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DISPERSION IN UNIT DISKS

ADRIAN DUMITRESCU [1] AND MINGHUI JIANG [2]

[1] Department of Computer Science, University of Wisconsin–Milwaukee, WI 53201-0784, USA
E-mail address: ad@cs.uwm.edu

[2] Department of Computer Science, Utah State University, Logan, UT 84322-4205, USA
E-mail address: mjiang@cc.usu.edu

Abstract. We present two new approximation algorithms with (improved) constant ratios for selecting n points in n unit disks such that the minimum pairwise distance among the points is maximized.

(I) A very simple $O(n \log n)$-time algorithm with ratio 0.5110 for disjoint unit disks. In combination with an algorithm of Cabello [3], it yields a $O(n^2)$-time algorithm with ratio of 0.4487 for dispersion in n not necessarily disjoint unit disks.

(II) A more sophisticated LP-based algorithm with ratio 0.6495 for disjoint unit disks that uses a linear number of variables and constraints, and runs in polynomial time. The algorithm introduces a novel technique which combines linear programming and projections for approximating distances.

The previous best approximation ratio for disjoint unit disks was $\frac{1}{2}$. Our results give a partial answer to an open question raised by Cabello [3], who asked whether $\frac{1}{2}$ could be improved.

1. Introduction

Let $\mathcal{R}$ be a family of n subsets of a metric space. The problem of *dispersion in* $\mathcal{R}$ is that of selecting n points, one in each subset, such that the minimum inter-point distance is maximized. This dispersion problem was introduced by Fiala et al. [6] as "systems of distant representatives", generalizing the classic problem "systems of distinct representatives". An especially interesting version of the dispersion problem, which has natural applications to wireless networking and map labeling, is in a geometric setting where $\mathcal{R}$ is a set of unit disks in the plane.

Fiala et al. [6] showed that dispersion in (not necessarily disjoint) unit disks is NP-hard. It is not difficult to modify their construction, which gives a reduction from Planar-3SAT, to show that dispersion in disjoint unit disks is also NP-hard. Moreover, by a slackness argument [7, 8], the same construction also implies that the problem is APX-hard; i.e, unless

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P = NP, the problem does not admit any polynomial-time approximation scheme. On the positive side, Cabello [3] presented a quadratic-time approximation algorithm with ratio $0.4465\ldots$ $(1/2.2393\ldots)$ for dispersion in not necessarily disjoint unit disks. For dispersion in disjoint unit disks, Cabello [3] noticed that a naive algorithm called CENTERS, which simply selects the centers of the given disks as the points, gives a $\frac{1}{2}$-approximation.

We first introduce some preliminaries. For two points, $p = (x_p, y_p)$ and $q = (x_q, y_q)$, let $|pq|$ denote the Euclidean distance between them: $|pq| = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2}$. A *unit disk* is a disk of radius one. Let the *distance between two disks* be the distance between their centers; e.g., the distance between two tangent disks is 2. Let $\mathcal{D}$ be a set of n disjoint unit disks in the plane. Let δ be the minimum pairwise distance of the disks in $\mathcal{D}$; clearly $\delta \geq 2$. The algorithm CENTERS, by the obvious inequalities APX $\geq \delta$ and OPT $\leq \delta + 2$, achieves an approximation ratio

$$\frac{\text{APX}}{\text{OPT}} \geq \frac{\delta}{\delta + 2} \geq \frac{1}{2}.$$

Observe that the approximation ratio of CENTERS gets better as δ increases; in fact, it can get arbitrarily close to 1, if δ is large enough. Cabello asked whether this trivial $\frac{1}{2}$-approximation can be improved for disjoint unit disks [3, p. 72].

We start with a very simple and efficient algorithm that achieves a ratio better than $\frac{1}{2}$ for dispersion in disjoint unit disks, and a ratio slightly better than 0.4465 for dispersion in not necessarily disjoint unit disks:

Theorem 1.1. *There is an $O(n \log n)$-time approximation algorithm with ratio* 0.5110 *for dispersion in n disjoint unit disks. In combination with an algorithm of Cabello, it yields a $O(n^2)$-time algorithm with ratio of* 0.4487 *for dispersion in n not necessarily disjoint unit disks.*

Using linear programming, we then obtain the following substantially better approximation for dispersion in disjoint unit disks:

Theorem 1.2. *There is an LP-based approximation algorithm, with $O(n)$ variables and constraints, and running in polynomial time, that achieves approximation ratio* 0.6495*, for dispersion in n disjoint unit disks.*

It is likely that our method for proving Theorem 1.2, which uses projections for approximating distances, and linear programming for optimization, is also applicable to other optimization problems involving distances.

Related work. The problem studied in this paper, dispersion in unit disks, is related to a few other problems in computational geometry. We mention three results that are more closely related to ours:

(1) For labeling n points with n disjoint congruent disks, each point on the boundary of a distinct disk, such that radius of the disks is maximized, Jiang et al. [8] presented a $\frac{1}{2.98+\varepsilon}$-approximation algorithm, and proved that the problem is NP-hard to approximate with ratio more than $\frac{1}{1.0349}$.

