# Computable Separation in Topology, from $T_0$ to $T_3$

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**Abstract.** This article continues the study of computable elementary topology started in [7]. We introduce a number of computable versions of the topological $T_0$ to $T_3$ separation axioms and solve their logical relation completely. In particular, it turns out that computable $T_1$ is equivalent to computable $T_2$. The strongest axiom $SCT_3$ is used in [2] to construct a computable metric.

## 1 Preliminaries

We use the representation approach to computable analysis [6] as the basis for our investigation. In particular, we use the terminology and concepts introduced in [7] (which can be considered as a revision and extension of parts from [6]).

Let $\Sigma^*$ and $\Sigma^\omega$ be the sets of the finite and infinite sequences, respectively, of symbols from a finite alphabet $\Sigma$. A function mapping finite or infinite sequences of symbols from $\Sigma$ is computable, if it can be computed by a Type-2 machine, that is, a Turing machine with finite or infinite input and output tapes. On $\Sigma^*$ and $\Sigma^\omega$ we use canonical tupling functions $\langle \cdot \rangle$ that are computable and have computable inverses. Computability on finite or infinite sequences of symbols is transferred to other sets by representations, where elements of $\Sigma^*$ or $\Sigma^\omega$ are used as "concrete names" of abstract objects. For representations $\gamma_i :\subseteq Y_i \to M_i$ we consider the product representation defined by $[\gamma_1, \gamma_2]\langle p, q\rangle := (\gamma_1(p_1), \gamma_2(p_2))$. Let $Y = Y_1 \times \ldots \times Y_n$, $M = M_1 \times \ldots \times M_n$ and $\gamma : Y \to M$, $\gamma(y_1, \ldots, y_n) = \gamma_1(y_1) \times \ldots \times \gamma_n(y_n)$. A partial function $h :\subseteq Y \to Y_0$ realizes the multi-function $f : M \rightrightarrows M_0$ if $\gamma_0 \circ h(y) \in f(x)$ whenever $x = \gamma(y)$ and $f(x)$ exists. This means that $h(y)$ is a name of some $z \in f(x)$ if $y$ is a name of $x \in \mathrm{dom}(f)$. The function $f$ is $(\gamma, \gamma_0)$-computable, if it has a computable realization.

We will consider computable topological spaces as defined in [7]. Various similar definitions have been used, see, for example, [4, 3, 5] and the references in [7]. In particular, the definition in [6] is slightly different. A computable topological space is a 4-tuple $\mathbf{X} = (X, \tau, \beta, \nu)$ such that $(X, \tau)$ is a topological $T_0$-space, $\nu :\subseteq \Sigma^* \to \beta$ is a notation of a base $\beta$ of $\tau$, $\mathrm{dom}(\nu)$ is recursive and $\nu(u) \cap \nu(v) = \bigcup\{\nu(w) \mid (u, v, w) \in S\}$ for all $u, v \in \mathrm{dom}(\nu)$ for some r.e. set $S \subseteq (\mathrm{dom}(\nu))^3$.

For the points, the open sets and the closed sets we use the representations $\delta$, $\theta$ and $\psi^-$ that are defined as follows. For $p \in \Sigma^\omega$ and $x \in X$, $\delta(p) = x$ iff $p$ is

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a list of all $u \in \mathrm{dom}(\nu)$ such that $x \in \nu(u)$, $\theta(p)$ is the union of all $\nu(u)$ where $u$ is listed by $p$, and $\psi^-(p) := X \setminus \theta(p)$.

## 2 Axioms of Computable Separation

For a topological space $\mathbf{X} = (X, \tau)$ with set $\mathcal{A}$ of closed sets we consider the following classical separation properties:

**Definition 1 (separation axioms).**

$$\begin{aligned}
&\mathrm{T}_0 : (\forall x, y \in X,\ x \neq y)(\exists W \in \tau)((x \in W \wedge y \notin W) \vee (x \notin W \wedge y \in W))),\\
&\mathrm{T}_1 : (\forall x, y \in X,\ x \neq y)(\exists W \in \tau)(x \in W \wedge y \notin W),\\
&\mathrm{T}_2 : (\forall x, y \in X,\ x \neq y)(\exists U, V \in \tau)(U \cap V = \emptyset \wedge x \in U \wedge y \in V),\\
&\mathrm{T}_3 : (\forall x \in X, \forall A \in \mathcal{A}, x \notin A)(\exists U, V \in \tau)(U \cap V = \emptyset \wedge x \in U \wedge A \subseteq V),\\
&\mathrm{T}_4 : (\forall A, B \in \mathcal{A}, A \cap B = \emptyset)(\exists U, V \in \tau)(U \cap V = \emptyset \wedge A \subseteq U \wedge B \subseteq V).
\end{aligned}$$

