$.

Theorem 4.2 (Lucas, 2006). *There are TRSs that are polynomially terminating over* $\mathbb{R}$ *but not over* $\mathbb{Q}$.

Hence, the extension of the coefficient domain from the integers to the rational numbers entails the possibility to prove some rewrite systems polynomially terminating, which could not be proved polynomially terminating otherwise. Moreover, a similar statement holds for the extension of the coefficient domain from the rational numbers to the real numbers. Based on these results and the fact that we have the strict inclusions $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, it is tempting to believe that polynomial interpretations with real coefficients properly subsume polynomial interpretations with rational coefficients, which in turn properly subsume polynomial interpretations with integer coefficients. Indeed, the former proposition holds according to Theorem 3.2. However, the latter proposition does not hold, as will be shown in this section. In particular, we present a TRS that can be proved terminating by a polynomial interpretation with integer coefficients, but cannot be proved terminating by a polynomial interpretation with real or rational coefficients.

4.1. Motivation

In order to motivate the construction of this particular rewrite system, let us first observe that from the viewpoint of number theory there is a fundamental difference between the integers and the real or rational numbers. More precisely, the integers are an example of a discrete domain, whereas both the real and rational numbers are *dense*[2] domains. In the context of polynomial interpretations, the consequences of this major distinction are best explained by an example. To this end, we consider the polynomial function $x \mapsto 2x^2 - x$ depicted in Figure 1 and assume that we want to use it as the interpretation of some unary function symbol. Now the point is that this function is permissible in a polynomial interpretation over $\mathbb{N}$ as it is both non-negative and strictly monotone over the natural numbers. However, viewing it as a function over a real (rational) variable, we observe that non-negativity is violated in the open interval $(0, \frac{1}{2})$ (and monotonicity requires a properly chosen value for δ). Hence, the polynomial function $x \mapsto 2x^2 - x$ is not permissible in any polynomial interpretation over $\mathbb{R}$ ($\mathbb{Q}$).

[2]Given two distinct real (rational) numbers a and b, there exists a real (rational) number c in between.

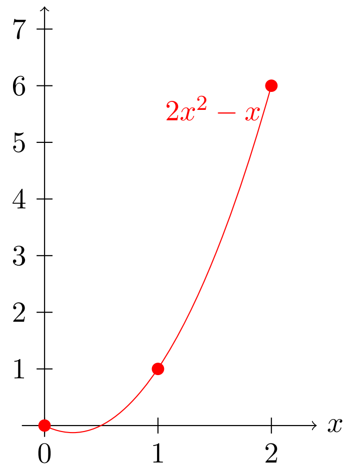

Figure 1: The polynomial function $x \mapsto 2x^2 - x$.

Thus, the idea is to design a rewrite system that enforces an interpretation of this shape for some unary function symbol, and the tool that can be used to achieve this is polynomial interpolation. To this end, let us consider the following scenario, which is fundamentally based on the assumption that some unary function symbol f is interpreted by a quadratic polynomial $\mathsf{f}(x) = ax^2 + bx + c$ with (unknown) coefficients a, b and c. Then, by polynomial interpolation, these coefficients are uniquely determined by the image of f at three pairwise different locations; in this way the interpolation constraints $\mathsf{f}(0) = 0$, $\mathsf{f}(1) = 1$ and $\mathsf{f}(2) = 6$ enforce the interpretation $\mathsf{f}(x) = 2x^2 - x$. Next we encode these constraints in terms of the TRS $\mathcal{R}$ consisting of the following rewrite rules, where $\mathsf{s}^n(x)$ abbreviates $\underbrace{\mathsf{s}(\mathsf{s}(\cdots\mathsf{s}}_{n\text{-times}}(x)\cdots))$,

$$
\begin{array}{ll}
\mathsf{s}(\mathsf{0}) \to \mathsf{f}(\mathsf{0}) & \\
\mathsf{s}^2(\mathsf{0}) \to \mathsf{f}(\mathsf{s}(\mathsf{0})) & \qquad \mathsf{f}(\mathsf{s}(\mathsf{0})) \to \mathsf{0} \\
\mathsf{s}^7(\mathsf{0}) \to \mathsf{f}(\mathsf{s}^2(\mathsf{0})) & \qquad \mathsf{f}(\mathsf{s}^2(\mathsf{0})) \to \mathsf{s}^5(\mathsf{0})
\end{array}
$$

and consider the following two cases: polynomial interpretations over $\mathbb{N}$ on the one hand and polynomial interpretations over $\mathbb{R}$ on the other hand.

In the context of polynomial interpretations over $\mathbb{N}$, we observe that if we equip the function symbols s and $\mathsf{0}$ with the (natural) interpretations $\mathsf{s}_{\mathbb{N}}(x) = x + 1$ and $\mathsf{0}_{\mathbb{N}} = 0$, then the TRS $\mathcal{R}$ indeed implements the above interpolation constraints.[3] For example, the constraint $\mathsf{f}_{\mathbb{N}}(1) = 1$ is expressed by $\mathsf{f}(\mathsf{s}(\mathsf{0})) \to \mathsf{0}$ and $\mathsf{s}^2(\mathsf{0}) \to \mathsf{f}(\mathsf{s}(\mathsf{0}))$. The former encodes $\mathsf{f}_{\mathbb{N}}(1) > 0$, whereas the latter encodes $\mathsf{f}_{\mathbb{N}}(1) < 2$. Moreover, the rule $\mathsf{s}(\mathsf{0}) \to \mathsf{f}(\mathsf{0})$ encodes $\mathsf{f}_{\mathbb{N}}(0) < 1$, which is equivalent to $\mathsf{f}_{\mathbb{N}}(0) = 0$ in the domain of the natural numbers. Thus, this interpolation constraint can be expressed by a single rewrite rule, whereas the other two constraints require two rules each. Summing up, by virtue of the method of polynomial interpolation, we have reduced the problem of enforcing a specific interpretation for some unary function symbol to the problem of enforcing natural semantics for the symbols s and $\mathsf{0}$.