(2) For packing of n axis-parallel congruent squares (congruent disks in the L_∞ metric) in the same rectilinear polygon such that the side length of the squares is maximized, Baur and Fekete [1] presented a $\frac{2}{3}$-approximation algorithm, and proved that the problem is NP-hard to approximate with ratio more than $\frac{13}{14}$.

(3) A $\frac{2}{3}$-approximation algorithm for a related problem of packing n unit disks in a rectangle without overlapping an existent set of m unit disks in the same rectangle, has been obtained by Benkert et al. [2].

(4) Given n points in the plane, Demaine et al. [4] considered the problem of moving them to an *independent set* in the unit disk graph metric: that is, each point has to move to a position such that all pairwise distances are at least 1, and such that the maximum distance a point moved is minimized. They presented an approximation algorithm, which achieves a good ratio if the points are initially "far from" an independent set. However the approximation ratio becomes unbounded for instances that are "very close to" an independent set. Observe that in this problem, the optimum may be arbitrarily small, i.e., arbitrarily close to 0.

2. A simple approximation algorithm for unit disks

In this section we present a very simple approximation algorithm **A1** for dispersion in (not necessarily disjoint) unit disks, and prove Theorem 1.1. The idea of the algorithm is as follows. Recall that δ is the minimum pairwise distance among the unit disks. Let $\sigma = \sigma(\delta)$ be a positive parameter to be specified; in particular, at the threshold distance $\delta = 2$ for disjoint unit disks, we have $\sigma(2) = 2.0883\ldots$, which is only slightly larger than δ. Consider the *distance graph* of the unit disks for the parameter σ, which has a vertex for each disk, and an edge between two vertices if and only if the corresponding disks have distance at most σ. If there is a vertex of degree at least two in the distance graph, that is, if there is a disk close to two other disks, then a packing argument shows that the minimum pairwise distance of any three points in the three disks must be small. Thus simply placing the points at the disk centers already achieves a good approximation ratio. Otherwise, every vertex in the distance graph has degree at most one, and the edges form a matching. In this case, the disks that are close to each other are grouped into pairs. The distance between the two points in each pair can be slightly increased by moving them away from the disk centers, at the cost of possibly decreasing the distances between points in different pairs.

Let $\mathcal{D}$ be a set of n (not necessarily disjoint) unit disks in the plane. The algorithm **A1** consists of three steps:

1. Compute the minimum pairwise distance δ of the disks in $\mathcal{D}$, and for each disk, find the two disks closest to it.
2. If the distance from some disk to its second closest disk is at most $\sigma = \sigma(\delta)$, return the n disk centers as the set of points. Otherwise, proceed to the next step.
3. Place a point at the center of each disk. Then, for each disk, if the distance from the disk to its closest disk is at most σ, move the point away from the closest disk for a distance of $(\sigma - \delta)/4$, so that the two points in each close pair of disks are moved in opposite directions; we will show that $\delta < \sigma < \delta + 4$, thus the distance $(\sigma - \delta)/4$ is between 0 and 1, and each point remains in its own disk. Finally, return the set of points.

Algorithm analysis. The bottleneck for the running time of the algorithm **A1** is simply the computation of the two closest disks from each disk in step 1, which takes $O(n \log n)$ time [5, p. 306]. The other two steps of the algorithm can clearly be done in $O(n)$ time.

For the proof of the approximation ratio, define the following function $f(s)$ for $s \geq 0$:

$$f(s) = \sqrt{(1+s)^2 + 1/2 + \sqrt{3(1+s)^2 - 3/4}}. \tag{2.1}$$

The function $f(\cdot)$ is increasing and $f(0) = \sqrt{3}$. The justification for step 2 of the algorithm **A1** is the following packing lemma (its proof is omitted). Here the disk with center O is close to two other disks with centers P and Q, respectively; see Figure 1.

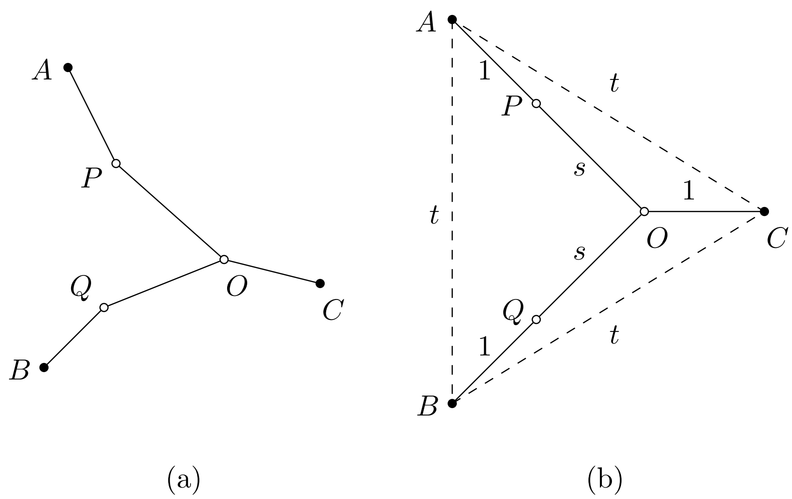

Figure 1: (a) A linkage of the five segments AP, BQ, CO, OP, OQ for three points A, B, C in three unit disks with centers P, Q, O, respectively. (b) The extreme configuration: A, P, O are collinear, B, Q, O are collinear, $|AP| = |BQ| = |CO| = 1$, $|OP| = |OQ| = s$, $|AC| = |BC| = |AB| = t$.

