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On the Tightening of the Standard SDP for Vertex Cover with ℓ_1 Inequalities

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ABSTRACT. We show that the integrality gap of the standard SDP for VERTEX COVER on instances of n vertices remains $2 - o(1)$ even after the addition of *all* hypermetric inequalities. Our lower bound requires new insights into the structure of SDP solutions behaving like ℓ_1 metric spaces when one point is removed. We also show that the addition of all ℓ_1 inequalities eliminates any solutions that are not convex combination of integral solutions. Consequently, we provide the strongest possible separation between hypermetrics and ℓ_1 inequalities with respect to the tightening of the standard SDP for VERTEX COVER.

1 Introduction

A *vertex cover* for a graph $G = (V, E)$ is a subset of the vertices touching all edges. The minimum VERTEX COVER problem (VC) is to find a minimal vertex cover for a graph. While the corresponding decision version for VERTEX COVER is a classic NP-hard problem, the exact approximability of the minimum VERTEX COVER problem remains one of the outstanding open problems in approximation algorithms.

In terms of lower bounds, Dinur and Safra [6] show that VERTEX COVER is NP-hard to approximate within a factor better than 1.36. Assuming Khot's [15] Unique Games conjecture holds, Khot and Regev [16] show that computing a $2 - \Omega(1)$ approximation is NP-hard. As for upper bounds, a very simple argument based on maximal matchings shows that VERTEX COVER admits a polynomial time 2 approximation. The best approximation algorithm known is due to Karakostas [13] and has an approximation ratio of $2 - \Omega(\sqrt{1/\log n})$.

Closing the gap between the known upper and lower bounds on VERTEX COVER's approximability (or obtaining a tight lower bound without relying on the Unique Games Conjecture) has proved particularly difficult. As a result researchers have focused on studying how well we can approximate VERTEX COVER using algorithmic techniques proven successful for other optimization problems. One such family of algorithms arises by first formulating the optimization problem as a (intractable) quadratic integer problem and then relaxing the integrality constraint to obtain a semidefinite program (SDP) that can be solved in polynomial time up to any desired precision. This approach was first introduced by Goemans and Williamson [11] and used to obtain a breakthrough 0.878-approximation algorithm for MAX CUT. Subsequently, many SDP-based algorithms have been discovered and which yield the best approximation algorithms known for several optimization problems [2, 14, 13].

The quality of an SDP relaxation is typically measured by its integrality gap, namely, the ratio between the true optimal solution and the relaxed SDP solution. It is generally accepted that a lower bound on the integrality gap is a lower bound on the approximation

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ratio achievable by any algorithm based on the SDP relaxation. Unfortunately, so far no SDP relaxation for VERTEX COVER has been found whose integrality gap is not $2 - o(1)$.

Indeed, Kleinberg and Goemans [10] show that the obvious "standard" SDP for VERTEX COVER (defined in Section 2.2) has integrality gap $2 - o(1)$. But can this standard SDP be tightened with further constraints to reduce the integrality gap? A series of papers studies whether so-called ℓ_1 inequalities can decrease the integrality gap. The use of ℓ_1 inequalities is motivated by the fact that solutions to the standard quadratic programming formulation for VERTEX COVER lie in an ℓ_1 metric space. Further motivation comes from a paper by Hatami et al. [12] showing that adding *all* ℓ_1 inequalities to the standard SDP for VERTEX COVER yields true optimal solutions. Now, adding all ℓ_1 inequalities yields an intractable SDP relaxation. The natural question that then emerges is whether there is a subset of ℓ_1 inequalities which decreases the integrality gap while keeping the program tractable. Indeed such subsets have been useful for other optimization problems: For instance, the simplest ℓ_1 inequality, the triangle inequality, is crucial in the Arora-Rao-Vazirani SDP algorithm for SPARSEST CUT [2] and subsequently in the best tractable SDP formulation for VERTEX COVER [13]. Avis and Umemoto [3] used k-gonal inequalities (a family of ℓ_1 inequalities generalizing the triangle inequality) to design a PTAS for MAX CUT on certain sparse graph families. However, results for VERTEX COVER have so far all been negative: A series of papers [4, 12, 9] culminates in showing that adding so-called hypermetric inequalities (the most well known canonical family of ℓ_1 inequalities, and a generalization of k-gonal inequalities) of *bounded support* does not reduce the integrality gap. The latter is also motivated by the fact that, as k grows, the k-gonal inequalities become increasingly stronger. This will be discussed in Section 2.1.

In this paper, we bring this series of results to its "completion" by showing, somewhat surprisingly, that hypermetrics *never* help for VERTEX COVER:

THEOREM 1. *The integrality gap of the standard SDP relaxation for* VERTEX COVER *tightened with* all *hypermetric inequalities is* $2 - o(1)$.

Theorem 1 may provide further evidence of the true inapproximability of the VERTEX COVER problem. It was consistent with previous results that tightening the standard SDP relaxation for VERTEX COVER with hypermetrics of sufficiently large support (note that such SDPs might not be "tractable": they would only be computable in time polynomial in the number of constraints added) might give an integrality gap of $2 - \Omega(1)$.

Our result extends several ideas from [9]. Indeed the graph instances and our SDP vector construction is similar to the one used in [9] (and related to those used in [8]). Our improvement relies on some new insights we develop for controlling the value of certain "hypermetric-like" inequalities on ℓ_1 embeddable metrics. (An in-depth comparison to previous constructions can be found in Section 3.4.)