[3] In fact, one can even show that $\mathsf{s}_{\mathbb{N}}(x) = x + 1$ is sufficient for this purpose.

Next we elaborate on the ramifications of considering the TRS $\mathcal{R}$ in the context of polynomial interpretations over $\mathbb{R}$. To this end, let us assume that the symbols s and $\mathsf{0}$ are interpreted by $\mathsf{s}_{\mathbb{R}}(x) = x + s_0$ and $\mathsf{0}_{\mathbb{R}} = 0$, so that s has some kind of successor function semantics. Then the TRS $\mathcal{R}$ translates to the following constraints:

$$\begin{aligned} s_0 - \delta &\geq_{\mathbb{R}} \mathsf{f}_{\mathbb{R}}(0) && \\ 2s_0 - \delta &\geq_{\mathbb{R}} \mathsf{f}_{\mathbb{R}}(s_0) & \mathsf{f}_{\mathbb{R}}(s_0) &\geq_{\mathbb{R}} 0 + \delta \\ 7s_0 - \delta &\geq_{\mathbb{R}} \mathsf{f}_{\mathbb{R}}(2s_0) & \mathsf{f}_{\mathbb{R}}(2s_0) &\geq_{\mathbb{R}} 5s_0 + \delta \end{aligned}$$

Hence, $\mathsf{f}_{\mathbb{R}}(0)$ is confined to the closed interval $[0, s_0 - \delta]$, whereas $\mathsf{f}_{\mathbb{R}}(s_0)$ is confined to $[0 + \delta, 2s_0 - \delta]$ and $\mathsf{f}_{\mathbb{R}}(2s_0)$ to $[5s_0 + \delta, 7s_0 - \delta]$. Basically, this means that these constraints do not uniquely determine the function $\mathsf{f}_{\mathbb{R}}$. In other words, the method of polynomial interpolation does not readily apply to the case of polynomial interpretations over $\mathbb{R}$. However, we can make it work. To this end, we observe that if $s_0 = \delta$, then the above system of inequalities actually turns into the following system of equations, which can be viewed as a set of interpolation constraints (parameterized by s_0) that uniquely determine $\mathsf{f}_{\mathbb{R}}$:

$$\mathsf{f}_{\mathbb{R}}(0) = 0 \qquad \mathsf{f}_{\mathbb{R}}(s_0) = s_0 \qquad \mathsf{f}_{\mathbb{R}}(2s_0) = 6s_0$$

Clearly, if $s_0 = \delta = 1$, then the symbol f is fixed to the interpretation $2x^2 - x$, as was the case in the context of polynomial interpretations over $\mathbb{N}$ (note that in the latter case $\delta = 1$ is implicit because of the equivalence $x >_{\mathbb{N}} y \iff x \geq_{\mathbb{N}} y + 1$). Hence, we conclude that once we can manage to design a TRS that enforces $s_0 = \delta$, we can again leverage the method of polynomial interpolation to enforce a specific interpretation for some unary function symbol. Moreover, we remark that the actual value of s_0 is irrelevant for achieving our goal. That is to say that s_0 only serves as a scale factor in the interpolation constraints determining $\mathsf{f}_{\mathbb{R}}$. Clearly, if $s_0 \neq 1$, then $\mathsf{f}_{\mathbb{R}}$ is not fixed to the interpretation $2x^2 - x$, however, it is still fixed to an interpretation of the same (desired) shape. But more on this later.

4.2. Main Theorem

In the previous subsection we have presented the basic method that we use in order to show that polynomial interpretations with real or rational coefficients do not properly subsume polynomial interpretations with integer coefficients. The construction presented there was based on several assumptions, the essential ones of which are:

(a) The symbol s had to be interpreted by a linear polynomial of the shape $x + s_0$.
(b) The condition $s_0 = \delta$ was required to hold.
(c) The function symbol f had to be interpreted by a quadratic polynomial.

Now the point is that one can get rid of all these assumptions by adding suitable rewrite rules to the TRS $\mathcal{R}$. The resulting TRS will be referred to as $\mathcal{S}$, and it consists of the following rewrite rules:

$$
\begin{array}{rclr@{\qquad}rclr}
\mathsf{s}(\mathsf{0}) & \to & \mathsf{f}(\mathsf{0}) & (1) & \mathsf{f}(\mathsf{g}(x)) & \to & \mathsf{g}(\mathsf{g}(\mathsf{f}(x))) & (7) \\
\mathsf{s}^2(\mathsf{0}) & \to & \mathsf{f}(\mathsf{s}(\mathsf{0})) & (2) & \mathsf{g}(\mathsf{s}(x)) & \to & \mathsf{s}(\mathsf{s}(\mathsf{g}(x))) & (8) \\
\mathsf{s}^7(\mathsf{0}) & \to & \mathsf{f}(\mathsf{s}^2(\mathsf{0})) & (3) & \mathsf{g}(x) & \to & \mathsf{h}(x,x) & (9) \\
\mathsf{f}(\mathsf{s}(\mathsf{0})) & \to & \mathsf{0} & (4) & \mathsf{s}(x) & \to & \mathsf{h}(\mathsf{0},x) & (10) \\
\mathsf{f}(\mathsf{s}^2(\mathsf{0})) & \to & \mathsf{s}^5(\mathsf{0}) & (5) & \mathsf{s}(x) & \to & \mathsf{h}(x,\mathsf{0}) & (11) \\
\mathsf{f}(\mathsf{s}^2(x)) & \to & \mathsf{h}(\mathsf{f}(x),\mathsf{g}(\mathsf{h}(x,x))) & (6) & \mathsf{h}(\mathsf{f}(x),\mathsf{g}(x)) & \to & \mathsf{f}(\mathsf{s}(x)) & (12)
\end{array}
$$