Lemma 2.1. *Let A, B, C be three points in three unit disks with centers P, Q, O, respectively. Let $s = \max\{|OP|, |OQ|\}$ and $t = \min\{|AC|, |BC|, |AB|\}$. Then $t \leq f(s)$.*

Consider the following equation in σ:

$$\frac{\delta}{f(\sigma)} = \frac{\sigma + \delta}{2(\delta + 2)}. \tag{2.2}$$

The next lemma (its proof is omitted) confirms that σ exists and lies in the desired range:

Lemma 2.2. *There is a unique solution σ to (2.2). Moreover, $\delta < \sigma < \delta + 4$.*

We now analyze the approximation ratio of the algorithm **A1**. Let APX be the minimum pairwise distance of the points returned by the algorithm. Let OPT be the minimum pairwise distance of the optimal set of points. Let

$$c = c(\delta) = \frac{\delta}{f(\sigma)} = \frac{\sigma + \delta}{2(\delta + 2)}. \tag{2.3}$$

We next prove that $\text{APX} \geq c \cdot \text{OPT}$ by considering two cases:

- If the algorithm returns the n disk centers as the set of points in step 2, then there is a disk such that the distances from the disk to its two closest disks are at most σ. By Lemma 2.1, we have $\text{OPT} \leq f(\sigma)$. Since $\text{APX} = \delta$, it follows that

$$\frac{\text{APX}}{\text{OPT}} \geq \frac{\delta}{f(\sigma)}. \tag{2.4}$$

- If the algorithm proceeds to step 3, then the distance from each disk to its second closest disk is more than σ. If two disks have distance at most σ, then they must be the closest disks of each other, and the movements of points in step 3 ensure that their two points have distance at least $\delta + 2(\sigma - \delta)/4 = (\sigma + \delta)/2$. On the other hand, if two disks have distance more than σ, then after the movements their two points have distance at least $\sigma - 2(\sigma - \delta)/4 = (\sigma + \delta)/2$. Thus APX $\geq (\sigma + \delta)/2$. Since OPT $\leq \delta + 2$, it follows that
$$\frac{\text{APX}}{\text{OPT}} \geq \frac{\sigma + \delta}{2(\delta + 2)}. \tag{2.5}$$

By (2.3), (2.4), and (2.5), the algorithm **A1** achieves an approximation ratio of $c(\delta)$ for $\delta \geq 0$. It can be verified that $c(\delta)$ is an increasing function of δ for $\delta \geq 0$. Thus, for dispersion in disjoint unit disks, the approximation ratio is
$$c(\delta) \geq c(2) = 0.5110\ldots, \quad \text{for } \delta \geq 2.$$

For dispersion in not necessarily disjoint unit disks, Cabello [3] presented a hybrid algorithm that applies two different algorithms Placement and Centers then returns the better solution. We now briefly review Cabello's analysis for the hybrid algorithm. Let $x = \text{OPT}/2$ (the scaling here is necessary because Cabello defined a unit disk as a disk of unit diameter instead of unit radius). The algorithm Placement, which runs in $O(n^2)$ time, achieves a ratio of
$$c_1(x) = \frac{-\sqrt{3} + \sqrt{3}x + \sqrt{3 + 2x - x^2}}{4x}, \quad \text{for } 1 \leq x \leq 2,$$
and a ratio of at least $\frac{1}{2}$ for $0 \leq x \leq 1$. The algorithm Centers achieves a ratio of
$$c_2(x) = \frac{x-1}{x}, \quad \text{for } x \geq 1,$$
which is at least $\frac{1}{2}$ for $x \geq 2$. Refer to Figure 2. Since $c_1(x)$ is decreasing in x and $c_2(x)$ is increasing in x, the minimum approximation ratio of the hybrid algorithm occurs at the intersection of the two curves $c_1(x)$ and $c_2(x)$ for $1 \leq x \leq 2$: precisely, $c_1(x) = c_2(x) = 0.4465\ldots(1/2.2393\ldots)$ for $x = 1.8068\ldots$.

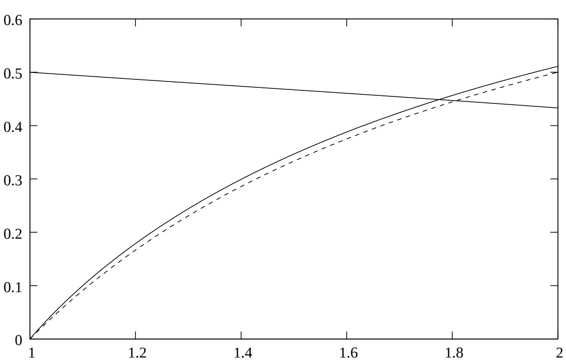

Figure 2: Approximation ratios $c_1(x)$, $c_2(x)$, and $c_3(x)$ for $1 \leq x \leq 2$. The solid decreasing curve is $c_1(x)$. The dashed increasing curve is $c_2(x)$. The solid increasing curve is $c_3(x)$.

