Real Computation with Least Discrete Advice: A Complexity Theory of Nonuniform Computability

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Abstract. It is folklore particularly in numerical and computer sciences that, instead of solving some general problem $f : A \to B$, additional structural information about the input $x \in A$ (that is any kind of promise that x belongs to a certain subset $A' \subseteq A$) should be taken advantage of. Some examples from real number computation show that such discrete advice can even make the difference between computability and uncomputability. We turn this into a both topological and combinatorial complexity theory of information, investigating for several practical problems how much advice is necessary and sufficient to render them computable.

Specifically, finding a nontrivial solution to a homogeneous linear equation $A \cdot \boldsymbol{x} = 0$ for a given singular real $n \times n$-matrix A is possible when knowing $\operatorname{rank}(A) \in \{0, 1, \ldots, n-1\}$; and we show this to be best possible. Similarly, diagonalizing (i.e. finding a basis of eigenvectors of) a given real symmetric $n \times n$-matrix A is possible when knowing the number of distinct eigenvalues: an integer between 1 and n (the latter corresponding to the nondegenerate case). And again we show that n–fold (i.e. roughly $\log n$ bits of) additional information is indeed necessary in order to render this problem (continuous and) computable; whereas finding *some single* eigenvector of A requires and suffices with $\Theta(\log n)$–fold advice.

1 Introduction

Recursive Analysis, that is Turing's [Turi36] theory of rational approximations with prescribable error bounds, is generally considered a very realistic model of real number computation [BrCo06]. Much research has been spent in 'effectivizing' classical mathematical theorems, that is replacing mere existence claims

i) "*for all x, there exists some y such that ...*" with
ii) "*for all* computable *x, there exists some* computable *y such that ...*"

Cf. e.g. the Intermediate Value Theorem in classical analysis [Weih00, Theorem 6.3.8.1] or the Krein-Milman Theorem from convex geometry [GeNe94].

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Note that Claim ii) is non-uniform: it asserts y to *be* computable whenever x is; yet, there may be no way of *converting* a Turing machine M computing x into a machine N computing y [Weih00, SECTION 9.6]. In fact, computing a function $f : x \mapsto y$ is significantly limited by the sometimes so-called Main Theorem, requiring that any such f be necessarily continuous: because finite approximations to the argument x do not allow to determine the value $f(x)$ up to absolute error smaller than the 'gap' $\limsup_{t\to x} f(t) - \liminf_{t\to x} f(t)$ in case x is a point of discontinuity of f. In particular any non-constant discrete-valued function on the reals is uncomputable—for information-theoretic (as opposed to recursion-theoretic) reasons. Thus, Recursive Analysis is sometimes criticized as a purely mathematical theory, rendering uncomputable even functions as simple as Gauß' staircase [Koep01].

1.1 Motivating Examples

On the other hand many applications do provide, in addition to approximations to the continuous argument x, also certain promise or discrete 'advice'; e.g. whether x is integral or not. And such additional information does render many otherwise uncomputable problems computable:

Example 1. *The Gauß staircase is discontinuous, hence uncomputable. Restricted to integers, however, it is simply the identity, thus computable. And restricted to* non-*integers, it is computable as well; cf.* [Weih00, EXERCISE 4.3.2]. *Thus, one bit of additional advice (*"integer or not"*) suffices to make* $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ *computable.*

Also many problems in analysis involving compact (hence bounded) sets are discontinuous unless provided with some integer bound; compare e.g. [Weih00, SECTION 5.2]. For a more involved illustration from computational linear algebra, we report from [ZiBr04, SECTION 3.5] the following

Example 2. *Given a real symmetric $d \times d$ matrix A (in form of approximations $A_n \in \mathbb{Q}^{d\times d}$ with $|A - A_n| \le 2^{-n}$), it is generally impossible, for lack of continuity and even in the multivalued sense, to compute (approximations to) any eigenvector of A.*
However when providing, in addition to A itself, the number of distinct eigenvalues (i.e. not *counting multiplicities) of A, finding the entire spectral resolution (i.e. an orthogonal basis of eigenvectors) becomes computable.*

1.2 Complexity Measure of Non-Uniform Computability

We are primarily interested in problems over real Euclidean spaces $\mathbb{R}^d$, $d \in \mathbb{N}$. Yet for reasons of general applicability to arbitrary spaces U of continuum cardinality, we borrow from Weihrauch's TTE framework [Weih00, SECTION 3] the concept of so-called *representations*, that is, encodings of all elements $u \in U$ as infinite binary strings; and a *realizer* of a function $f : U \to V$ maps encodings of $u \in U$ to encodings of $f(u) \in V$. A *notation* is basically a representation of a merely countable set.