In this system the rewrite rules (7) and (8) serve the purpose of ensuring the first of the above items. Informally, (8) constrains the interpretation of the symbol s to a linear polynomial by simple reasoning about the degrees of the left- and right-hand side polynomials, and (7) does the same thing with respect to g. Because both interpretations are linear, compatibility with (8) can only be achieved if the leading coefficient of the interpretation of s is one.

Concerning item (c) above, we remark that the tricky part is to enforce the upper bound of two on the degree of the polynomial $\mathsf{f}_\mathbb{R}$ that interprets the symbol f. To this end, we make the following observation. If $\mathsf{f}_\mathbb{R}$ is at most quadratic, then the function $\mathsf{f}_\mathbb{R}(x+s_0)-\mathsf{f}_\mathbb{R}(x)$ is at most linear; that is, there is a linear function $\mathsf{g}_\mathbb{R}(x)$ such that $\mathsf{g}_\mathbb{R}(x) > \mathsf{f}_\mathbb{R}(x+s_0)-\mathsf{f}_\mathbb{R}(x)$, or equivalently, $\mathsf{f}_\mathbb{R}(x)+\mathsf{g}_\mathbb{R}(x) > \mathsf{f}_\mathbb{R}(x+s_0)$, for all values of x. This can be encoded in terms of rule (12) as soon as the interpretation of h corresponds to addition of two numbers. And this is exactly the purpose of rules (9), (10) and (11). More precisely, by linearity of the interpretation of g, we infer from (9) that the interpretation of h must have the linear shape $h_2x+h_1y+h_0$. Furthermore, compatibility with (10) and (11) implies $h_2=h_1=1$ due to item (a) above. Hence, the interpretation of h is $x+y+h_0$, and it really models addition of two numbers (modulo adding a constant).

Next we comment on how to enforce the second of the above assumptions. To this end, we remark that the hard part is to enforce the condition $s_0 \leq \delta$. The idea is as follows. First, we consider rule (2), observing that if f is interpreted by a quadratic polynomial $\mathsf{f}_\mathbb{R}$ and s by the linear polynomial $x+s_0$, then (the interpretation of) its right-hand side will eventually become larger than its left-hand side with growing s_0, thus violating compatibility. In this way, s_0 is bounded from above, and the faster the growth of $\mathsf{f}_\mathbb{R}$, the lower the bound. The problem with this statement, however, is that it is only true if $\mathsf{f}_\mathbb{R}$ is fixed (which is a priori not the case); otherwise, for any given value of s_0, one can always find a quadratic polynomial $\mathsf{f}_\mathbb{R}$ such that compatibility with (2) is satisfied. The parabolic curve associated with $\mathsf{f}_\mathbb{R}$ only has to be flat enough. So, in order to prevent this, we have to somehow control the growth of $\mathsf{f}_\mathbb{R}$. Now that is where rule (6) comes into play, which basically expresses that if you increase the argument of $\mathsf{f}_\mathbb{R}$ by a certain amount (i.e., $2s_0$), then the value of the function is guaranteed to increase by a certain minimum amount, too. Thus, this rule establishes a lower bound on the growth of $\mathsf{f}_\mathbb{R}$. And it turns out that if $\mathsf{f}_\mathbb{R}$ has just the right amount of growth, then we can readily establish the desired upper bound δ for s_0.

Finally, having presented all the relevant details of our construction, it remains to formally prove our main claim that the TRS $\mathcal{S}$ is polynomially terminating over $\mathbb{N}$, but not over $\mathbb{R}$ or $\mathbb{Q}$.

Lemma 4.3. *The TRS $\mathcal{S}$ is polynomially terminating over $\mathbb{N}$.*

Proof. We consider the following interpretation:

$$\mathsf{0}_\mathbb{N} = 0 \qquad \mathsf{s}_\mathbb{N}(x) = x+1 \qquad \mathsf{f}_\mathbb{N}(x) = 2x^2 - x \qquad \mathsf{g}_\mathbb{N}(x) = 4x+4 \qquad \mathsf{h}_\mathbb{N}(x,y) = x+y$$

Note that the polynomial $2x^2 - x$ is a permissible interpretation function as it is both non-negative and strictly monotone over the natural numbers (cf. Figure 1). The rewrite rules of $\mathcal{S}$ are compatible with this interpretation because the resulting inequalities

$$
\begin{array}{cc}
1 >_{\mathbb{N}} 0 & 32x^2 + 60x + 28 >_{\mathbb{N}} 32x^2 - 16x + 20 \\
2 >_{\mathbb{N}} 1 & 4x + 8 >_{\mathbb{N}} 4x + 6 \\
7 >_{\mathbb{N}} 6 & 4x + 4 >_{\mathbb{N}} 2x \\
1 >_{\mathbb{N}} 0 & x + 1 >_{\mathbb{N}} x \\
6 >_{\mathbb{N}} 5 & x + 1 >_{\mathbb{N}} x \\
2x^2 + 7x + 6 >_{\mathbb{N}} 2x^2 + 7x + 4 & 2x^2 + 3x + 4 >_{\mathbb{N}} 2x^2 + 3x + 1
\end{array}
$$

are clearly satisfied for all natural numbers x. ∎

Lemma 4.4. *The TRS $\mathcal{S}$ is not polynomially terminating over $\mathbb{R}$.*