*For $i = 0, 1, 2, 3$, we call $\mathbf{X} = (X, \tau)$ a $T_i$-space iff $\mathrm{T_i}$ is true.*

For the four axioms, $\mathrm{T}_2 \Longrightarrow \mathrm{T}_1 \Longrightarrow \mathrm{T}_0$ and $\mathrm{T}_0 + \mathrm{T}_3 \Longrightarrow \mathrm{T}_2$, where all the implications are proper [1]. $T_2$-spaces are called *Hausdorff spaces* and $T_3$-spaces are called *regular*. (Many authors, for example [1], call a space $T_3$-space or regular iff $\mathrm{T}_1 + \mathrm{T}_3$.) We mention that $(X, \tau)$ is a $T_1$-space, iff all sets $\{x\}$ $(x \in X)$ are closed [1]. For computable topological spaces $\mathbf{X} = (X, \tau, \beta, \nu)$, which are countably based $T_0$-spaces (also called *second countable*), $T_3 \Longrightarrow T_2$.

We introduce computable versions $\mathrm{CT_i}$ of the conditions $\mathrm{T_i}$ by requiring that the existing open neighborhoods can be computed. For the points we compute basic neighborhoods.

**Definition 2 (axioms of computable separation).** *For $i \in \{0, 1, 2, 3\}$ define conditions $\mathrm{CT_i}$ as follows.*

$\mathrm{CT}_0$ : *The multi-function $t_0$ is $(\delta, \delta, \nu)$-computable where $t_0$ maps each $(x, y) \in X^2$ such that $x \neq y$ to some $U \in \beta$ such that*

$$(x \in U \text{ and } y \notin U) \text{ or } (x \notin U \text{ and } y \in U). \tag{1}$$

$\mathrm{CT}_1$ : *The multi-function $t_1$ is $(\delta, \delta, \nu)$-computable, where $t_1$ maps each $(x, y) \in X^2$ such that $x \neq y$ to some $U \in \beta$ such that $x \in U$ and $y \notin U$.*

$\mathrm{CT}_2$ : *The multi-function $t_2$ is $(\delta, \delta, [\nu, \nu])$-computable, where $t_2$ maps each $(x, y) \in X^2$ such that $x \neq y$ to some $(U, V) \in \beta^2$ such that $U \cap V = \emptyset$, $x \in U$ and $y \in V$.*

$\mathrm{CT}_3$ : *The multi-function $t_3$ is $(\delta, \psi^-, [\nu, \theta])$-computable, where $t_3$ maps each $(x, A)$ such that $x \in X$, $A \subseteq X$ closed, and $x \notin A$ to some $(U, V) \in \beta \times \tau$ such that $U \cap V = \emptyset$, $x \in U$ and $A \subseteq V$.*

Obviously, $\mathrm{CT_i}$ implies $\mathrm{T_i}$. We introduce some further computable $T_i$-conditions.

**Definition 3 (further axioms of computable separation).**
$\mathrm{WCT}_0$ : *There is an r.e. set* $H \subseteq \mathrm{dom}(\nu) \times \mathrm{dom}(\nu)$ *such that*

$$(\forall x, y,\ x \neq y)(\exists (u,v) \in H)(x \in \nu(u)\ \wedge\ y \in \nu(v)) \quad \textit{and} \tag{2}$$

$$(\forall (u,v) \in H) \begin{cases} \quad \nu(u) \cap \nu(v) = \emptyset \\ \vee\ (\exists x)\, \nu(u) = \{x\} \subseteq \nu(v) \\ \vee\ (\exists y)\, \nu(v) = \{y\} \subseteq \nu(u)\,. \end{cases} \tag{3}$$

$\mathrm{SCT}_0$ : *The multi-function* $t_0^s$ *is* $(\delta, \delta, [\nu_{\mathbb{N}}, \nu])$*-computable where* $t_0^s$ *maps each* $(x,y) \in X^2$ *such that* $x \neq y$ *to some* $(k, U) \in \mathbb{N} \times \beta$ *such that* ($k = 1$, $x \in U$ *and* $y \notin U$) *or* ($k = 2$, $x \notin U$ *and* $y \in U$).