Definition 3. *a) A function $f :\subseteq A \to B$ between topological spaces A and B is k-*wise continuous *if there exists a covering (equivalently: a partition) Δ of* $\mathrm{dom}(f) = \bigcup_{D\in\Delta} D$ *with* $\mathrm{Card}(\Delta) = k$ *such that $f|_D$ is continuous for each $D \in \Delta$. Call $\mathfrak{C}_{\mathrm{t}}(f) := \inf\{k : f$ is k-wise continuous$\}$ the* cardinal of discontinuity of *f.*

b) A function $f :\subseteq A \to B$ between represented spaces (A, α) and (B, β) is (α, β)–computable with k-wise advice *if there exists an at most countable partition Δ of* $\mathrm{Card}(\Delta) = k$ *and a notation δ of Δ such that the mapping f_Δ : $(a, D) \mapsto f(a)$ is (α, δ, β)–computable on* $\mathrm{dom}(f_\Delta) := \{(a, D) : a \in D \in \Delta\}$. *Call $\mathfrak{C}_{\mathrm{c}}(f) = \mathfrak{C}_{\mathrm{c}}(f, \alpha, \beta) := \min\{k : f$ is (α, β)–computable with k-wise advise$\}$ the* complexity of non-uniform (α, β)–computability of *f.*

c) A function $f :\subseteq A \to B$ is nonuniformly *(α, β)–computable if, for every α–computable $a \in \mathrm{dom}(f)$, $f(a)$ is β-computable.*

So continuous functions are exactly the 1-wise continuous ones; and computability is equivalent to (weak or strong) computability with 1-wise advice. Also we have, as an extension of the Main Theorem of Recursive Analysis, the following immediate

Observation 4. *If α, β are* admissible *representations in the sense of* [Weih00, DEFINITION 3.2.7]*, then every k-wise (α, β)–computable function is k-wise continuous (but not vice versa); that is $\mathfrak{C}_{\mathrm{t}}(f) \le \mathfrak{C}_{\mathrm{c}}(f)$ holds.*
More precisely, every k-wise (α, β)–computable possibly multivalued function $f :\subseteq A \rightrightarrows B$ has a k-wise continuous (α, β)–realizer in the sense of [Weih00, DEFINITION 3.1.3.4].

The above examples illustrate some interesting discontinuous functions to be computable with k-wise advice for some $k \in \mathbb{N}$. Specifically Example 2, diagonalization of real symmetric $n \times n$–matrices is n–wise computable; and Theorem 20 below will show this value n to be optimal.

Remark 5. *We advertise Computability with Finite Advice as a generalization of classical Recursive Analysis:*
a) It constitutes a hybrid approach to both discrete and continuous computation.
b) It complements Type-2 oracle computation: In the discrete realm, every *function $f : \mathbb{N} \to \mathbb{N}$ becomes computable when employing an appropriate oracle; whereas in the Type-2 case, exactly the* continuous *functions $f : \mathbb{R} \to \mathbb{R}$ are computable relative to some oracle. On the other hand, 2-wise advice can make a continuous function computable which without advice has unbounded degree of uncomputability; see Proposition 6d).*
c) Discrete advice avoids a common major point of criticism against Recursive Analysis, namely that it denounces even simplest discontinuous functions as uncomputable;
d) and such kind of advice is very practical: In applications additional discrete information about the input is often actually available and should be used. For instance a given real matrix may be known to be non-degenerate (as is often exploited in numerics) or, slightly more generally, to have k eigenvalues coincide

for some known $k \in \mathbb{N}$.
e) The topology of the members of the collection Δ from Definition 3 can usually be chosen not too wild: compare the examples considered below. In practice, we consider the discrete advice to arise with the input itself. For instance the band-width of a given matrix A may be known as 3 because A comes from a finite element triangular grid approach. Hence the collection Δ need not even be explicit (since it is usually far from unique, anyway; compare Remark 11), nor required effective in any sense.