But if hypermetrics don't help, can we hope that the family of all ℓ_1 inequalities helps? The answer seems to depend on the problem; for example, in the MINIMUM MULTICUT problem [1] the addition of all ℓ_1 inequalities does not yield integrality gap 1. In contrast, we show that for VERTEX COVER the opposite is true:

THEOREM 2. *Consider a vector solution of the standard SDP for* VERTEX COVER *that along with (at least) one antipode vector satisfies all ℓ_1 inequalities. Then the solution is in the* *integral hull, and therefore the integrality gap is 1.*

In particular, Theorems 1 and 2 together show that to reduce the integrality gap one *must* employ "unnatural" ℓ_1-inequalities. As mentioned above, Hatami et al. [12] prove a similar result to Theorem 2 showing that strengthening the standard SDP for VERTEX COVER with all ℓ_1 inequalities yields an SDP with no integrality gap. However, we emphasize that their result, which is essentially proved by exploiting the optimality of the SDP solution, does not rule out the possibility of a feasible SDP solution outside the integral hull.

Relations to Lift-and-project systems Lift-and-project procedures, such as those defined by Lovász and Schrijver [18] and Lasserre [17], take an initial LP or SDP relaxation and then systematically derive (over successive rounds) *all* inequalities valid for the integral hull. Relaxations for VERTEX COVER in the Lovász-Schrijver hierachy are incomparable to those studied here (see [9]); the VERTEX COVER SDP relaxation produced after k rounds of Lasserre's tightening satisfies all ℓ_1 inequalities of support k. Strong integrality gaps for lift-and-project derived SDPs (but incomparable to those proved here) are proved by Georgiou et al. [8] and Schoenebeck [19] for the Lovász-Schrijver and Lasserre systems, respectively.

2 Preliminaries

2.1 Metric Spaces, and ℓ_1 and Hypermetric Inequalities

A finite metric space (X, d) is ℓ_1 *embeddable*, or simply an ℓ_1-*metric*, if there exists a mapping $f : X \to \mathbb{R}^n$ such that for all $x, y \in X$ we have $d(x, y) = \|f(x) - f(y)\|_1$. The mapping f is called an *isometry*. We now survey those facts about ℓ_1 metric spaces we will need. For proofs see [5].

Fix a finite set of points X of size n which we will denote by $[n]$. For each $S \subseteq X$ define the *cut metric* $\delta_S : [n] \times [n] \to \{0, 1\}$ such that $\delta_S(i, j) = 1$ if $|S \cap \{i, j\}| = 1$ and 0 otherwise. Cut metrics are clearly ℓ_1 embeddable, and moreover, every ℓ_1 embeddable metric space d can be represented as a convex combination of cut metrics, namely $d(i, j) = \sum_S \lambda_S \delta_S(i, j)$, where $\lambda_S \geq 0$. We then say that (X, d) is *realized* by $\{\lambda_S\}_{S \subseteq X}$ (realization is not unique in general). An ℓ_1 inequality is an inequality $\sum_{ij} B_{ij} x_{ij} \leq 0$ that holds for all ℓ_1 embeddable metrics d, that is $\sum_{ij} B_{ij} d(i, j) \leq 0$. It is possible to show that $\sum_{ij} B_{ij} x_{ij} \leq 0$ is an ℓ_1 inequality if and only if it satisfies $\sum_{ij} B_{ij} d(i, j) \leq 0$ for all cut metrics d.

A canonical discrete class of ℓ_1 inequalities is the class of *hypermetric inequalities*.

DEFINITION 3. *For any* $\mathbf{b} \in \mathbb{Z}^n$ *with* $\sum_{i=1}^n b_i = 1$, *the inequality* $\sum_{ij} b_i b_j x_{ij} \leq 0$ *is a* hypermetric inequality. *The support of a hypermetric inequality is the support of* $\mathbf{b} = (b_1, \ldots, b_n)$.

It is well known that hypermetric inequalities are ℓ_1-inequalities, that is $\sum_{ij} b_i b_j d(i, j) \leq 0$ for all ℓ_1-metrics d (this also follows as a corollary from Lemma 5 below). Note that the hypermetrics include the triangle inequality (by taking $b_i = b_j = 1$, $b_k = -1$ and $\mathbf{b}$ is 0 elsewhere), and all other k-gonal inequalities (e.g., the pentagonal inequality) which are simply those hypermetrics where each b_i is ± 1 or 0.

Both hypermetric inequalities and ℓ_1 inequalities define convex cones. The cone of hypermetric inequalities is contained in the cone of ℓ_1 inequalities, and the containment is strict for dimension at least 7. The cone of hypermetric inequalities is polyhedral, and many of its facets define facets in the ℓ_1 cone. Since no hypermetric inequality is a positive multiple of another, it follows that only finitely many hypermetrics define facets of the cut cone. A canonical example of such facets are the k-gonal inequalities defined above. It is important to note that k-gonal inequalities are stronger the larger k is in the following sense: for every $k > 1$ there exists a metric on n-points satisfying all $(2t+1)$-gonal inequalities, $0 < t < k$ while the $(2k+1)$-inequality is violated (Corollary 28.3.3 in [5]).

2.2 SDP Formulations for Vertex Cover and Integrality Gap Constructions

Let $G = (V, E)$ be a graph with $V = [n]$. The standard SDP relaxation for VERTEX COVER is

$$\begin{array}{lll} \min & \sum_{i \in V} \|\mathbf{z}_i + \mathbf{z}_0\|_2^2/4 & \\ \text{s.t.} & \|\mathbf{z}_i - \mathbf{z}_0\|_2^2 + \|\mathbf{z}_j - \mathbf{z}_0\|_2^2 = \|\mathbf{z}_i - \mathbf{z}_j\|_2^2 & \forall ij \in E \\ & \|\mathbf{z}_i\|_2^2 = 1 & \forall i \in \{0\} \cup V, \end{array} \tag{1}$$

where the $\mathbf{z}_i$ are vectors*. Note that any vector solution $\{\mathbf{z}_i\}_{i \in \{0\} \cup V}$ of (1) induces a distance function $d(i,j) = \|\mathbf{z}_i - \mathbf{z}_j\|_2^2$.