$\mathrm{CT}_0'$ : *There is an r.e. set* $H \subseteq \mathrm{dom}(\nu_{\mathbb{N}}) \times \mathrm{dom}(\nu) \times \mathrm{dom}(\nu)$ *such that*

$$(\forall x, y,\ x \neq y)(\exists (w,u,v) \in H)(x \in \nu(u)\ \wedge\ y \in \nu(v)) \quad \textit{and} \tag{4}$$

$$(\forall (w,u,v) \in H) \begin{cases} \quad \nu(u) \cap \nu(v) = \emptyset \\ \vee\ \nu_{\mathbb{N}}(w) = 1 \wedge (\exists x)\, \nu(u) = \{x\} \subseteq \nu(v) \\ \vee\ \nu_{\mathbb{N}}(w) = 2 \wedge (\exists y)\, \nu(v) = \{y\} \subseteq \nu(u)\,. \end{cases} \tag{5}$$

$\mathrm{CT}_1'$ : *There is an r.e. set* $H \in \Sigma^* \times \Sigma^*$ *such that*

$$(\forall x, y,\ x \neq y)(\exists (u,v) \in H)(x \in \nu(u) \wedge y \in \nu(v)) \quad \textit{and} \tag{6}$$

$$(\forall (u,v) \in H) \begin{cases} \quad \nu(u) \cap \nu(v) = \emptyset \\ \vee\ (\exists x)\, \nu(u) = \{x\} \subseteq \nu(v)\,. \end{cases} \tag{7}$$

$\mathrm{CT}_2'$ : *There is an r.e. set* $H \in \Sigma^* \times \Sigma^*$ *such that*

$$(\forall x, y,\ x \neq y)(\exists (u,v) \in H)(x \in \nu(u) \wedge y \in \nu(v)) \quad \textit{and} \tag{8}$$

$$(\forall (u,v) \in H) \begin{cases} \quad \nu(u) \cap \nu(v) = \emptyset \\ \vee\ (\exists x)\, \nu(u) = \{x\} = \nu(v)\,. \end{cases} \tag{9}$$

$\mathrm{SCT}_2$ : *There is an r.e. set* $H \in \Sigma^* \times \Sigma^*$ *such that*

$$(\forall x, y,\ x \neq y)(\exists (u,v) \in H)(x \in \nu(u) \wedge y \in \nu(v)) \quad \textit{and} \tag{10}$$

$$(\forall (u,v) \in H)\ \nu(u) \cap \nu(v) = \emptyset\,. \tag{11}$$

$\mathrm{CT}_3'$ : *The multi-function* $t_3'$ *is* $(\delta, \nu, [\nu, \psi^-])$*-computable where* $t_3'$ *maps each* $(x, W) \in X \times \beta$ *such that* $x \in W$ *to some* $(U, B)$ *such that* $U \in \beta$, $B \subseteq X$ *is closed and* $x \in U \subseteq B \subseteq W$.

$\mathrm{WCT}_3$ : *The multi-function* $t_3^w$ *is* $(\delta, \nu, \nu)$*-computable where* $t_3^w$ *maps each* $(x, W) \in X \times \beta$ *such that* $x \in W$ *to some* $U$ *such that* $U \in \beta$ *and* $x \in U \subseteq \overline{U} \subseteq W$.

$\mathrm{SCT}_3$: *There are an r.e. set* $R \subseteq \mathrm{dom}(\nu) \times \mathrm{dom}(\nu)$ *and a computable function* $r :\subseteq \Sigma^* \times \Sigma^* \to \Sigma^\omega$ *such that for all* $u, w \in \mathrm{dom}(\nu)$,

$$\nu(w) = \bigcup \{\nu(u) \mid (u,w) \in R\}\,, \tag{12}$$

$$(u,w) \in R \Longrightarrow\ \nu(u) \subseteq \psi^- \circ r(u,w) \subseteq \nu(w)\,. \tag{13}$$