Proof. Let us assume that $\mathcal{S}$ is polynomially terminating over $\mathbb{R}$ and derive a contradiction. Compatibility with rule (8) implies

$$\deg(\mathsf{g}_{\mathbb{R}}(x)) \cdot \deg(\mathsf{s}_{\mathbb{R}}(x)) \geq \deg(\mathsf{s}_{\mathbb{R}}(x)) \cdot \deg(\mathsf{s}_{\mathbb{R}}(x)) \cdot \deg(\mathsf{g}_{\mathbb{R}}(x))$$

As a consequence, $\deg(\mathsf{s}_{\mathbb{R}}(x)) \leq 1$, and because $\mathsf{s}_{\mathbb{R}}$ and $\mathsf{g}_{\mathbb{R}}$ must be strictly monotone, we conclude $\deg(\mathsf{s}_{\mathbb{R}}(x)) = 1$. The same reasoning applied to rule (7) yields $\deg(\mathsf{g}_{\mathbb{R}}(x)) = 1$. Hence, the symbols s and g must be interpreted by linear polynomials. So $\mathsf{s}_{\mathbb{R}}(x) = s_1 x + s_0$ and $\mathsf{g}_{\mathbb{R}}(x) = g_1 x + g_0$ with $s_0, g_0 \in \mathbb{R}_0$ and, due to Lemma 2.6, $s_1 \geq_{\mathbb{R}} 1$ and $g_1 \geq_{\mathbb{R}} 1$. Then the compatibility constraint imposed by rule (8) gives rise to the inequality

$$g_1 s_1 x + g_1 s_0 + g_0 >_{\mathbb{R}_0,\delta} s_1^2 g_1 x + s_1^2 g_0 + s_1 s_0 + s_0 \tag{13}$$

which must hold for all non-negative real numbers x. This implies the following condition on the respective leading coefficients: $g_1 s_1 \geq_{\mathbb{R}} s_1^2 g_1$. Because of $s_1 \geq_{\mathbb{R}} 1$ and $g_1 \geq_{\mathbb{R}} 1$, this can only hold if $s_1 = 1$. Hence, $\mathsf{s}_{\mathbb{R}}(x) = x + s_0$. This result simplifies (13) to $g_1 s_0 >_{\mathbb{R}_0,\delta} 2s_0$, which implies $g_1 s_0 >_{\mathbb{R}} 2s_0$. From this, we conclude that $s_0 >_{\mathbb{R}} 0$ and $g_1 >_{\mathbb{R}} 2$.

Now suppose that the function symbol f were also interpreted by a linear polynomial $\mathsf{f}_{\mathbb{R}}$. Then we could apply the same reasoning to rule (7) because it is structurally equivalent to (8), thus inferring $g_1 = 1$. However, this would contradict $g_1 >_{\mathbb{R}} 2$; therefore, $\mathsf{f}_{\mathbb{R}}$ cannot be linear.

Next we turn our attention to the rewrite rules (9), (10) and (11). Because $\mathsf{g}_{\mathbb{R}}$ is linear, compatibility with (9) constrains the function $h\colon \mathbb{R}_0 \to \mathbb{R}_0, x \mapsto \mathsf{h}_{\mathbb{R}}(x,x)$ to be at most linear. This can only be the case if $\mathsf{h}_{\mathbb{R}}$ contains no monomials of degree two or higher. In other words, $\mathsf{h}_{\mathbb{R}}(x,y) = h_1 \cdot x + h_2 \cdot y + h_0$, where $h_0 \in \mathbb{R}_0$, $h_1 \geq_{\mathbb{R}} 1$ and $h_2 \geq_{\mathbb{R}} 1$ (cf. Lemma 2.6). Because of $\mathsf{s}_{\mathbb{R}}(x) = x + s_0$, compatibility with (11) implies $h_1 = 1$, and compatibility with (10) implies $h_2 = 1$; thus, $\mathsf{h}_{\mathbb{R}}(x,y) = x + y + h_0$.

Using the obtained information in the compatibility constraint associated with rule (12), we get

$$\mathsf{g}_{\mathbb{R}}(x) + h_0 >_{\mathbb{R}_0,\delta} \mathsf{f}_{\mathbb{R}}(x + s_0) - \mathsf{f}_{\mathbb{R}}(x) \quad \text{for all } x \in \mathbb{R}_0$$

This implies that $\deg(\mathsf{g}_{\mathbb{R}}(x) + h_0) \geq \deg(\mathsf{f}_{\mathbb{R}}(x + s_0) - \mathsf{f}_{\mathbb{R}}(x))$, which simplifies to $1 \geq \deg(\mathsf{f}_{\mathbb{R}}(x)) - 1$ because $s_0 \neq 0$. Consequently, $\mathsf{f}_{\mathbb{R}}$ must be a quadratic polynomial. Without loss of generality, let $\mathsf{f}_{\mathbb{R}}(x) = ax^2 + bx + c$, subject to the constraints: $a >_{\mathbb{R}} 0$ and $c \geq_{\mathbb{R}} 0$ because of non-negativity (for all $x \in \mathbb{R}_0$), and $a\delta + b \geq_{\mathbb{R}} 1$ because $\mathsf{f}_{\mathbb{R}}(\delta) >_{\mathbb{R}_0,\delta} \mathsf{f}_{\mathbb{R}}(0)$ due to strict monotonicity of $\mathsf{f}_{\mathbb{R}}$.