The SDP relaxation (1) is in general stronger than the standard LP relaxation for VERTEX COVER. Unlike the standard LP, showing that (1) has an integrality gap of $2 - o(1)$ is non-trivial [10]. The graph instances witnessing the integrality gap rely on a powerful combinatorial theorem due to Frankl and Rödl [7] that shows that there cannot be a large family of sets of certain cardinality, all of whose pairwise intersections satisfy a certain condition.

DEFINITION 4. *Given $\gamma > 0$, the* Frankl-Rödl graph G_m^γ *is the graph on the 2^m vertices of the m-dimensional hypercube $\{-1,1\}^m$ having edges between those vertices with Hamming distance exactly $(1-\gamma)m$.*

The theorem of Frankl and Rödl [7] implies that for any constant $\gamma > 0$, a vertex cover of the graphs G_m^γ has size $2^m - o(2^m)$. In fact, it follows from their work that G_m^γ enjoys these properties even for sufficiently large subconstant γ; this was made explicit in [8] showing that one can set $\gamma = \Omega(\sqrt{\log m/m})$ to ensure that no small vertex covers exist.

To appreciate the theorem, notice that for $\gamma = 0$ the graph G_m^γ is just a perfect matching, and hence has a vertex cover of size only half the graph. But by making γ only slightly positive the minimum vertex cover of the obtained graph "jumps" in size to be almost all the vertices!

Frankl-Rödl graphs have been used to prove all the tight integrality gap results [10, 4, 12, 8, 9] for VERTEX COVER SDPs mentioned in the introduction. Most of these papers study (implicitly or explicitly) whether there exists some small enough subset of ℓ_1-inequalities that can be added to the standard SDP relaxation (1) to reduce the integrality gap. Let us briefly explain the role of ℓ_1 inequalities in this context. The metric induced by an integral solution of (1) is (a scalar multiple of) the cut metric associated with the vertex cover. Therefore, ℓ_1 inequalities are valid for all integral solutions. In the extreme, adding *all* ℓ_1-inequalities eliminates the integrality gap [12], and thus focusing on this family of inequalities seems natural.

*Note that the edge constraint can be equivalently written (and is perhaps more well-known) as $(\mathbf{z}_0 - \mathbf{z}_i) \cdot (\mathbf{z}_0 - \mathbf{z}_j) = 0$.

In this paper we analyze the performance of the standard SDP for VERTEX COVER strengthened with hypermetric inequalities, namely, the SDP (1) strengthened by

$$\sum_{ij} b_i b_j \|\mathbf{z}_i - \mathbf{z}_j\|_2^2 \le 0, \qquad \forall \mathbf{b} \in \mathbb{Z}^{n+1} \text{ such that } \sum_i^{n+1} b_i = 1. \tag{2}$$

In the above, if we only use integer vectors $\mathbf{b}$ of support k we obtain the SDP for VERTEX COVER strengthened by all hypermetrics of support k.

Charikar [4] was the first to show tight integrality gaps when we add triangle inequality (a hypermetric of support three) to SDP (1). In [12] a similar result was shown when pentagonal inequalities are added. The strongest negative result analyzing the effect of ℓ_1-inequalities on the standard SDP for VERTEX COVER is due to Georgiou et al. [9] where it is shown that the addition of hypermetrics with support $O(\sqrt{\log n / \log\log n})$ cannot reduce the integrality gap below $2 - o(1)$.

3 Hypermetrics Cannot Strengthen the Standard SDP for VC

3.1 Preparatory Observations about Hypermetric Inequalities

How can we show that a certain metric d satisfies all hypermetric inequalities? Of course the simplest way would be to take an ℓ_1 embeddable metric d that "automatically" satisfies all such inequalities. But by [12] we know that if the solution metric is ℓ_1 embeddable then the value of the SDP will be the same as the integral optimum. However, this type of reasoning is still useful: our solution d will be "almost" ℓ_1 embeddable: if we remove the point associated with v_0 the rest of the points will in fact be ℓ_1 embeddable; nevertheless, we will pick our solution so that we have an integrality gap as large as $2 - o(1)$. Next we present some simple lemmas that will help in analyzing hypermetric inequalities for such "almost-ℓ_1" metrics.

We start by analyzing a generalization of the notion of hypermetric inequalities (hypermetrics correspond to the case $q = 1$).

LEMMA 5. *Let (X, d), be an ℓ_1-metric on n points realized by $\{\lambda_S\}_{S \subseteq X}$. Let $b_1, \dots, b_n \in \mathbb{Z}$ be such that $\sum_i^n b_i = q$. Then $\sum_{1 \le i < j \le n} b_i b_j d(i,j) \le \lfloor (q/2)^2 \rfloor \sum_S \lambda_S$.*

PROOF.

$$\begin{aligned}\sum_{1\le i<j\le n} b_i b_j d(i,j) &= \sum_{1\le i<j\le n} b_i b_j \sum_S \lambda_S \delta_S(i,j) = \sum_S \lambda_S \sum_{1\le i<j\le n} b_i b_j \delta_S(i,j)\\ &= \sum_S \lambda_S \sum_{i\in S, j\notin S} b_i b_j = \sum_S \lambda_S \left(\sum_{i\in S} b_i\right)\left(q - \sum_{i\in S} b_i\right) \le \sum_S \lambda_S (\lfloor (q/2)^2 \rfloor).\end{aligned}$$

The last inequality follows from the geometric-mean arithmetic-mean inequality for integers.