1.3 Related Work, in particular Kolmogorov Complexity

Several approaches have been pursued in literature to make also discontinuous functions accessible for computability investigations.

Exact Geometric Computation considers the arguments $\boldsymbol{x}$ as exact rational numbers [LPY05].

Special encodings of discontinuous functions motivated by spaces in Functional Analysis, are treated e.g. in [ZhWe03]; however these do not admit evaluation.

Weakened notions of computability may refer to stronger models of computation [ChHo99]; provide more information on (e.g. the binary encoding of, rather than rational approximations with error bounds to) the argument x [Mori02,MTY05]; or expect less information on (e.g. no error bounds for approximations to) the value $f(x)$ [WeZh00].

A taxonomy of discontinuous functions, namely their *degrees* of Borel measurability, is investigated in [Brat05,Zie07a,Zie07b]:
Specifically, a function $f :\subseteq A \to B$ is continuous ($=\Sigma_1$–measurable) iff, for every closed $T \subseteq B$, its preimage $f^{-1}[T]$ is closed in $\mathrm{dom}(f) \subseteq A$; and f is computable iff this mapping $T \mapsto f^{-1}[T]$ on closed sets is $(\psi^d_>, \psi^d_>)$–computable. A degree relaxation, f is called Σ_2–measurable iff, for every closed $T \subseteq B$, $f^{-1}[T]$ is an F_δ-set.

Wadge degrees of discontinuity are an (immense) refinement of the above, namely with respect to so-called *Wadge reducibility*; cf. e.g. [Weih00, SECTION 8.2].

Levels of discontinuity are studied in [HeWe94,Her96a,Her96b]:
Take the set $X_0 \subseteq \mathrm{dom}(f)$ of points of discontinuity of f; then the set $X_1 \subseteq X_0$ of points of discontinuity of $f|_{X_0}$ and so on: the least index k for which X_k is empty is f's level of discontinuity.

Our approach superficially resembles the third and last ones above. A minor difference, they correspond to *ordinal* measures whereas the size of the partition considered in Definition 3 is a *cardinal*. As a major difference we now establish these measures as logically largely independent:

Proposition 6. *a) There exists a 2-wise computable function $f : [0,1] \to \{0,1\}$ which is not measurable nor on any level of discontinuity.*
b) There exists a Δ_2–measurable function $f : [0,1] \to [0,1]$ with is not k-wise continuous for any finite k.

c) *If f is on the k-th level of discontinuity, it is $(k+1)$-wise continuous.*
d) *There exists a continuous, 2-wise computable function $f :\subseteq [0,1] \to [0,1]$ which is not computable, even relative to any prescribed oracle.*
e) *Every k-wise computable function is nonuniformly computable; whereas there are nonuniformly computable functions not k-wise computable for any $k \in \mathbb{N}$.*

Conditions where nonuniform computability does imply (even) 1-wise computability have been devised in [Brat99]. Further related research includes

Computational Complexity of real functions; see e.g. [Ko91] and [Weih00, SECTION 7]. Note, however, that Definition 3 refers to a purely *information*-theoretic notion of complexity of a function and is therefore more in the spirit of

Information-based Complexity in the sense of [TWW88]. There, on the other hand, inputs are considered as real number entities given exactly; whereas we consider approximations to real inputs enhanced with discrete advice.

Finite Continuity is being studied for *Darboux Functions* in [MaPa02,Marc07]. It amounts to d-wise continuity for some $d \in \mathbb{N}$ according to Definition 3a).

Kolmogorov Complexity has been investigated for finite strings and, asymptotically, for infinite ones; cf. e.g. [LiVi97, SECTION 2.5] and [Stai99]. Also a kind of advice is part of that theory in form of *conditional* complexity [LiVi97, DEFINITION 2.1.2].