$\mathrm{CT}_0'$, $\mathrm{CT}_1'$ and $\mathrm{CT}_2'$ are versions of $\mathrm{CT}_0$, $\mathrm{CT}_1$ and $\mathrm{CT}_2$, respectively, where base sets are used instead of points (see Theorem 1 below). Similarly, $SCT_3$ is a pointless version of $CT_3'$. In contrast to $\mathrm{CT}_0$, in $\mathrm{SCT}_0$ the separating function gives immediate information about the direction of the separation. Also in $\mathrm{CT}_0'$ some information about the direction of the separation is included while no such information is given in its weak version $\mathrm{WCT}_0$. The strong version $\mathrm{SCT}_2$ results from $\mathrm{CT}_2'$ by excluding the case $(\exists x)\,\nu(u) = \{x\} = \nu(v)$. Notice that $\mathrm{SCT}_2$ results also from $\mathrm{WCT}_0$, $\mathrm{CT}_0'$ and $\mathrm{CT}_1'$ by excluding the corresponding cases. The following examples illustrate the definitions. Further examples can be found in Section 4.

*Example 1.* 1. Consider the computable real line $\mathbf{R} := (\mathbb{R}, \tau_{\mathbb{R}}, \beta, \nu)$ such that $\tau_{\mathbb{R}}$ is the real line topology and $\nu$ is a canonical notation of the set of all open intervals with rational endpoints. $\mathbf{R}$ is $SCT_3$ (easy proof).

2. ($T_0$ but not $WCT_0$) Consider the computable lower real line $\mathbf{R}_< := (\mathbb{R}, \tau_<, \beta_<, \nu_<)$, defined by $\nu_<(w) := (\nu_{\mathbb{Q}}; \infty)$, which is $T_0$ but not $T_1$. Suppose $\mathbf{R}_<$ is $WCT_0$. Since for any two base elements $U, V$, $U$ is not a singleton and $U \cap V \neq \emptyset$, $H = \emptyset$ by (3). But $H \neq \emptyset$ by (2).
3. ($T_1$ but not $T_2$ or $WCT_0$) Let $\mathbf{X} = (\mathbb{N}, \tau, \beta, \nu)$ such that $\tau = \beta$ is the set of cofinite subsets of $\mathbb{N}$ and $\nu$ is a canonical notation of $\nu$. Then $\mathbf{X}$ is a computable topological space. It is $T_1$ since singletons $\{x\}$ are closed. Suppose $\mathbf{X}$ is $WCT_0$. Since the intersection of base elements cannot be empty and singletons are not open the set $H$ in (3) must be empty. But then (2) cannot be true. The space is not $T_2$ since the intersection of any two non-empty open set is not empty.

By the next lemma the above computable separation axioms are robust, that is, they do not depend on the notation $\nu$ of the base explicitly but only on the computability concept on the points induced by it. Call the computable topological spaces $\mathbf{X} = (X, \tau, \beta, \nu)$ and $\widetilde{\mathbf{X}} = (X, \tau, \widetilde{\beta}, \widetilde{\nu})$ equivalent, iff $\delta \equiv \widetilde{\delta}$ [7, Definition 21 and Theorem 22].

**Lemma 1.** *1. For $i \in \{0,1,2,3\}$ let $\overline{\mathrm{CT}_\mathrm{i}}$ be the condition obtained from $\mathrm{CT}_\mathrm{i}$ and let $\overline{\mathrm{SCT}_0}$ be the condition obtained from $SCT_0$ by replacing $\beta$ and $\nu$ by $\tau$ and $\theta$, respectively. Then $\overline{\mathrm{CT}_\mathrm{i}} \iff \mathrm{CT}_\mathrm{i}$ and $\overline{\mathrm{SCT}_0} \iff \mathrm{SCT}_0$.*

*2. Let $\widetilde{\mathbf{X}} = (X, \tau, \widetilde{\beta}, \widetilde{\nu})$ be a computable topological space equivalent to $\mathbf{X} = (X, \tau, \beta, \nu)$. Then each separation axiom from Definitions 2 and 3 for $\mathbf{X}$ is equivalent to the corresponding axiom for $\widetilde{\mathbf{X}}$.*

The proofs are straightforward. In particular, apply [7, Theorem 22] by which "equivalence" is equivalent to ($\nu \leq \widetilde{\theta}$ and $\widetilde{\nu} \leq \theta$).