We next show that when an ℓ_1-metric has a unit representation, that is, points are vectors in $\mathbb{R}^n$ of unit ℓ_2^2 norm, then it is sometimes possible to bound the sum of the cut coefficients. We say that an ℓ_1-metric with a unit representation has *large diameter* if it has diameter 4. (Notice that the diameter of any metric with unit representation is at most 4.)

LEMMA 6. *Let d be an ℓ_1-metric with unit representation that has large diameter. Then $\sum_S \lambda_S = 4$.*

PROOF. Having a large diameter is equivalent to having two unit vectors in the representation that are antipodes. Without loss of generality, let $\mathbf{z}_1 = -\mathbf{z}_2$. Also, since any $S \subseteq X$ induces a cut, we may assume that $\{\lambda_S\}_{S \subseteq X}$ are non-zero only for sets S that contain 1. Now note that $4 = \|\mathbf{z}_1 - \mathbf{z}_2\|^2 = d_{12} = \sum_{S \not\ni 2} \lambda_S$ so our task is to show that $\lambda_S = 0$ whenever $2 \in S$. Let $i \in [n]$. Then $\|\mathbf{z}_1 - \mathbf{z}_i\|^2 + \|\mathbf{z}_i - \mathbf{z}_2\|^2 = 4 - 2(\mathbf{z}_1\mathbf{z}_i + \mathbf{z}_2\mathbf{z}_i) = 4 = \|\mathbf{z}_1 - \mathbf{z}_2\|^2$. Since for every S, $\delta_S(1,i) + \delta_S(2,i) \geq \delta_S(1,2)$ and since $\|\mathbf{z}_1 - \mathbf{z}_i\|^2 + \|\mathbf{z}_i - \mathbf{z}_2\|^2 = \|\mathbf{z}_1 - \mathbf{z}_2\|^2$, we know that whenever $\lambda_S > 0$ we must have $\delta_S(1,i) + \delta_S(2,i) = \delta_S(1,2)$. But for S that contains 1 and 2 the right hand side is 0, and hence the left hand side is too and $i \in S$. This is true for all i, and hence $S = X$ which makes it a trivial cut that can be ignored.

COROLLARY 7. *Let $\tilde{d}$ be an ℓ_1-metric space with large diameter unit representation, and let d be the restriction of $\tilde{d}$ on a subset of the points. Further, let $b_1, \ldots, b_n \in \mathbb{Z}$ be such that $\sum_i^n b_i = q$. Then $\sum_{i,j} b_i b_j d(i,j) \leq 4\lfloor (q/2)^2 \rfloor$.*

PROOF. Let λ_S be a realization of $\tilde{d}$. By Lemma 6 we have $\sum_S \lambda_S = 4$. We now apply Lemma 5 to $\tilde{d}$ to get $\sum_{i,j} b_i b_j \tilde{d}(i,j) \leq \sum_S \lambda_S \left(\lfloor q/2 \rfloor\right)^2 = 4\left(\lfloor q/2 \rfloor\right)^2$. Since d is a restriction of $\tilde{d}$ the corollary follows.

3.2 The Vector Solution

Our construction is based on tensored vectors. Recall that the tensor product $\mathbf{u} \otimes \mathbf{v}$ of vectors $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^m$ is the vector in $\mathbb{R}^{nm}$ indexed by ordered pairs from $n \times m$ and assuming the value $u_i v_j$ at coordinate (i,j). Define $\mathbf{u}^{\otimes d}$ to be the vector in $\mathbb{R}^{n^d}$ obtained by tensoring $\mathbf{u}$ with itself d times. Let $P(x) = c_1 x^{t_1} + \ldots + c_q x^{t_q}$ be a polynomial with nonnegative coefficients. Then T_P is the function that maps a vector $\mathbf{u}$ to the vector $T_P(\mathbf{u}) = (\sqrt{c_1}\mathbf{u}^{\otimes t_1}, \ldots, \sqrt{c_q}\mathbf{u}^{\otimes t_q})$. Polynomial tensoring can be used to manipulate inner products in the sense that $T_P(\mathbf{u}) \cdot T_P(\mathbf{v}) = P(\mathbf{u} \cdot \mathbf{v})$.

Recall Definition 4 of the graphs G_m^γ for which we want to build a vector solution for SDP (1) strengthened by (2). For $\gamma > 0$ where $1/\gamma$ is even, our SDP solution will be the result of the tensoring polynomial $P(x) = c_2(x+1)x^{2m/\gamma} + c_1 x^{1/\gamma} + (1 - (c_1 + 2c_2))x$ applied on the normalized m-dimensional hypercube $\{-1,1\}^m$, where all c_1, c_2 and $1 - (c_1 + 2c_2)$ are non-negative. Note that regardless of c_1, c_2, we have $P(1) = 1$. Let $\mathbf{u}_i$ be the normalized vectors of the hypercube, namely $\{\pm 1/\sqrt{m}\}^m$. Our solution vectors are then

$$\mathbf{w}_i = (18\gamma, \sqrt{1 - (18\gamma)^2} T_P(\mathbf{u}_i)),\ i = 1, \ldots, 2^m, \tag{3}$$
$$\mathbf{w}_0 = (1, 0, \ldots, 0).$$

Regardless of the exact choice of P, the value of the objective with the vectors $\{\mathbf{w}_i\}$ in (1) is $2^m(1/2 + 9\gamma)$. To achieve a big integrality gap, we will use the smallest possible value of γ that ensures that no small vertex covers exist, namely $\gamma = \Theta(\sqrt{\log m / m})$.