Now define
$$c_3(x) = c(2x - 2), \quad \text{for } x \geq 1.$$

From the obvious inequality $\mathrm{OPT} \le \delta + 2$, we have $\delta \ge \mathrm{OPT} - 2 = 2x - 2$. Recall that the function $c(\delta)$ is increasing in δ. Thus our algorithm **A1** achieves an approximation ratio of at least $c(\delta) \ge c(2x-2) = c_3(x)$ for $x \ge 1$. It can be verified that $c_2(x) = c_3(x) = 0$ for $x = 1$ and $0 < c_2(x) < c_3(x) < 1$ for $x > 1$. Refer back to Figure 2. Replace the algorithm Centers by our algorithm **A1** in the hybrid algorithm. Then the two curves $c_1(x)$ and $c_3(x)$ intersects at $x = 1.7750\ldots$ and, correspondingly, the minimum approximation ratio of the new hybrid algorithm is $0.4487\ldots$ $(1/2.2284\ldots)$. This completes the proof of Theorem 1.1.

3. An LP-based approximation algorithm for disjoint unit disks

In this section we present and analyze approximation algorithm **A2**. We first introduce some definitions and notations. Let $\Omega_1, \ldots, \Omega_n$ be n pairwise disjoint unit disks, and let o_i be the center of Ω_i. Denote by δ the minimum pairwise distance among the disks; clearly, $\delta \ge 2$. The algorithm computes δ in $O(n \log n)$ time in a preliminary step.

Let $r = r(\delta)$, where $0 < r \le 1$, be a parameter that will be chosen later, in order to maximize the approximation ratio. For $i = 1, \ldots, n$, let $\omega_i \subset \Omega_i$ be a concentric disk of radius r. Let $\alpha_{ij} \in [-\pi/2, \pi/2)$ be the direction (or angle) of the line determined by o_i and o_j. For $\alpha \in [-\pi/2, \pi/2)$, let ℓ_α be any line of direction α. For two vectors $\overline{u} = (u_1, u_2)$, and $\overline{v} = (v_1, v_2)$, their dot product is $\langle \overline{u} \cdot \overline{v} \rangle = u_1 v_1 + u_2 v_2$. The scalar projection of $\overline{v}$ onto $\overline{u}$ is given by the formula

$$\mathrm{proj}_{\overline{u}} \overline{v} = \frac{\langle \overline{u} \cdot \overline{v} \rangle}{|\overline{u}|}. \tag{3.1}$$

For two points, p and q, let $\mathrm{proj}_\alpha(p, q)$ denote the length of the projection of the segment pq onto a line ℓ_α of direction α, i.e., onto the vector $(\cos\alpha, \sin\alpha)$.

Our approximation algorithm can be viewed as a two step process: Step 1. We first restrict the feasible region of each point p_i, from the given unit disk Ω_i to a smaller concentric disk ω_i of radius r, $0 < r < 1$. Further, we approximate each smaller disk ω_i by an inscribed regular polygon with sufficiently many sides (say, 64). For convenience however, we still use “disks” when referring to the convex polygons approximating (inscribed in) the smaller disks. Note that this first step is only conceptual. Step 2. We find a good approximation for the dispersion problem constrained to the smaller size disks.

The idea is as follows: Observe that after Step 1, the centers of the original disks Ω_i are still in the feasible regions for each of the n points. So the $\frac{1}{2}$ approximation that we could easily achieve earlier, is still attainable. Secondly, observe that if r is sufficiently small, then the distance between two points (in two smaller disks) can be well approximated by the projection of the segment connecting the two points onto the line connecting the centers of the two disks. The length of each such projection can be expressed as a linear combination of the coordinates of the two points, and we can use linear programming in order to maximize the smallest projection length of an inter-point distance. So all the constraints in the dispersion problem will be expressed as linear inequalities, at the cost of finding only an approximate solution. The resulting approximation ratio of the algorithm is the product of the ratios achievable in Step 1 and Step 2. In the end, we select r so as to maximize the overall ratio. We now present the technical details.

We start with a technical lemma that guarantees that a large fraction of the distance between two points in two smaller disks is preserved by projection onto the line through the two disk centers (Step 2).