We quote from [LiVi97, EXERCISE 2.3.4abe] the following

Fact 7. *An infinite string $\bar\sigma = (\sigma_n)_{n\in\omega} \in \Sigma^\omega$ is computable (e.g. printed onto a one-way output tape by some so-called Type-2 or monotone machine; cf.* [Weih00,Schm02]*)*

a) *iff its initial segments $\bar\sigma_{1:n} := (\sigma_1, \ldots \sigma_n)$ have Kolmogorov complexity $\mathcal{O}(1)$ conditionally to n, i.e., iff $C(\bar\sigma_{1:n}|n)$ is bounded by some $c = c(\bar\sigma) \in \mathbb{N}$ independent of n.*
b) *Equivalently: the* uniform *complexity $C_{\mathrm{u}}(\bar\sigma_{1:n}) := C(\bar\sigma_{1:n}; n)$ in the sense of* [LiVi97, EXERCISE 2.3.3] *(that is the complexity of the function $\{1, \ldots, n\} \ni i \mapsto \sigma_i$ from* [LiVi97, EXERCISE 2.1.12] *but additionally relativized to the size n of the domain) is bounded by some c for infinitely many n.*

Definition 8. *a) For $\bar\sigma \in \Sigma^\omega$, write $C(\bar\sigma) := \sup_n C\big(\bar\sigma_{1:n}|n\big)$ and $C(\bar\sigma|\bar\tau) := \sup_n C\big(\bar\sigma_{1:n}|n, \bar\tau\big)$, where the Kolmogorov complexity conditional to an* in*finite string is defined literally as for a finite one* [LiVi97, DEFINITION 2.1.1].
b) *Similarly, let $C_{\mathrm{u}}(\bar\sigma|\bar\tau) := \sup_n C_{\mathrm{u}}\big(\bar\sigma_{1:n}|\bar\tau\big)$.*
c) *For a represented space (A, α) and $a \in A$, write $C(a) := \inf\{\, C(\bar\sigma) : \alpha(\bar\sigma) = a\}$ and $C_{\mathrm{u}}(a) := \inf\{\, C_{\mathrm{u}}(\bar\sigma) : \alpha(\bar\sigma) = a\}$.*

Note that we purposely do not consider some *normalized* form like $\mathrm{C}(\bar\sigma_{1:n}|n)\boldsymbol{/n}$ in order to establish the following

Proposition 9. *A function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ is computable with finite advice iff the Kolmogorov complexity $C_{\mathrm{u}}\big(F(\bar\sigma)|\bar\sigma\big)$ is bounded by some c independent of $\bar\sigma \in \mathrm{dom}(F)$.*

2 Complexity of Nonuniform Computability

Lemma 10. *a) Let $f : A \to B$ be d-wise continuous (computable) and $A' \subseteq A$. Then the restriction $f|_{A'}$ is again d-wise continuous (computable).*

b) Let $f : A \to B$ be d-wise continuous (computable) and $g : B \to C$ be k-wise continuous (computable). Then $g \circ f : A \to C$ is $d \cdot k$-wise continuous (computable).

c) If $f : A \to B$ is (α, β)–computable with d-wise advice and $\alpha' \preceq \alpha$ and $\beta \preceq \beta'$, then f is also (α', β')–computable with d-wise advice.

A minimum size partition Δ of $\mathrm{dom}(f)$ to make f computable on each $D \in \Delta$ need not be unique: Alternative to Example 1, we

Remark 11. *Given a ρ–name of $x \in \mathbb{R}$ and indicating whether $\lfloor x \rfloor \in \mathbb{Z}$ is even or odd suffices to compute $\lfloor x \rfloor$:*
Suppose $\lfloor x \rfloor = 2k \in 2\mathbb{Z}$ (the odd case proceeds analogously). Then $x \in [2k, 2k+1)$. Conversely, $x \in [2k-1, 2k+2)$, together with the promise $\lfloor x \rfloor \in 2\mathbb{Z}$, implies $\lfloor x \rfloor = 2k$. Hence, given $(q_n) \in \mathbb{Q}$ with $|x - q_n| \leq 2^{-n}$, $k := 2 \cdot \lfloor q_1/2 + \frac{1}{4} \rfloor$ (calculated in exact rational arithmetic) will yield the answer. □

2.1 Witness of *k*-wise Discontinuity

Recall that the partition Δ in Definition 3 need not satisfy any (e.g. topological regularity) conditions. The following notion turns out as useful in lower bounding the cardinality of such a partition:

Definition 12. *a) A d-dimensional flag $\mathcal{F}$ in a topological Hausdorff space X is a collection*

$$x, \quad (x_n)_n \ , \quad (x_{n,m})_{n,m} \ , \quad (x_{n,m,\ell})_{n,m,\ell} \ , \quad \ldots, \quad (x_{n_1,\ldots,n_d})_{n_1,\ldots,n_d}$$

of a point and of (multi-)sequences[†] in X such that, for each (possibly empty) multi-index $\bar{n} \in \mathbb{N}^k$ $(0 \leq k < d)$, it holds $x_{\bar{n}} = \lim_{m\to\infty} x_{\bar{n},m}$.

b) $\mathcal{F}$ is uniform if furthermore, again for each $\bar{n} \in \mathbb{N}^k$ $(0 \leq k < d)$ and for each $1 \leq \ell \leq d-k$, it holds $x_{\bar{n}} = \lim_{m\to\infty} x_{\bar{n},\underbrace{m,\ldots,m}_{\ell \text{ times}}}$.

c) For $f :\subseteq X \to Y$ and $x \in \mathrm{dom}(f)$ a witness of discontinuity of f at x is a sequence $x_n \in \mathrm{dom}(f)$ such that $\lim_{n\to\infty} f(x_n)$ exists but differs from $f(x)$.

d) For $f :\subseteq X \to Y$, a witness of d-wise discontinuity of f is a uniform d-dimensional flag $\mathcal{F}$ in $\mathrm{dom}(f)$ such that, for each $k = 0, 1, \ldots, d-1$ and for each $\bar{n} \in \mathbb{N}^k$ and for each $1 \leq \ell \leq d-k$, $\big(x_{\bar{n},\underbrace{m,\ldots,m}_{\ell \text{ times}}}\big)_m$ is a witness of discontinuity of f at $x_{\bar{n}}$.

[†] The generally more appropriate concept is that of a *Moore-Smith* sequence or *net*. However, being interested in second countable spaces, we may and shall restrict to ordinary sequences. Similarly, the Hausdorff condition is invoked for mere convenience.

Observe that, since d is finite, we may always (although not effectively) proceed from a flag to a uniform one by iteratively taking appropriate subsequences. In fact, sub(multi)sequences of d-flags and of witnesses of discontinuity are again d-flags and witnesses of discontinuity.

Lemma 13. *Let X, Y be Hausdorff, $f : X \to Y$ a function, and suppose there exists a witness of d-wise discontinuity of f. Then $\mathfrak{C}_t(f) > d$.*

2.2 First Example: Matrix Rank

Observe that for an $N \times M$-matrix A and $d := \min(N, M)$, $\mathrm{rank}(A)$ is an integer between 0 and d; and knowing this number makes rank trivially computable. Conversely, such $(d+1)$–fold information is necessary by Lemma 13 and

Example 14. *Consider the space $\mathbb{R}^{N\times M}$ of rectangular matrices and let $d := \min(N, M)$. For $i \in \{0, 1, \ldots, d\}$ write*

$$E_i \quad := \quad \sum_{j=1}^{i} \big((0, \cdots, 0, \underbrace{1}_{j\text{-}th}, 0, \cdots, \underbrace{0}_{n\text{-}th})^\dagger \otimes (0, \cdots, 0, \underbrace{1}_{j\text{-}th}, 0, \cdots, \underbrace{0}_{m\text{-}th})\big) \ .$$

$$X := 0, \qquad X_{n_1,\ldots,n_i} \quad := \quad E_1/n_1 \ + \ E_2/n_2 \ + \ \cdots \ + \ E_i/n_i$$

has $\lim_{m\to\infty} X_{n_1,\ldots,n_i,m,\ldots,m} = X_{n_1,\ldots,n_i}$, hence constitutes a uniform d-dimensional flag. Moreover, $\mathrm{rank}(E_i) = i = \mathrm{rank}(X_{n_1,\ldots,n_i}) \neq i + \ell = \mathrm{rank}(X_{n_1,\ldots,n_i,\underbrace{m,\ldots,m}_{\ell \text{ times}}})$ *shows it is a witness of d-wise discontinuity of* $\mathrm{rank} : \mathbb{R}^{N\times M} \to \{0, 1, \ldots, d\}$. □