The following lemma whose proof is deferred to the appendix shows that there exist appropriate constants c_1, c_2 such that the vector solution both satisfies the standard SDP and the triangle inequality.

LEMMA 8. *For sufficiently big m, there exist positive c_1, c_2 (both of order $\Theta(\gamma)$), such that for G_m^γ, the vectors* (3) *satisfy the standard SDP* (1) *strengthened with the triangle inequality. Moreover, $c_2 > 9\gamma$.*

An analogous lemma (Lemma 3) with different bounds on the constants c_1 and c_2 was proved in [8], and the proof is very similar. Indeed, the precise constraints on c_1, c_2 given by Lemma 8 will be crucial for our analysis here and are not implied by Lemma 3 in [8].

Lemma 8 immediately implies that the integrality gap of SDP (1) is at least $\frac{2^m - o(2^m)}{2^m(1/2+9\gamma)}$, which is of course $2 - o(1)$. Therefore Theorem 1 will follow if we additionally show that the vectors (3) satisfy any hypermetric inequality (2). This is taken care of in Section 3.3.

3.3 Proof of Theorem 1

Let $\sum_{ij} B_{ij}x_{ij} \le 0$ be a hypermetric inequality, with $B_{ij} = b_i b_j$, $b_i \in \mathbb{Z}$, $i = 0, \dots, n$. Our goal is to show that for the vectors (3), $\sum_{0\le i<j\le n} B_{ij}\|\mathbf{w}_i - \mathbf{w}_j\|_2^2 \le 0$. By definition, for $i, j \ge 1$,

$$\|\mathbf{w}_i - \mathbf{w}_j\|_2^2 = 2 - 2((18\gamma)^2 + (1-(18\gamma)^2)P(\mathbf{u}_i \cdot \mathbf{u}_j)) = (1-(18\gamma)^2)\|T_P(\mathbf{u}_i) - T_P(\mathbf{u}_j)\|_2^2 \ ,$$

and $\|\mathbf{w}_i - \mathbf{w}_0\|_2^2 = 2(1-18\gamma)$. Hence,

$$\sum_{0\le i<j\le n} B_{ij}\|\mathbf{w}_i - \mathbf{w}_j\|_2^2 = 2(1-18\gamma)\sum_{i=1}^{n} B_{0i} + (1-(18\gamma)^2)\sum_{1\le i<j\le n} B_{ij}\|T_P(\mathbf{u}_i) - T_P(\mathbf{u}_j)\|_2^2$$

Therefore, we need to show

$$\sum_{i=1}^{n} B_{0i} + (1+18\gamma)\frac{1}{2}\sum_{1\le i<j\le n} B_{ij}\|T_P(\mathbf{u}_i) - T_P(\mathbf{u}_j)\|_2^2 \le 0 \ . \tag{4}$$

Let now (Y, d) be a metric defined as $Y = \{1, \dots, n\}$, and $d(i,j) = \|T_P(\mathbf{u}_i) - T_P(\mathbf{u}_j)\|_2^2$. All points $T_P(\mathbf{u}_i)$ are normalized sign vectors. By considering all points $T_P(\mathbf{u}_i)$ along with their antipodes $-T_P(\mathbf{u}_i)$ we can obtain the metric $(\tilde{Y}, \tilde{d})$, where again $\tilde{d}$ is the square Euclidean distance of the vectors. Clearly, d is a restriction of $\tilde{d}$ on a subset of points (recall that the tensoring polynomial P is not odd).

CLAIM 9. *The metric $(\tilde{Y}, \tilde{d})$ is ℓ_1 with large diameter unit representation.*

PROOF. $(\tilde{Y}, \tilde{d})$ has large diameter because all antipodes are present. Now, the vectors $\mathbf{u}_i$ have unit ℓ_2^2 norm, and so do the vectors $T_P(\mathbf{u}_i)$, $i = 1, \dots, n$. Notice that in $\tilde{Y}$ we have excluded the point that corresponds to $\mathbf{z}_0$ in the SDP. Now, applying the tensor operation on a ± 1 vector results in a, say, M-dimensional, ± 1 vector, and hence applying a polynomial on such a vector yields a vector which assumes one of two values in each of the coordinates, and further, one of the values, say x_i, $i = 1, \dots, M$, is the negation of the other. The same holds by including all their antipodes. In other words, all points $\pm T_P(\mathbf{u}_i)$ are vertices of a box centered at the origin. It is easy to see that the ℓ_2^2-metric associated with such a box is ℓ_1 embeddable: any vector $T_P(\mathbf{u}_i)$ (or its antipode) has the form $\mathbf{u}'_i = (s_1^{(i)}x_1, \dots, s_M^{(i)}x_M)$ where $s_t^{(i)} \in \{\pm 1\}$ and it can be mapped by f to $(2s_1^{(i)}x_1^2, \dots, 2s_M^{(i)}x_M^2)$. Hence for any two $i, j \in V$

$$\|\mathbf{u}'_i - \mathbf{u}'_j\|_2^2 = \sum_{t=1}^{M}(s_t^{(i)}x_t - s_t^{(j)}x_t)^2 = \sum_{t=1}^{M}|2s_t^{(i)}x_t^2 - 2s_t^{(j)}x_t^2| = \|f(\mathbf{u}'_i) - f(\mathbf{u}'_j)\|_1.$$

Therefore, for the metric (Y, d) we can apply Corollary 7 with $q = \sum_{i=1}^n b_i = 1 - b_0$ and conclude that the left hand side of expression (4) is upper-bounded by

$$b_0(1-b_0) + (1+18\gamma)2\lfloor(1-b_0)^2/4\rfloor \leq \begin{cases} 0 & \text{if } b_0 \geq 0, \\ \frac{1}{2}(1-b_0)((1-18\gamma)b_0 + (1+18\gamma)) < 0 & \text{if } b_0 \leq -2. \end{cases}$$

Therefore, we have shown that all hypermetrics are satisfied except perhaps those for which $b_0 = -1$ (like the triangle inequality, pentagonal inequality, etc.).