Lemma 3.1. *Let ω_i, ω_j be two congruent disjoint disks of radius r, where $0 < r \le 1$, at distance $d \ge \delta \ge 2$. Let ℓ_{ij} be the line determined by o_i and o_j, and ℓ be a line that intersects both ω_i and ω_j. Let α be the (nonnegative) angle between ℓ_{ij} and ℓ. Then $\cos\alpha \ge \frac{\sqrt{d^2-4r^2}}{d} \ge \frac{\sqrt{\delta^2-4r^2}}{\delta}$.*

Proof. We can assume w.l.o.g. that ℓ_{ij} is horizontal; see Figure 3. By symmetry, we can

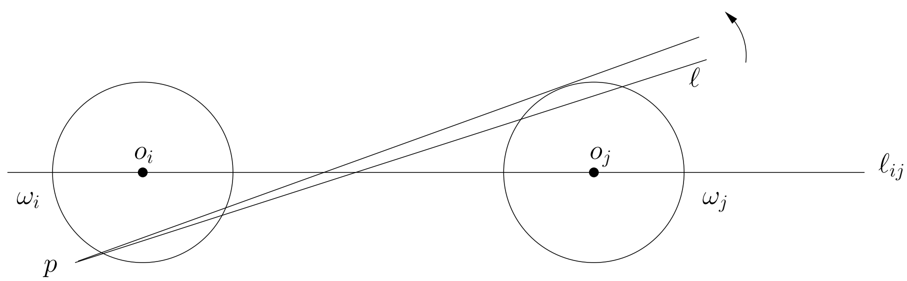

Figure 3: Lemma 3.1.

assume that ℓ has positive slope. We claim that if $\alpha \in [0, \pi/2]$ is maximized, then ℓ must be tangent to ω_i and ω_j. Assume for instance that ℓ is not tangent to ω_j, as illustrated in the figure. Select a point p on ℓ left of the intersections points of ℓ with $\partial\omega_i$, and $\partial\omega_j$, and rotate ℓ counterclockwise around p until ℓ becomes tangent to ω_j. The angle α increases in this operation, a contradiction of the assumed maximality. We conclude that ℓ must be tangent to ω_i and ω_j in the first place, as desired. The angle formula $\cos\alpha = \frac{\sqrt{d^2-4r^2}}{d}$ is now easily verified to hold in the tangent case. ■

The next two lemmas guarantee that a large fraction of OPT survives after restricting the feasible regions to smaller disks (STEP 1).

Lemma 3.2. *Consider two disjoint unit disks Ω_i and Ω_j at distance $|o_i o_j| = d$. Let $p_i \in \Omega_i$ and $p_j \in \Omega_j$ be two points. Let $q_i \in \omega_i$ be the point on $o_i p_i$ at distance $r|o_i p_i|$ from o_i. Similarly define $q_j \in \omega_j$ as the point on $o_j p_j$ at distance $r|o_j p_j|$ from o_j. Then*

$$\frac{|q_i q_j|}{|p_i p_j|} \ge \frac{d+2r}{d+2}. \tag{3.2}$$

This inequality is tight.

Proof. We can assume w.l.o.g. that $o_i = (0,0)$ and $o_j = (d,0)$, where $d \ge 2$. To represent points, we use complex numbers in the proof. The point p_i is represented by z_1, where $z_1 \in \mathbb{C}$, with $|z_1| \le 1$; hence q_i is represented by rz_1. The point p_j is represented by $d + z_2$, where $z_2 \in \mathbb{C}$, with $|z_2| \le 1$; hence q_j is represented by $d + rz_2$. With this notation, the claimed inequality is

$$\frac{|d + rz_2 - rz_1|}{|d + z_2 - z_1|} \ge \frac{d+2r}{d+2}. \tag{3.3}$$

Write $z = z_2 - z_1$, and note that $|z| \le |z_1| + |z_2| \le 2$. Inequality (3.3) can be written now as

$$\frac{|d + rz|}{|d + z|} \ge \frac{d+2r}{d+2}. \tag{3.4}$$

Let $z = a(\cos\alpha + i\sin\alpha)$, be the complex number representation of z, where $0 \le a \le 2$, and $\alpha \in [0, 2\pi]$. We have

$$|d+z|^2 = (a\cos\alpha + d)^2 + a^2\sin^2\alpha = a^2 + d^2 + 2ad\cos\alpha.$$

$$|d+rz|^2 = (ar\cos\alpha + d)^2 + a^2r^2\sin^2\alpha = a^2r^2 + d^2 + 2adr\cos\alpha.$$

Inequality (3.4) is thus equivalent to the following inequality:

$$(d+2)^2(a^2r^2 + d^2 + 2adr\cos\alpha) \ge (d+2r)^2(a^2 + d^2 + 2ad\cos\alpha). \tag{3.5}$$

After performing the multiplications, canceling the same terms, and simplifying by $(1-r)$, this amounts to verifying that

$$4d^3 + 4d^2(1+r) + 8adr\cos\alpha \ge a^2d^2(1+r) + 2ad^3\cos\alpha + 4a^2dr. \tag{3.6}$$

Observe that

$$4d^2(1+r) \ge a^2d^2(1+r).$$

It remains to show that (after simplifying by $2d$):

$$2d^2 + 4ar\cos\alpha \ge ad^2\cos\alpha + 2a^2r. \tag{3.7}$$

This last inequality is equivalent to

$$2(d^2 - a^2r) \ge a(d^2 - 4r)\cos\alpha. \tag{3.8}$$

Inequality (3.8) is clearly satisfied when $\cos\alpha < 0$, so assume now that $\cos\alpha \ge 0$. Obviously $2 \ge a\cos\alpha$, and from $a^2 \le 4$, we also get

$$d^2 - a^2r \ge d^2 - 4r.$$

Putting these two inequalities together (taking the product) gives inequality (3.8), hence inequality (3.2) is proved.