3 Multivalued Functions, i.e. Relations

Many applications involve functions which are 'non-deterministic' in the sense that, for a given input argument x, several values y are acceptable as output; recall e.g. Items i) and ii) in Section 1. Also in linear algebra, given a singular matrix A, we want to find *some* (say normed) vector $\boldsymbol{v}$ such that $A \cdot \boldsymbol{v} = 0$. This is reflected by relaxing the mapping $f : x \to y$ to be not a function but a relation (also called multivalued function); writing $f : X \rightrightarrows Y$ instead of $f : X \to 2^Y \setminus \{\emptyset\}$ to indicate that for an input $x \in X$, *any* output $y \in f(x)$ is acceptable. Many practical problems have been shown computable as multivalued functions but admit no computable single-valued so-called *selection*; cf. e.g. [Weih00, Exercise 5.1.13], [ZiBr04, Lemma 12 or Proposition 17]. On the other hand, even relations often lack computability merely for reasons of continuity—and appropriate additional discrete advice renders them computable, recall Example 2.

3.1 Dis-/Continuity for Multivalued Mappings

Like single-valued computable functions (recall the Main Theorem), also computable relations satisfy certain topological conditions. However for such multivalued mappings, literature knows a variety of easily confusable notions [ScNe07].

Hemicontinuity for instance is not necessary for real computability. It may be tempting to regard *computing* a multivalued mapping f as the task of calculating, given x, the set-value $f(x)$ [Spre08]. In our example applications, however, one wants to capture that a machine is permitted, given x, to 'nondeterministically' choose and output *some* value $y \in f(x)$. Note that this coincides with [Weih00, DEFINITION 3.1.3]. In particular we do not insist that, upon input x, *all* $y \in f(x)$ occur as output for *some* nondeterministic choice—as required in [Brat03, SECTION 7]. Instead, let us generalize Definition 12 as follows:

Definition 15. *Fix some possibly multivalued mapping $f :\subseteq X \rightrightarrows Y$ and write $\mathrm{dom}(f) := \{x \in X : f(x) \neq \emptyset\}$. Call f continuous at $x \in X$ if there is some $y \in f(x)$ such that for every open neighbourhood V of y there exists a neighbourhood U of x such that $f(z) \cap V \neq \emptyset$ for all $z \in U$.*

For ordinary (i.e. single-valued) functions f, $\mathrm{dom}(f)$ amounts to the usual notion; and such f is obviously continuous (at x) iff it is continuous (at x) in the original sense. Indeed, Lemma 18a) below is an immediate extension of the Main Theorem of Recursive Analysis, showing that any computable *multi*valued mapping is necessarily continuous.

Lemma 10a) literally applies also to multivalued mappings $f : A \rightrightarrows B$. We failed to similarly fully generalize Lemma 10b); but already the following partial generalization turns out as useful:

Lemma 16. *a) Let $f : A \to B$ be single-valued and $g : B \rightrightarrows C$ multivalued. If f is d-wise continuous (computable) and g is k-wise continuous (computable), then $g \circ f : A \rightrightarrows C$ is $d \cdot k$-wise continuous (computable).*

b) Let $f : A \rightrightarrows B$ and $g : B \rightrightarrows C$ be multivalued. If f is d-wise continuous (computable) and g is continuous (computable), then $g \circ f : A \rightrightarrows C$ is again d-wise continuous (computable).

Definition 17. *a) For $x \in \mathrm{dom}(f)$, a witness of discontinuity of f at x is a sequence $(x_n) \in \mathrm{dom}(f)$ converging to x such that, for every $y \in f(x)$ there is some open neighbourhood V of y disjoint from $f(x_n)$ for infinitely many $n \in \mathbb{N}$.*

b) A uniform d-dimensional flag $\mathcal{F}$ in X is a witness of d-wise discontinuity of f if, for each $0 \leq k < d$ and for each $\bar{n} \in \mathbb{N}^k$ and for each $1 \leq \ell \leq d-k$ and for each $y \in f(x_{\bar{n}})$, $\big(x_{\bar{n},\underbrace{m,\ldots,m}_{\ell \text{ times}}}\big)_m$ is a witness of discontinuity of f at $x_{\bar{n}}$.