In order to deal with the case $b_0 = -1$ we look deeper into the structure of $T_P(\mathbf{u}_i)$. To start, we simplify our notation by abbreviating $(\mathbf{u}_i \cdot \mathbf{u}_j + 1)(\mathbf{u}_i \cdot \mathbf{u}_j)^{2m/\gamma}$, $(\mathbf{u}_i \cdot \mathbf{u}_j)^{1/\gamma}$, and $\mathbf{u}_i \cdot \mathbf{u}_j$ by H_{ij}, M_{ij} and L_{ij}, respectively, the "high", "medium" and "low" order terms. Then, $P(\mathbf{u}_i \cdot \mathbf{u}_j) = c_2 H_{ij} + c_1 M_{ij} + (1 - (c_1 + 2c_2))L_{ij}$. Note that for distinct $\mathbf{u}_i, \mathbf{u}_j$, we have $|\mathbf{u}_i \cdot \mathbf{u}_j| \leq 1 - 1/m$, and hence H_{ij} is negligible. We therefore omit it in what follows. As $b_0 = -1$, it follows that $\sum_{i=1}^n B_{0i} = b_0(1-b_0) = -2$ and hence that the left hand side of (4) is

$$\begin{aligned} &-2 + (1+18\gamma) \sum_{1 \leq i < j \leq n} B_{ij}(1 - P(\mathbf{u}_i \cdot \mathbf{u}_j)) \\ \approx &-2 + (1+18\gamma) \sum_{1 \leq i < j \leq n} B_{ij}(1 - c_1 M_{ij} - (1 - (c_1 + 2c_2))L_{ij}) \ . \end{aligned} \tag{5}$$

Now we make some simple observations. We have $(\sum_{i=1}^n b_i)^2 = \sum_{i=1}^n b_i^2 + 2\sum_{1 \leq i < j \leq n} b_i b_j$, and since $\sum_{i=1}^n b_i = 1 - b_0 = 2$ we get

$$\sum_{1 \leq i < j \leq n} B_{ij} = \frac{1}{2}(4 - \sum_{i=1}^n b_i^2). \tag{6}$$

Now, note that for unit vectors $\mathbf{u}, \mathbf{v}$ we have $\|\mathbf{u} - \mathbf{v}\|_2^2 = 2(1 - \mathbf{u} \cdot \mathbf{v})$. Hence the values $2(1 - M_{ij})$, $1 \leq i < j \leq n$, are the ℓ_2^2 distances of unit vectors that have undergone the polynomial tensoring transformation using some monomial (similarly for the values $2(1 - L_{ij})$). Arguing exactly as in Claim 9, the vectors form an ℓ_1-metric that has a large diameter unit representation, and so by Corollary 7 we have

$$\sum_{1 \leq i < j \leq n} B_{ij}(1 - M_{ij}) \leq 2, \text{ and } \sum_{1 \leq i < j \leq n} B_{ij}(1 - L_{ij}) \leq 2. \tag{7}$$

We now use (6), (7), to conclude that

$$\begin{aligned} &\sum_{1 \leq i < j \leq n} B_{ij}(1 - c_1 M_{ij} - (1 - c_1 - 2c_2)L_{ij}) \\ &= 2c_2 \sum_{1 \leq i < j \leq n} B_{ij} + c_1 \sum_{1 \leq i < j \leq n} B_{ij}(1 - M_{ij}) + (1 - c_1 - 2c_2) \sum_{1 \leq i < j \leq n} B_{ij}(1 - L_{ij}) \\ &\leq c_2 \left(4 - \sum_{i=1}^n b_i^2\right) + 2c_1 + 2(1 - c_1 - 2c_2) = -c_2 \sum_{i=1}^n b_i^2 + 2. \end{aligned}$$

Recall here that by Lemma 8, $c_2 > 9\gamma$, and the SDP vectors $\{\mathbf{w}_i\}$ satisfy the triangle inequality. Therefore when $b_0 = -1$ we may assume that $\sum_{i=1}^n b_i^2 \geq 4$. Theorem 1 now follows since (5) is upper-bounded by

$$-2 + (1 + 18\gamma) \left(-c_2 \sum_{i=1}^{n} b_i^2 + 2 \right) \leq -2 + (1 + 18\gamma)\,(-36\gamma + 2) = -648\gamma^2. \tag{8}$$

3.4 Discussion and a strengthened version of Theorem 1

In this section we look a bit more carefully at how our result differs from previous work and use the resulting observations to obtain a strengthened version of Theorem 1.