To see that (3.2) is tight, take $p_i = (-1, 0)$, and $p_j = (d+1, 0)$, i.e., all six points $p_i, p_j, o_i, o_j, q_i, q_j$ are on the same line. The proof of Lemma 3.2 is now complete. ∎

Lemma 3.3. *Let $p_1, \ldots, p_n$ be n points, where $p_i \in \Omega_i$, such that for any $i \ne j$, $|p_ip_j| \ge d$, for some $d > 0$. Then there exist n points, $q_1, \ldots, q_n$, such that $q_i \in \omega_i$, and for any $i \ne j$, $|q_iq_j| \ge \frac{\delta+2r}{\delta+2} \cdot d$.*

Proof. Let q_i be defined as in Lemma 3.2. It suffices to show that

$$\frac{|q_iq_j|}{|p_ip_j|} \ge \frac{\delta+2r}{\delta+2}.$$

By Lemma 3.2,

$$\frac{|q_iq_j|}{|p_ip_j|} \ge \frac{|o_io_j|+2r}{|o_io_j|+2}.$$

Since $|o_io_j| \ge \delta$, we obviously have

$$\frac{|o_io_j|+2r}{|o_io_j|+2} \ge \frac{\delta+2r}{\delta+2}.$$

By combining the two inequalities the lemma follows. ∎

For $\delta \geq 2$, and $0 < r \leq 1$, let

$$c_1(\delta, r) = \frac{\delta + 2r}{\delta + 2}, \quad c_2(\delta, r) = \frac{\sqrt{\delta^2 - 4r^2}}{\delta}.$$

Observe that $c_1(\delta, r) \leq 1$, and $c_2(\delta, r) \leq 1$. We will show that STEP 1 and STEP 2 can be implemented as to achieve approximation ratios $c_1(\delta, r)$ and $c_2(\delta, r)$, respectively. The resulting overall approximation ratio is then

$$c(\delta, r) = c_1(\delta, r) \cdot c_2(\delta, r),$$

and it remains to choose $r = r(\delta)$ over the whole range $\delta \geq 2$, so as to maximize $c(\delta, r)$.

Selecting $r(\delta)$. For a fixed $\delta \geq 2$, let

$$f(r) = c(\delta, r) = c_1(\delta, r) \cdot c_2(\delta, r) = \frac{\delta + 2r}{\delta + 2} \cdot \frac{\sqrt{\delta^2 - 4r^2}}{\delta}.$$

Note that $r \leq 1 \leq \frac{\delta}{2}$, hence $f(r)$ is well defined.

Consider first the case $2 \leq \delta < 4$. Assume further that $r < 1$, so that $\sqrt{\delta^2 - 4r^2}$ and $f(r)$ are strictly positive. The derivative of $f(r)$ is

$$f'(r) = \frac{2(\delta + 2r)(\delta - 4r)}{\delta(\delta + 2)\sqrt{\delta^2 - 4r^2}}. \tag{3.9}$$

The function $f(r)$ is maximized by setting $f'(r)$ to zero, which yields $r = \frac{\delta}{4}$, (note that $r < 1$), and correspondingly,

$$c\left(\delta, \frac{\delta}{4}\right) = c_1\left(\delta, \frac{\delta}{4}\right) \cdot c_2\left(\delta, \frac{\delta}{4}\right) = \frac{3\delta}{2} \cdot \frac{1}{\delta + 2} \cdot \sqrt{\frac{3}{4}} = \frac{3\sqrt{3}}{4} \cdot \frac{\delta}{\delta + 2}.$$

Observe that $c(\delta, \frac{\delta}{4}) \geq c(2, \frac{1}{2}) = 0.6495\ldots$, in our interval $2 \leq \delta < 4$.

Consider now the case $\delta \geq 4$, and assume further that $r \leq 1$. Since $\delta \geq 4 > 2$, the expression of the derivative $f'(r)$ in equation (3.9) is still valid. We have $f'(r) > 0$, hence $f(r)$ is an increasing function, so

$$c(\delta, r) = f(r) \leq f(1) = \frac{\sqrt{\delta^2 - 4}}{\delta}.$$

Thus for $\delta \geq 4$, we set $r = 1$. To summarize, we set

$$r = r(\delta) = \begin{cases} \frac{\delta}{4} & \text{if } 2 \leq \delta \leq 4, \\ 1 & \text{if } \delta \geq 4. \end{cases} \tag{3.10}$$

Note that $r(\delta)$ is a continuous function over the entire range $\delta \geq 2$. The resulting overall approximation ratio of the algorithm, denoted by $c = c(\delta)$, is at least

$$c(\delta) = \begin{cases} \frac{3\sqrt{3}}{4} \cdot \frac{\delta}{\delta+2} & \text{if } 2 \leq \delta \leq 4, \\ \frac{\sqrt{\delta^2-4}}{\delta} & \text{if } \delta \geq 4. \end{cases} \tag{3.11}$$