If multivalued f admits a witness of discontinuity at x, then f is not continuous. Conversely, if X is first-countable, discontinuity of f at x yields the existence of a witness of discontinuity at x. Also, witnesses of 1-wise discontinuity coincide with witnesses of discontinuity; and they generalize the definition from the single-valued case. Lemma 18 below extends Lemma 13 in showing that a witness of d-wise discontinuity of f inhibits d-wise computability.

Lemma 18. *Let (A,α) and (B,β) be effective metric spaces*[‡] *with corresponding Cauchy representations and $f :\subseteq A \rightrightarrows B$ a possibly multivalued mapping.*

a) If f admits a witness of discontinuity, then it is not (α,β)–continuous.
b) If f admits a witness of d-wise discontinuity, f is not d-wise (α,β)–continuous.

4 Applications to Effective Linear Algebra

Based on Lemma 13b), we now determine the complexity of non-uniform computability for several concrete standard problems in linear algebra and in particular of Example 2. But first consider the problem of solving a system of linear equations; more precisely of finding a nonzero vector in the kernel of a given singular matrix. It is for mere notational convenience that we formulate for the case of real matrices: complex ones work just as well.

Theorem 19. *Fix $n,m\in\mathbb{N}$, $d:=\min(n,m-1)$, and consider the space $\mathbb{R}^{n\times m}$ of $n\times m$ matrices, considered as linear mappings from $\mathbb{R}^m$ to $\mathbb{R}^n$. Then the multivalued mapping*

$$\mathrm{LinEq}: A \;\mapsto\; \mathrm{kernel}(A)\setminus\{0\}, \qquad \mathrm{dom}(\mathrm{LinEq}) \;:=\; \{A\in\mathbb{R}^{n\times m} : \mathrm{rank}(A)\le d\}$$

is well-defined and has complexity $\mathfrak{C}_{\mathrm{t}}(\mathrm{LinEq}) = \mathfrak{C}_{\mathrm{c}}(\mathrm{LinEq},\rho^{n\times m},\rho^m) = d+1$.

Concerning diagonalization of symmetric real matrices, we can prove

Theorem 20. *Fix $d\in\mathbb{N}$ and consider the space $\mathbb{R}^{\binom{d}{2}}$ of real symmetric $d\times d$ matrices. Then the multivalued mapping*

$$\mathrm{Diag}: \mathbb{R}^{\binom{d}{2}} \;\ni\; A \;\mapsto\; \big\{(\boldsymbol{w}_1,\ldots,\boldsymbol{w}_d) \text{ basis of } \mathbb{R}^d \text{ of eigenvectors to } A\big\}$$

has complexity $\mathfrak{C}_{\mathrm{t}}(\mathrm{Diag}) = \mathfrak{C}_{\mathrm{c}}\big(\mathrm{Diag},\rho^{\binom{d}{2}},\rho^{d\times d}\big) = d$.

The lack of continuity of the mapping Diag is closely related to inputs with degenerate eigenvalues [ZiBr04, Example 18]. In fact our below proof yields a witness of d-wise discontinuity by constructing an iterated sequence of *symmetry breakings* in the sense of Mathematical Physics. On the other hand even in the non-degenerate case, Diag is inherently multivalued since any permutation of a basis constitutes again a basis.

4.1 Finding Some Eigenvector

Instead of computing an entire basis of eigenvectors, we now turn to the problem of determining just one arbitrary eigenvector to a given real symmetric matrix. This turns out to be considerably less 'complex':

‡ Cf. [Weih00, Section 8.1] for a formal definition and imagine Euclidean spaces $\mathbb{R}^k$ as major examples and focus of interest for our purpose.

Theorem 21. *For a real symmetric $n \times n$-matrix A, consider the quantity*

$$m \quad := \quad \min\left\{ \dim \operatorname{kernel}(A - \lambda\,\mathrm{id}) : \lambda \in \sigma(A) \right\} \quad \in \quad \{1, \ldots, n\} \ .$$

Given $d := \lfloor \log_2 m \rfloor \in \{0, 1, \ldots, \lfloor \log_2 n \rfloor\}$ and a $\rho^{\binom{n}{2}}$–name of A, one can ρ^n–compute (i.e. effectively find) some eigenvector of A.

The proof employs the following tool about computability of finite multi-sets.