Next we consider the compatibility constraint associated with rule (6), from which we deduce an important auxiliary result. After unraveling the definitions of $>_{\mathbb{R}_0,\delta}$ and the interpretation functions, this constraint simplifies to

$$4as_0x + 4as_0^2 + 2bs_0 \geq_{\mathbb{R}} 2g_1x + g_1h_0 + g_0 + h_0 + \delta \quad \text{for all } x \in \mathbb{R}_0,$$

which implies the following condition on the respective leading coefficients: $4as_0 \geq_{\mathbb{R}} 2g_1$; from this and $g_1 >_{\mathbb{R}} 2$, we conclude

$$as_0 >_{\mathbb{R}} 1 \tag{14}$$

and note that $as_0 = \mathsf{f}'_{\mathbb{R}}(s_0/2) - \mathsf{f}'_{\mathbb{R}}(0)$. Hence, as_0 expresses the change of the slopes of the tangents to $\mathsf{f}_{\mathbb{R}}$ at the points $(0, \mathsf{f}_{\mathbb{R}}(0))$ and $(s_0/2, \mathsf{f}_{\mathbb{R}}(s_0/2))$, and thus (14) actually sets a lower bound on the growth of $\mathsf{f}_{\mathbb{R}}$.

Now let us consider the combined compatibility constraint imposed by rule (2) and rule (4), namely $\mathsf{0}_{\mathbb{R}} + 2s_0 >_{\mathbb{R}_0,\delta} \mathsf{f}_{\mathbb{R}}(\mathsf{s}_{\mathbb{R}}(\mathsf{0}_{\mathbb{R}})) >_{\mathbb{R}_0,\delta} \mathsf{0}_{\mathbb{R}}$, which implies $\mathsf{0}_{\mathbb{R}} + 2s_0 \geq_{\mathbb{R}} \mathsf{0}_{\mathbb{R}} + 2\delta$ by definition of $>_{\mathbb{R}_0,\delta}$. Thus, we conclude $s_0 \geq_{\mathbb{R}} \delta$. In fact, we even have $s_0 = \delta$, which can be derived from the compatibility constraint of rule (2) using the conditions $s_0 \geq_{\mathbb{R}} \delta$, $a\delta + b \geq_{\mathbb{R}} 1$ and $as_0 + b \geq_{\mathbb{R}} 1$, the combination of the former two conditions:

$$\begin{array}{rcl}
\mathsf{0}_{\mathbb{R}} + 2s_0 & >_{\mathbb{R}_0,\delta} & \mathsf{f}_{\mathbb{R}}(\mathsf{s}_{\mathbb{R}}(\mathsf{0}_{\mathbb{R}})) \\
\mathsf{0}_{\mathbb{R}} + 2s_0 - \delta & \geq_{\mathbb{R}} & \mathsf{f}_{\mathbb{R}}(\mathsf{s}_{\mathbb{R}}(\mathsf{0}_{\mathbb{R}})) \\
& = & a(\mathsf{0}_{\mathbb{R}} + s_0)^2 + b(\mathsf{0}_{\mathbb{R}} + s_0) + c \\
& = & a\mathsf{0}_{\mathbb{R}}^2 + \mathsf{0}_{\mathbb{R}}(2as_0 + b) + as_0^2 + bs_0 + c \\
& \geq_{\mathbb{R}} & a\mathsf{0}_{\mathbb{R}}^2 + \mathsf{0}_{\mathbb{R}} + as_0^2 + bs_0 + c \\
& \geq_{\mathbb{R}} & \mathsf{0}_{\mathbb{R}} + as_0^2 + bs_0 \\
& \geq_{\mathbb{R}} & \mathsf{0}_{\mathbb{R}} + as_0^2 + (1 - a\delta)s_0 \\
& = & \mathsf{0}_{\mathbb{R}} + as_0(s_0 - \delta) + s_0
\end{array}$$

Hence, $\mathsf{0}_{\mathbb{R}} + 2s_0 - \delta \geq_{\mathbb{R}} \mathsf{0}_{\mathbb{R}} + as_0(s_0 - \delta) + s_0$, or equivalently, $s_0 - \delta \geq_{\mathbb{R}} as_0(s_0 - \delta)$. But because of (14) and $s_0 \geq_{\mathbb{R}} \delta$, this inequality can only be satisfied if:

$$s_0 = \delta \tag{15}$$

This result has immediate consequences concerning the interpretation of the constant $\mathsf{0}$. To this end, we consider the compatibility constraint of rule (10), which simplifies to $s_0 \geq_{\mathbb{R}} \mathsf{0}_{\mathbb{R}} + h_0 + \delta$. Because of (15) and the fact that $\mathsf{0}_{\mathbb{R}}$ and h_0 must be non-negative, we conclude $\mathsf{0}_{\mathbb{R}} = h_0 = 0$.