Define also for future reference the approximation ratios achieved in STEP 1 and STEP 2 of the algorithm, based on our previous choice of r, depending on δ.

$$c_1 = c_1(\delta) = \begin{cases} \frac{3}{2} \cdot \frac{\delta}{\delta+2} & \text{if } 2 \leq \delta \leq 4, \\ 1 & \text{if } \delta \geq 4. \end{cases} \tag{3.12}$$

$$c_2 = c_2(\delta) = \begin{cases} \frac{\sqrt{3}}{2} & \text{if } 2 \le \delta \le 4, \\ \frac{\sqrt{\delta^2-4}}{\delta} & \text{if } \delta \ge 4. \end{cases} \tag{3.13}$$

In particular, for $\delta = 2$, we have

$$r = \frac{1}{2}, \quad c_1 = \frac{3}{4}, \quad c_2 = \frac{\sqrt{3}}{2},$$

hence the overall ratio for Step 1 and Step 2 is $c_1 c_2 = \frac{3\sqrt{3}}{8}$.

To implement Step 2, we are lead to the following linear program, with the constraints expressed symbolically at this point. LP1 maximizes the minimum projection on the set of lines connecting the centers of the disks; that is, for each pair of disks, the length of the projection of the segment connecting the corresponding two points on the line connecting the two disk centers.

$$\begin{array}{ll} \text{maximize} & z \\ \text{subject to} & \begin{cases} p_i \in \omega_i, & 1 \le i \le n \\ \text{proj}_{\alpha_{ij}}(p_i, p_j) \ge z, & 1 \le i < j \le n \end{cases} \end{array} \tag{LP1}$$

Approximating the small disks by regular polygons. Let $\lambda > 0$ be small. Recall that $r = r(\delta)$ is a fixed precomputed value. Select k large enough so that the apothem of the regular k-gon inscribed in a circle of radius r is at least $r(1-\lambda)$. Recall that the apothem length a is given by the formula: $a = r\cos\frac{\pi}{k}$, so we need to choose k so that

$$\cos\frac{\pi}{k} \ge 1 - \lambda. \tag{3.14}$$

The symbolic constraint $p_i \in \omega_i$ is replaced by the k linear constraints defining the sides of the regular polygon (the polygon is the intersection of k half-planes). Let $\varepsilon > 0$ be small. By setting $\lambda = \lambda(\varepsilon)$ sufficiently small, we can ensure that the approximation ratio remains at least $(1-\varepsilon)\frac{3\sqrt{3}}{8}$, say at least 0.649. Let now

$$c_3(\delta, r) = \frac{\delta + 2r(1-\lambda)}{\delta + 2r}. \tag{3.15}$$

Replacing the small disks of radius r by regular polygons with k sides incurs only a slight loss in the approximation ratio for k sufficiently large, since the disks of radii a are contained in the regular polygons with k sides, and a is close to r. Analogous to inequality (3.2) in Lemma 3.2, the setting in (3.15) is justified, and the overall approximation ratio of the algorithm is at least $c_3(\delta, r) \cdot c(\delta, r)$. Recall the setting of $r(\delta)$ given by (3.10). For $2 \le \delta \le 4$, we have

$$c_3\left(\delta, \frac{\delta}{4}\right) = \frac{\delta + 2\frac{\delta}{4}(1-\lambda)}{\delta + 2\frac{\delta}{4}} = 1 - \frac{\lambda}{3}.$$

For $\delta \ge 4$, we have

$$c_3(\delta, 1) = \frac{\delta + 2(1-\lambda)}{\delta + 2} = 1 - \frac{2\lambda}{\delta + 2} \ge 1 - \frac{\lambda}{3}.$$

Consequently, to ensure that the approximation ratio of the algorithm is at least $(1-\varepsilon)\cdot c(\delta)$ over the entire range $\delta \geq 2$, let $\lambda = 3\varepsilon$, and choose k such that (recall (3.14)):

$$\cos\frac{\pi}{k} \geq 1 - 3\varepsilon.$$

For instance, setting $\varepsilon = \frac{7}{10000}$, and $k = 50$ satisfies the above inequality and ensures that the approximation ratio remains at least $(1-\varepsilon)\frac{3\sqrt{3}}{8} \geq 0.649$.

Writing the linear constraints. Implement each symbolic constraint $\mathrm{proj}_{\alpha_{ij}}(p_i, p_j) \geq z$ as follows: Let $o_i = (\xi_i, \eta_i)$ be coordinates of o_i, for $i = 1, \ldots, n$ (part of the input). For simplicity, assume that the disk centers are non-decreasing order of their x-coordinates: $\xi_1 \leq \xi_2 \leq \ldots \leq \xi_n$. Consider a pair i, j, where $i < j$. Recall that $\alpha_{ij} \in (-\pi/2, \pi/2)$ is the angle of the line determined by o_i and o_j. We have

$$\cos\alpha_{ij} = \frac{\xi_j - \xi_i}{|o_i o_j|}, \quad \sin\alpha_{ij} = \frac{\eta_j - \eta_i}{|o_i o_j|}. \tag{3.16}$$