Lemma 22. *Let $(x_1, \ldots, x_n)$ denote an n-tuple of real numbers and consider the induced partition $\mathcal{I} := \big\{\{1 \le i \le n : x_i = x_j\} : 1 \le j \le n\big\}$ of the index set $\{1, \ldots, n\} =: [n]$ according to the equivalence relation $i \equiv j :\Leftrightarrow x_i = x_j$. Furthermore let $m := \min\left\{ \mathrm{Card}(I) : I \in \mathcal{I}\right\}$.*

a) Consider $I \subseteq [n]$ with $1 \le \mathrm{Card}(I) < 2m$. Then the following implies $I \in \mathcal{I}$:

$$x_i \neq x_j \quad \text{for all} \quad i \in I \quad \text{and all} \quad j \in [n] \setminus I \ . \tag{1}$$

b) Suppose $k \in \mathbb{N}$ is such that $k \le m < 2k$. Then there exists $I \in \mathcal{I}$ with $k \le \mathrm{Card}(I) < 2k$ satisfying (1). Conversely every $I \subseteq [n]$ with $k \le \mathrm{Card}(I) < 2k$ satisfying (1) has $I \in \mathcal{I}$.
c) Given a ρ^n–name of $(x_1, \ldots, x_n)$ and given $k \in \mathbb{N}$ with $k \le m < 2k$, one can computably find some $I \in \mathcal{I}$.
d) Given a ρ^n–name of $(x_1, \ldots, x_n)$ and given $\mathrm{Card}(\mathcal{I})$, one can compute $\mathcal{I}$.

Claim c) can be considered a weakening of Claim d) which had been established in [ZiBr04, Proposition 20].

Proof (Theorem 21). Compute according to [ZiBr04, Proposition 17] some (ρ^n–name of an) n-tuple of eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of A, repeated according to their multiplicities. Now due to [ZiBr04, Theorem 11], (some eigenvector in) the eigenspace $\operatorname{kernel}(A - \lambda_i\,\mathrm{id})$ can be computably found when knowing $\operatorname{rank}(A - \lambda_i\,\mathrm{id})$ (recall Theorem 19), that is the multiplicity of λ_i in the multi-set $(\lambda_1, \ldots, \lambda_n)$. To this end we apply Lemma 22c), observing $k := 2^d \le m < 2k$ since $d = \lfloor \log_2 m \rfloor$. □

Theorem 23. *The multivalued mapping*

$$\mathrm{EVec}_n : \mathbb{R}^{\binom{n}{2}} \quad \ni \quad A \quad \mapsto \quad \{\boldsymbol{w} \text{ eigenvector of } A\}$$

has complexity $\mathfrak{C}_{\mathrm{t}}(\mathrm{EVec}_n) = \mathfrak{C}_{\mathrm{c}}\big(\mathrm{EVec}_n, \rho^{\binom{n}{2}}, \rho^n\big) = \lfloor \log_2 n \rfloor + 1$.

5 Conclusion and Perspectives

We claim that a major source of criticism against Recursive Analysis misses the point: Although computable functions f *are* necessarily continuous when given approximations to the argument x *only*, most practical f's do become computable when providing in addition some discrete information about x. Such 'advice' usually consists of some very natural and mathematically explicit integer value from a bounded range (e.g. the rank of the matrix under consideration) and is readily available in practical applications.

We have then turned this observation into a complexity theory, investigating the minimum *size* (=cardinal) of the range this discrete information comes from. And we have devised mathematical tools and used them to determine this quantity for several simple and natural problems from linear algebra: calculating the rank of a given matrix, solving a system of linear equalities, diagonalizing a symmetric matrix, and finding some eigenvector to a given symmetric matrix. The latter three are inherently multivalued. And they exhibit a considerable difference in complexity: for input matrices of format $n \times n$, usually discrete advice of order $\Theta(n)$ is necessary and sufficient; whereas some single eigenvector can be found using only $\Theta(\log n)$–fold advice: namely the quantity $\lfloor \log_2 \min \{ \dim \operatorname{kernel}(A - \lambda \operatorname{id}) : \lambda \in \sigma(A) \} \rfloor$. The algorithm exploits this data based on some combinatorial considerations—which nicely complement the heavily analytical and topological arguments usually dominant in proofs in Recursive Analysis.

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