Moreover, condition (15) is the key to the proof of this lemma. To this end, we consider the compatibility constraints associated with the five rewrite rules (1)–(5):

$$\begin{array}{rcl}
s_0 >_{\mathbb{R}_0,s_0} \mathsf{f}_{\mathbb{R}}(0) & & \\
2s_0 >_{\mathbb{R}_0,s_0} \mathsf{f}_{\mathbb{R}}(s_0) & \qquad & \mathsf{f}_{\mathbb{R}}(s_0) >_{\mathbb{R}_0,s_0} 0 \\
7s_0 >_{\mathbb{R}_0,s_0} \mathsf{f}_{\mathbb{R}}(2s_0) & \qquad & \mathsf{f}_{\mathbb{R}}(2s_0) >_{\mathbb{R}_0,s_0} 5s_0
\end{array}$$

By definition of $>_{\mathbb{R}_0,s_0}$, these inequalities give rise to the following system of equations:

$$\mathsf{f}_{\mathbb{R}}(0) = 0 \qquad \mathsf{f}_{\mathbb{R}}(s_0) = s_0 \qquad \mathsf{f}_{\mathbb{R}}(2s_0) = 6s_0$$

After unraveling the definition of $\mathsf{f}_{\mathbb{R}}$ and substituting $z := as_0$, we get a system of linear equations in the unknowns z, b and c

$$c = 0 \qquad z + b = 1 \qquad 4z + 2b = 6$$

which has the unique solution $z = 2$, $b = -1$ and $c = 0$. Hence, $\mathsf{f}_{\mathbb{R}}$ must have the shape $\mathsf{f}_{\mathbb{R}}(x) = ax^2 - x = ax(x - \frac{1}{a})$ in every compatible polynomial interpretation over $\mathbb{R}$. However, this function is not a permissible interpretation for the function symbol f because it is not non-negative for all $x \in \mathbb{R}_0$. In particular, it is negative in the open interval $(0, \frac{1}{a})$; e.g., $\mathsf{f}_{\mathbb{R}}(\frac{1}{2a}) = -\frac{1}{4a}$. Hence, $\mathcal{S}$ is not compatible with any polynomial interpretation over $\mathbb{R}$. ∎

Remark 4.5. In this proof the interpretation of f is fixed to $\mathsf{f}_{\mathbb{R}}(x) = ax^2 - x$, which violates well-definedness in $\mathbb{R}_0$. However, this function is obviously well-defined in $\mathbb{R}_m$ for a properly chosen negative real number m. So, what happens if we take this $\mathbb{R}_m$ instead of $\mathbb{R}_0$ as carrier of a polynomial interpretation? To this end, we observe that $\mathsf{f}_{\mathbb{R}}(0) = 0$ and $\mathsf{f}_{\mathbb{R}}(\delta) = \delta(a\delta - 1) = \delta(as_0 - 1) = \delta$. Now let us consider some negative real number $x_0 \in \mathbb{R}_m$. Then $\mathsf{f}_{\mathbb{R}}(x_0) >_{\mathbb{R}} 0$ such that $\mathsf{f}_{\mathbb{R}}(\delta) - \mathsf{f}_{\mathbb{R}}(x_0) <_{\mathbb{R}} \delta$, which means that $\mathsf{f}_{\mathbb{R}}$ violates monotonicity with respect to the order $>_{\mathbb{R}_m,\delta}$.

The previous lemma, together with Theorem 3.2, yields the following corollary.

Corollary 4.6. *The TRS $\mathcal{S}$ is not polynomially terminating over $\mathbb{Q}$.*

Finally, combining the results presented in this section, we establish the main theorem of this paper.

Theorem 4.7. *There are TRSs that can be proved polynomially terminating over $\mathbb{N}$, but cannot be proved polynomially terminating over $\mathbb{R}$ or $\mathbb{Q}$.*

5. Conclusion and Future Work

In this paper, we investigated the relationship of polynomial interpretations with real, rational and integer coefficients with respect to termination proving power. In particular, we presented two new results, the first of which shows that polynomial interpretations with real coefficients subsume polynomial interpretations with rational coefficients, and the second of which shows that polynomial interpretations with real or rational coefficients do not properly subsume polynomial interpretations with integer coefficients, a result that comes somewhat unexpected. Together with the results of Lucas [Luc06], our results imply that polynomial interpretations with real or rational coefficients are incomparable to polynomial interpretations with integer coefficients with respect to termination proving power. Notwithstanding all these facts, the overall picture is not quite complete yet, there is still an open question: Are there TRSs that are polynomially terminating over $\mathbb{N}$ and $\mathbb{R}$, but not over $\mathbb{Q}$? Graphically, this question amounts to the inhabitation of the area depicted in red in Figure 2, which summarizes our results and the results of Lucas [Luc06].

We conclude this paper with two additional observations. First, we show that for polynomial interpretations over $\mathbb{R}$ it suffices to consider real *algebraic*[4] numbers as interpretation domain. Second, we present an alternative proof of Theorem 4.1, which shows the inhabitation of the area with the symbol $\mathbb{Q}$ in Figure 2.

Concerning the use of real algebraic numbers in polynomial interpretations, in [Luc07, Section 6] it is shown that it suffices to consider polynomials with real algebraic coefficients as interpretations of function symbols. Now the obvious question is whether it is also sufficient to consider only the (non-negative) real algebraic numbers $\mathbb{R}_{\mathsf{alg}}$ instead of the

[4]A real number is said to be algebraic if it is a root of a non-zero polynomial in one variable with rational coefficients.