Let $\overline{a_{ij}} = (\cos\alpha_{ij}, \sin\alpha_{ij})$, so that $|\overline{a_{ij}}| = 1$. Let $\overline{s_{ij}} = (x_j - x_i, y_j - y_i)$. According to (3.1),

$$\mathrm{proj}_{\alpha_{ij}}(p_i, p_j) = \frac{\langle \overline{a_{ij}} \cdot \overline{s_{ij}} \rangle}{|\overline{a_{ij}}|} = \langle \overline{a_{ij}} \cdot \overline{s_{ij}} \rangle = (x_j - x_i)\cos\alpha_{ij} + (y_j - y_i)\sin\alpha_{ij}.$$

Consequently, for each pair i, j, where $i < j$, generate the constraint:

$$(x_j - x_i)\cos\alpha_{ij} + (y_j - y_i)\sin\alpha_{ij} \geq z;$$

where $\cos\alpha_{ij}$ and $\sin\alpha_{ij}$ are as in (3.16).

Establishing the approximation ratio.

Lemma 3.4. *Let $p_1, \ldots, p_n$ be n points, where $p_i \in \omega_i$, such that for any $i \neq j$, $|p_i p_j| \geq d$, for some $d > 0$. Then for any $i \neq j$, $\mathrm{proj}_{\alpha_{ij}}(p_i, p_j) \geq c_2 \cdot d$.*

Proof. Observe that the line determined by the points p_i and p_j intersects both disks ω_i and ω_j. The claimed inequality is now immediate from Lemma 3.1. ∎

Lemma 3.5. *Let $p_1, \ldots, p_n$ be n points, where $p_i \in \omega_i$, such that for any $i \neq j$, $\mathrm{proj}_{\alpha_{ij}}(p_i, p_j) \geq d$, for some $d > 0$. Then, for any $i \neq j$, $|p_i p_j| \geq d$.*

Proof. Obviously, $|p_i p_j| \geq \mathrm{proj}_{\alpha_{ij}}(p_i, p_j) \geq d$, as required. ∎

Lemma 3.6. *The ratio of the approximation algorithm* **A2** *is at least $(1-\varepsilon)\frac{3\sqrt{3}}{8}$, for any given $\varepsilon > 0$. ($\frac{3\sqrt{3}}{8} = 0.6495\ldots$) Moreover, if $\delta \geq 2$ is the minimum distance among the unit disk centers, the approximation ratio is at least $(1-\varepsilon)\cdot c(\delta) \geq (1-\varepsilon)\frac{3\sqrt{3}}{8}$, where $c(\delta)$ is given by* (3.11).

Proof. Let $p_1, \ldots, p_n$ be n points, where $p_i \in \Omega_i$, such that for any $i \neq j$, $|p_i p_j| \geq d$, for some $d > 0$. In other words, assume that $\mathrm{OPT} \geq d$. By Lemma 3.3, there exist n points, $q_1, \ldots, q_n$, such that $q_i \in \omega_i$, and for any $i \neq j$, $|q_i q_j| \geq c_1 \cdot d$. (This inequality is trivial for $\delta \geq 4$, since we set $r = 1$, and $c_1 = 1$ in that case; refer to (3.12).) By Lemma 3.4, for any $i \neq j$, $\mathrm{proj}_{\alpha_{ij}}(q_i, q_j) \geq c_2 \cdot c_1 \cdot d = c(\delta)\cdot d$. Recall that the linear program (LP1) finds a point set $\{p_i = (x_i, y_i), i = 1, \ldots, n\}$, for which the minimum projection is maximized. However, the feasible regions for each point are the slightly smaller inscribed regular polygons rather than the small disks. By Lemma 3.5, and the preceding discussion, the computed point set satisfies that, for any $i \neq j$, $|p_i p_j| \geq (1-\varepsilon) \cdot c(\delta) \cdot d$. Hence the approximation algorithm has ratio at least $(1-\varepsilon) \cdot c(\delta) \geq (1-\varepsilon)\frac{3\sqrt{3}}{8}$, as claimed. ■

Reducing the number of constraints to $O(n)$. Recall that OPT $\leq \delta + 2$. So there is no need to write any constraints for pairs of disks at distance $\delta + 4$ or more, since the distance between the corresponding points is at least $\delta + 2$. An easy packing argument shows that the number of pairs of disks at distance at most $\delta + 4$ is only $O(n)$.

Solving the LP. The constraints of the LP involve irrational numbers, and hence it cannot be claimed that the original LP is solvable in polynomial time. However, it is enough to solve the LP up to some precision. For this, it is enough to approximate the numbers involved in the constraints up to some precision, which is polynomial in the error of the output. There are bounds on how many bits of precision are needed in the constraints to obtain a bound on the precision of the solution, and they are polynomially related [9]. Consequently, since we are dealing with ε-approximation anyway, we can encode each coefficient into a rational number with $(1/\varepsilon)^{O(1)}$ bits. Then, by our choice of ε, each coefficient has a constant number of bits. Thus the LP algorithm runs in polynomial time; e.g., $O(n^4)$ or $O(n^{3.5})$ using interior point methods.

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