As mentioned in Section 2.2 all previous works [10, 4, 12, 9] studying integrality gaps for VERTEX COVER SDPs use Frankl-Rödl graphs G_m^γ on $n = 2^m$ vertices. Moreover, they all employ tensoring polynomials of some sort to construct their vector solutions. Perhaps the most useful parameter differentiating the vector solutions amongst these papers (including the current paper) is each solution's minimal distance $\Delta = \min_{i \neq j} \|\mathbf{w}_i - \mathbf{w}_j\|_2^2$. In [4, 12] Δ behaves like $1/m$. To a large degree, what allowed the improvement of [9] was a modification of the tensoring polynomials thereby increasing the minimal distance Δ to a constant (an arbitrary small one). The analysis of [9] then showed that the resulting solution satisfies all hypermetrics of support $O(\Delta/\gamma)$ with an integrality gap of $2 - \Theta(\Delta)$ (in particular, taking the smallest possible γ, namely $\gamma = \Theta(\sqrt{m/\log m})$, the analysis in [9] shows that the solution satisfies all hypermetrics of support $O(\sqrt{\log\log n/\log n})$).

In the present work we use similar vectors as the one used in [9] but get more mileage by more carefully analyzing the structure of the ℓ_1-metric that emerges from the solution. In particular, while both [12] and [9] use the fact that removing $\mathbf{w}_0$ from the vector solution gives an ℓ_1-metric, in the current paper we crucially use the fact that our vectors arise by applying tensoring polynomials to "sign" vectors. More precisely, we exploit the fact that the ℓ_1-metric corresponding to the vectors $\{\mathbf{w}_i\}_{i\geq 1}$ has a unit representation with large diameter. The bottom line is that our new analysis allows us to show that *any* hypermetric (not just those with support $O(\Delta/\gamma)$) is satisfied as long as Δ/γ is a sufficiently large constant (our argument does not work for the vector construction of [4] but also does not rule out that same vector construction satisfying SDP (1) strengthened by (2)). But note now that if we take $\gamma = \Theta(\sqrt{\log m/m})$ when defining our Frankl-Rödl instances, then for Δ/γ to be constant it suffices to use a tensoring construction where the minimum distance Δ is of order up to $O(\sqrt{\log m/m})$. In particular, the integrality gap obtained by our analysis is $2 - O(\gamma)$; so taking $\gamma = \Theta(\sqrt{\log m/m})$ gives the following strengthened version of Theorem 1:

THEOREM 10. *The integrality gap of the standard SDP relaxation for* VERTEX COVER *on instances of n vertices tightened with* all *hypermetric inequalities is* $2 - O(\sqrt{\log\log n/\log n})$.

Interestingly, the lower bound in Theorem 10 almost matches the upper bound given by Karakostas [13] who gives an SDP for VERTEX COVER tightened with the triangle inequality and which has integrality gap $2 - \Omega(\sqrt{1/\log n})$.

4 ℓ_1 Embeddability Implies Integrality

This section is devoted to proving Theorem 2 which is based on the following simple observation. Let the metric induced by SDP (1) be ℓ_1, realized by some $\{\lambda_S\}_{S\subseteq X}$. Since every subset S induces a cut, we may restrict ourselves only to subsets S that contain the element 0, corresponding to $\mathbf{z}_0$. Now let $\Lambda = \sum_S \lambda_S$ and consider an orthonormal basis $\{\mathbf{e}_Y\}_{Y\subseteq X}$ ($\mathbf{e}_Y$ is indexed by all subsets of X, and is 1 in the Y-th coordinate and 0 elsewhere). For every $A \subseteq X$ we define $\mathbf{u}_A = \sum_{S:A\subseteq S} \sqrt{\lambda_S}\mathbf{e}_S$. Associate also the singleton $\{0\}$ with the vector $\mathbf{u}_\varnothing$ corresponding to the empty set $\varnothing$. The key observation is that the mapping $\mathbf{z}_i \mapsto \mathbf{u}_{\{i\}}$ is an isometry. This is because $\mathbf{u}_{\{i\}} - \mathbf{u}_{\{j\}}\|_2^2$ equals

$$\begin{aligned}\sum_{S:i\in S}\lambda_S + \sum_{S:j\in S}\lambda_S - 2\sum_{S:\{i,j\}\subseteq S}\lambda_S &= \sum_{S:i\in S}\lambda_S - \sum_{S:\{i,j\}\subseteq S}\lambda_S + \sum_{S:j\in S}\lambda_S - \sum_{S:\{i,j\}\subseteq S}\lambda_S\\ &= \sum_{S:j\notin S\&i\in S}\lambda_S + \sum_{S:i\notin S\&j\in S}\lambda_S = \sum_S \lambda_S\delta_S(i,j)\end{aligned}$$

The last expression is exactly $\|\mathbf{z}_i - \mathbf{z}_j\|_2^2$. Theorem 2 now follows from Lemma 11 below.

LEMMA 11. *Let $G = (V, E)$ be a graph for which the metric induced by the solution of the standard SDP* (1) *is an ℓ_1-metric with unit representation $\{\mathbf{z}_i\}$ that has large diameter. Then the vector solution is a convex combination of vertex covers.*

PROOF. For every $S \subseteq \{1, \ldots, n\}$ consider the characteristic vector $\mathbf{y}^S \in \{0,1\}^n$ with $y_i^S = 1$ if and only if $i \in S$. We prove (A) If $\lambda_S > 0$ then S is a vertex cover; and (B) $\frac{1}{4}\left(\|\mathbf{z}_1 + \mathbf{z}_0\|^2, \ldots, \|\mathbf{z}_n + \mathbf{z}_0\|^2\right) = \sum_S \frac{1}{\Lambda}\lambda_S\mathbf{y}^S$.