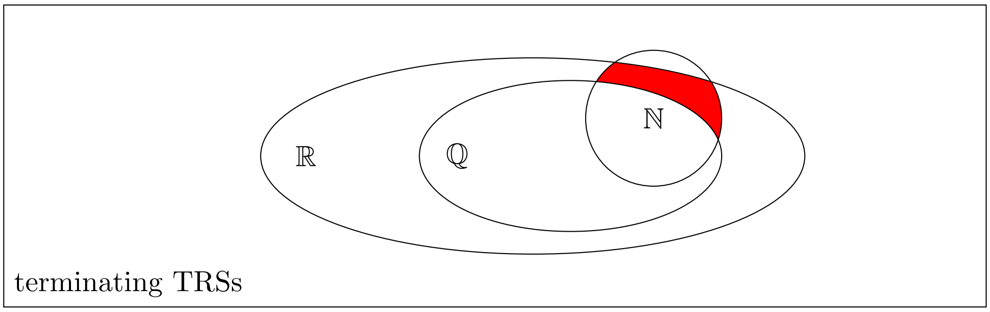

Figure 2: Comparison.

entire set $\mathbb{R}$ of real numbers as interpretation domain. We give an affirmative answer to this question by extending the result of [Luc07]. To this end, let us assume that $\mathcal{R}$ is a TRS that is polynomially terminating over $\mathbb{R}$. So, using the result of [Luc07], there exist a positive real number δ and a polynomial $f_{\mathbb{R}} \in \mathbb{R}_{\mathsf{alg}}[x_1, \ldots, x_n]$ for every n-ary function symbol $f \in \mathcal{F}$ such that:

(a) for all n-ary $f \in \mathcal{F}$, $f_{\mathbb{R}}(x_1, \ldots, x_n) \geq 0$ for all $x_1, \ldots, x_n \in \mathbb{R}_0$,
(b) for all $f \in \mathcal{F}$, $f_{\mathbb{R}}$ is strictly monotone with respect to $>_{\mathbb{R}_0,\delta}$ in all arguments,
(c) for every rewrite rule $l \to r \in \mathcal{R}$, $P_l >_{\mathbb{R}_0,\delta} P_r$ for all $x_1, \ldots, x_m \in \mathbb{R}_0$.

Treating δ as a variable (which we will later quantify existentially), we note that, similarly to the proof of Theorem 3.2, all three conditions can be phrased as quantified polynomial inequalities of the shape "$P(x_1, \ldots, x_k, \delta) \geq 0$ for all $x_1, \ldots, x_k \in \mathbb{R}_0$" for some polynomial P with real algebraic coefficients. Moreover, we note that there are finitely many of them if we assume $\mathcal{R}$ to be a finite TRS over a finite signature. Next we observe that any of these quantified inequalities can readily be expressed as a formula in the first order theory of ordered fields (where the atoms are polynomial (in)equalities, cf. [Bas06]) with δ as only free variable. Taking the conjunction of all these formulas and existentially quantifying δ and adding the conjunct $\delta > 0$, we obtain a sentence S in the first order theory of ordered fields, where all coefficients are real algebraic numbers. By assumption, this sentence holds in $\mathbb{R}$, and since both $\mathbb{R}$ and $\mathbb{R}_{\mathsf{alg}}$ are real closed fields with $\mathbb{R}_{\mathsf{alg}} \subset \mathbb{R}$ and all coefficients in S are from $\mathbb{R}_{\mathsf{alg}}$, we may apply the Tarski-Seidenberg transfer principle ([Bas06, Theorem 2.80]), from which we infer that S holds in $\mathbb{R}$ if and only if it holds in $\mathbb{R}_{\mathsf{alg}}$. Hence S also holds in $\mathbb{R}_{\mathsf{alg}}$ and therefore the TRS $\mathcal{R}$ is polynomially terminating over $\mathbb{R}_{\mathsf{alg}}$ (whose formal definition is the obvious specialization of Definition 2.5). This shows that polynomial termination over $\mathbb{R}$ implies polynomial termination over $\mathbb{R}_{\mathsf{alg}}$. As the reverse implication can be shown to hold by the same technique, we conclude that polynomial termination over $\mathbb{R}$ is equivalent to polynomial termination over $\mathbb{R}_{\mathsf{alg}}$.

Finally, we present our proof of Theorem 4.1, which is both shorter and simpler than the original proof in [Luc06, pp. 62–67]. Moreover, it shows that the strict inclusion holds even for *ground* TRSs.

Proof of Theorem 4.1. Consider the TRS $\mathcal{T}$ comprising the two rewrite rules

$$\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{b}) \qquad\qquad \mathsf{g}(\mathsf{b}) \to \mathsf{g}(\mathsf{a})$$

We claim that $\mathcal{T}$ is polynomially terminating over $\mathbb{Q}$, but not over $\mathbb{N}$. We start with the latter. In every compatible polynomial interpretation over $\mathbb{N}$, we have $\mathsf{a}_{\mathbb{N}} > \mathsf{b}_{\mathbb{N}}$ or $\mathsf{a}_{\mathbb{N}} \leq \mathsf{b}_{\mathbb{N}}$.

Strict monotonicity of $\mathsf{f}_{\mathbb{N}}$ and $\mathsf{g}_{\mathbb{N}}$ yields $\mathsf{g}_{\mathbb{N}}(\mathsf{a}_{\mathbb{N}}) > \mathsf{g}_{\mathbb{N}}(\mathsf{b}_{\mathbb{N}})$ or $\mathsf{f}_{\mathbb{N}}(\mathsf{a}_{\mathbb{N}}) \leq \mathsf{f}_{\mathbb{N}}(\mathsf{b}_{\mathbb{N}})$. In both cases compatibility is violated. It remains to show that $\mathcal{T}$ is polynomially terminating over $\mathbb{Q}$. The following interpretation applies:

$$\delta := 1 \qquad \mathsf{a}_{\mathbb{Q}} := 0 \qquad \mathsf{b}_{\mathbb{Q}} := \tfrac{1}{2} \qquad \mathsf{g}_{\mathbb{Q}}(x) := 2x \qquad \mathsf{f}_{\mathbb{Q}}(x) := 6x^2 - 5x + 2$$