For (A) note that the SDP edge constraints simply require that the triangle inequality $\|\mathbf{z}_0 - \mathbf{z}_i\|^2 + \|\mathbf{z}_0 - \mathbf{z}_j\|^2 - \|\mathbf{z}_i - \mathbf{z}_j\|^2 \geq 0$ is tight. The same is true for the vectors $\mathbf{u}_{\{i\}}, \mathbf{u}_{\{j\}}$, since the mapping $\mathbf{z}_i \mapsto \mathbf{u}_{\{i\}}$ is an isometry. It follows that for every edge $ij \in E$ we have

$$\begin{aligned}(\mathbf{u}_\varnothing - \mathbf{u}_{\{i\}})(\mathbf{u}_\varnothing - \mathbf{u}_{\{j\}}) &= \mathbf{u}_\varnothing^2 - \mathbf{u}_\varnothing\cdot\mathbf{u}_{\{i\}} - \mathbf{u}_\varnothing\cdot\mathbf{u}_{\{i\}} + \mathbf{u}_{\{i\}}\cdot\mathbf{u}_{\{j\}}\\ &= \left(\sum_S \lambda_S - \sum_{S\ni i}\lambda_S\right) - \left(\sum_{S\ni j}\lambda_S - \sum_{S\supseteq\{i,j\}}\lambda_S\right)\\ &= \sum_{S\not\ni i}\lambda_S - \sum_{j\in S\not\ni i}\lambda_S = \sum_{j\notin S\not\ni i}\lambda_S,\end{aligned}$$

and the last expression equals 0. Since cut coefficients are non-negative, claim 1 follows.

For (B) it suffices to show that for every $i \in V$, $\frac{1}{4}\|\mathbf{z}_i + \mathbf{z}_0\|^2 = \frac{1}{\Lambda}\sum_{S:i\in S}\lambda_S$. To that end, recall that $\|\mathbf{z}_0 - \mathbf{z}_i\|^2 = \sum_S \delta_S(0,i)$ and $0 \in S$. Hence, $2\mathbf{z}_i\mathbf{z}_0 = 2 - \sum_{S\not\ni i}\lambda_S$, and $\frac{1}{4}\|\mathbf{z}_0 + \mathbf{z}_i\|^2 = \frac{1}{4}\left(4 - \sum_{S\not\ni i}\lambda_S\right)$. The latter equals $\frac{1}{\Lambda}\sum_{S:i\in S}\lambda_S$ iff $\sum_S \lambda_S = 4$. This is guaranteed by Lemma 6, since the ℓ_1-metric induced by SDP (1) has large diameter.

5 Discussion - Open Problems

Our work raises two natural questions. Theorem 1 implies that the most interesting ℓ_1 inequalities are those that are not hypermetric. Given that hypermetrics are the most natural inequalities to consider, can we identify another family of interesting yet natural inequalities that could potentially strengthen the standard SDP for VERTEX COVER? Since such inequalities are produced by the Lasserre system, it seems we must better characterize the constraints derived by that system. Second, it is interesting to investigate to what extent our arguments apply to general ℓ_1 inequalities. A positive answer could potentially give a first step towards showing tight integrality gaps for VERTEX COVER in the Lasserre system.

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Appendix

PROOF. [Lemma 8 - sketch] Let $G_m^\gamma = (V, E)$. The standard SDP (1) strengthened with the triangle inequality requires for our vectors $\mathbf{w}_i$ that

$$\|\mathbf{w}_i - \mathbf{w}_0\|^2 + \|\mathbf{w}_j - \mathbf{w}_0\|^2 = \|\mathbf{w}_i - \mathbf{w}_j\|^2 \quad , \quad \forall ij \in E \text{ (the edge constraints)} \tag{9}$$

$$\|\mathbf{w}_i - \mathbf{w}_0\|^2 + \|\mathbf{w}_j - \mathbf{w}_0\|^2 \geq \|\mathbf{w}_i - \mathbf{w}_j\|^2 \quad , \quad \forall i,j \in V \text{ (the triangle inequality)} \tag{10}$$

and that all vectors have unit norm. For an edge $ij \in E$ we have $\mathbf{u}_i \cdot \mathbf{u}_j = -1 + 2\gamma$. Recalling that $\mathbf{w}_i = (18\gamma, \sqrt{1-(18\gamma)^2} T_P(\mathbf{u}_i))$ where P is our "tensoring" polynomial (see section 3.2), it is easy to see that for the above constraints to hold it suffices to have

$$-\frac{1-18\gamma}{1+18\gamma} = P(-1+2\gamma) \leq P(x), \ \forall x \in [-1,1], \tag{11}$$

where the left equality takes care of the edge constraints and the right inequality implies the triangle inequality. Set $c_1 = \eta_1\gamma$ and $c_2 = \eta_2\gamma$.

For any distinct points of the hypercube, the high order term of P is negligible so we can disregard it. Recall that $1/\gamma$ is even. For the edge constraint, i.e. the right inequality in (11), we require $P'(-1+2\gamma) = 0$, and $P''(-1+2\gamma) > 0$. The former requires that $\eta_1(1-2\gamma)^{1/\gamma-1} = 1-(\eta_1+2\eta_2)\gamma$. The left constraint of condition (11) requires that $-\frac{1-18\gamma}{1+18\gamma} = (1-(\eta_1+2\eta_2)\gamma)(-1+2\gamma) + \eta_1\gamma(1-2\gamma)^{1/\gamma}$. Solving the system of inequalities with respect to η_1, η_2 and taking the limit $\gamma \to 0$ (or equivalently $m \to \infty$) we get that $\eta_1 = e^2$ and $\eta_2 = (36-3-e^2)/2 > 9$. Finally it is easy to check that the second derivative is positive as required.