First, we show compatibility of this interpretation with the rules of $\mathcal{T}$. To this end, we observe that the inequalities

$$\mathsf{f}_{\mathbb{Q}}(\mathsf{a}_{\mathbb{Q}}) >_{\mathbb{Q}_0,\delta} \mathsf{f}_{\mathbb{Q}}(\mathsf{b}_{\mathbb{Q}}) \qquad\qquad \mathsf{g}_{\mathbb{Q}}(\mathsf{b}_{\mathbb{Q}}) >_{\mathbb{Q}_0,\delta} \mathsf{g}_{\mathbb{Q}}(\mathsf{a}_{\mathbb{Q}})$$

which simplify to $2 >_{\mathbb{Q}_0,1} 1$ and $1 >_{\mathbb{Q}_0,1} 0$, do indeed hold by definition of $>_{\mathbb{Q}_0,1}$. Next we show well-definedness (non-negativity) and monotonicity of $\mathsf{f}_{\mathbb{Q}}$ and $\mathsf{g}_{\mathbb{Q}}$.

For well-definedness we have to show $\mathsf{f}_{\mathbb{Q}}(x) \geq 0$ and $\mathsf{g}_{\mathbb{Q}}(x) \geq 0$ for all non-negative rational numbers x. While $\mathsf{g}_{\mathbb{Q}}$ obviously satisfies this condition, $\mathsf{f}_{\mathbb{Q}}$ requires further reasoning. To this end, it suffices to observe that $\mathsf{f}_{\mathbb{Q}}$ has a global minimum at $x_0 = \frac{5}{12}$, namely $\mathsf{f}_{\mathbb{Q}}(x_0) = \frac{23}{24}$, which is positive.

The strict monotonicity of $\mathsf{g}_{\mathbb{Q}}$ follows from Lemma 2.7. The function $\mathsf{f}_{\mathbb{Q}}$ is strictly monotone with respect to $>_{\mathbb{Q}_0,\delta}$ if and only if $\mathsf{f}_{\mathbb{Q}}(x+h) - \mathsf{f}_{\mathbb{Q}}(x) \geq \delta$ for all non-negative rational numbers x and $h \geq \delta$. Thus, we have to show that $h(6h - 5 + 12x) \geq 1$ for all non-negative rational numbers x and $h \geq 1$. As x is non-negative and occurs only with a positive sign, this is equivalent to showing that $h(6h-5) \geq 1$ for all non-negative rational numbers $h \geq 1$, which is easy. Note that $\mathsf{f}_{\mathbb{Q}}$ is not strictly monotone with respect to the standard order $>_{\mathbb{Q}}$ on $\mathbb{Q}$. ∎

References

[Art00] T. Arts and J. Giesl. Termination of term rewriting using dependency pairs. *Theoretical Computer Science*, 236(1-2):133–178, 2000.

[Baa98] F. Baader and T. Nipkow. *Term Rewriting and All That*. Cambridge University Press, 1998.

[Bas06] S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in Real Algebraic Geometry*. Springer, 2nd edn., 2006.

[Con05] E. Contejean, C. Marché, A.-P. Tomás, and X. Urbain. Mechanically proving termination using polynomial interpretations. *Journal of Automated Reasoning*, 34(4):325–363, 2005.

[Der79] N. Dershowitz. A note on simplification orderings. *Information Processing Letters*, 9(5):212–215, 1979.

[Gie05] J. Giesl, R. Thiemann, and P. Schneider-Kamp. The dependency pair framework: Combining techniques for automated termination proofs. In *Proc. 11th International Conference on Logic for Programming, Artificial Intelligence, and Reasoning (LPAR 2004)*, *Lecture Notes in Artificial Intelligence*, vol. 3452, pp. 301–331. 2005.

[Hir05] N. Hirokawa and A. Middeldorp. Automating the dependency pair method. *Information and Computation*, 199(1-2):172–199, 2005.

[Hof01] D. Hofbauer. Termination proofs by context-dependent interpretations. In *Proc. 12th International Conference on Rewriting Techniques and Applications (RTA 2001)*, *Lecture Notes in Computer Science*, vol. 2051, pp. 108–121. 2001.

[Kor03] K. Korovin and A. Voronkov. Orienting rewrite rules with the Knuth-Bendix order. *Information and Computation*, 183:165–186, 2003.

[Lan79] D. Lankford. On proving term rewrite systems are noetherian. Tech. Rep. MTP-3, Louisiana Technical University, Ruston, 1979.

[Lep01] I. Lepper. Derivation lengths and order types of Knuth-Bendix orders. *Theoretical Computer Science*, 269(1-2):433–450, 2001.

[Luc05] S. Lucas. Polynomials over the reals in proofs of termination: From theory to practice. *Theoretical Informatics and Applications*, 39(3):547–586, 2005.

[Luc06] S. Lucas. On the relative power of polynomials with real, rational, and integer coefficients in proofs of termination of rewriting. *Applicable Algebra in Engineering, Communication and Computing*, 17(1):49–73, 2006.

[Luc07] S. Lucas. Practical use of polynomials over the reals in proofs of termination. In *Proc. 9th International Conference on Principles and Practice of Declarative Programming (PPDP 2007)*, pp. 39–50. Association of the Computing Machinery, 2007.

[Ter03] Terese. *Term Rewriting Systems*, *Cambridge Tracts in Theoretical Computer Science*, vol. 55. Cambridge University Press, 2003.