## 3 Implications

In this section we prove the implications between the separation properties, in the next section we give counterexamples for the proper ones.

**Theorem 1.**

1. $\mathrm{SCT}_3 \Longrightarrow \mathrm{CT}_3 \Longrightarrow \mathrm{SCT}_2 \Longrightarrow \mathrm{CT}_2 \Longrightarrow \mathrm{CT}_0 \Longrightarrow \mathrm{WCT}_0$,
2. $CT_3 \iff CT_3' \Longrightarrow \mathrm{WCT}_3$,
3. $CT_2 \iff CT_2' \iff CT_1 \iff CT_1'$,
4. $CT_0 \iff SCT_0 \iff CT_0'$,

The proofs of $SCT_0 \Longrightarrow CT_0'$ and $\mathrm{CT}_3' \Longrightarrow \mathrm{SCT}_2$ need some care. They are based on the observation that a realizing machine needs only finitely many steps for finding an appropriate base element for the result. We omit the details (approximately 2 pages).

Surprisingly, computable $T_1$-spaces are exactly computable $T_2$. We add some further interesting results. Let "D" be the axiom stating that the topological space is discrete.

**Theorem 2.** *For computable topological spaces,*

1. *if* $\{x\}$ *is not open for all* $x \in X$ *then* $\mathrm{WCT}_0 \Longrightarrow \mathrm{SCT}_2$,
2. $\mathrm{SCT}_2$ *if* $\mathrm{T}_2$ *and* $\{(u,v) \mid \nu(u) \cap \nu(v) = \emptyset\}$ *is r.e.,*
3. $\mathrm{SCT}_2 \iff (x \neq y \;\; \text{is } (\delta,\delta)\text{-r.e.})$,
4. $\mathrm{CT}_3 \Longrightarrow \mathrm{SCT}_3$ *if the set* $\{w \in \Sigma^* \mid \nu(w) \neq \emptyset\}$ *is r.e.*
5. $\mathrm{D} \Longrightarrow \mathrm{WCT}_3$

We include only the proof of 4. For the terminology see [7].

**Proof:** Since finite intersection is computable, there is a computable function $g$ such that $\bigcap \nu^{\mathrm{fs}}(w) = \theta \circ g(w)$. Therefore, the set $\{w \in \Sigma^* \mid \bigcap \nu^{\mathrm{fs}}(w) \neq \emptyset\}$ is r.e. There is a machine $M$ such that $f_M$ realizes the multi-function $t_3'$. If $x = \delta(p) \in \nu(w)$ then for some $u_1 \in \mathrm{dom}(\nu)$ and $q \in \mathrm{dom}(\psi^-)$, $f_M(p,w) = \langle u_1, q\rangle = \iota(u_1)q$ such that

$$x \in \nu(u_1) \subseteq \psi^-(q) \subseteq \nu(w)\,. \tag{14}$$

For computing $\iota(u_1)$ some prefix $u_0 \in \mathrm{dom}(\nu^{\mathrm{fs}}) \cap \Sigma^*11$ of $p$ suffices. Since $\delta(p) \in \nu(w)$ we may assume $w \ll u_0$. Since $x \in \delta[u_0 11 \Sigma^\omega] = \bigcap \nu^{\mathrm{fs}}(u_0)$, $\bigcap \nu^{\mathrm{fs}}(u_0) \neq \emptyset$. We will compute $\bigcap \nu^{\mathrm{fs}}(u_0) \cap \nu(u_1)$ as a union $\bigcup\{\nu(u) \mid u \in L\}$ of base sets and add all these $(u,w)$ to $R$.

There is a machine $N$ that works on input $(u,w)$ as follows:
(S1) If $u, w \in \mathrm{dom}(\nu)$, $\nu(u) \neq \emptyset$ and $\nu(w) \neq \emptyset$ then
(S2) $N$ searches for words $u_0 \in \mathrm{dom}(\nu^{\mathrm{fs}}) \cap \Sigma^*11$ and $u_1 \in \mathrm{dom}(\nu)$ such that $w \ll u_0$, $M$ on input $(u_0 1^\omega, w)$ writes $\iota(u_1)$ in at most $|u_0|$ steps and $u \ll g(u_0 \iota(u_1))$,
(S3) and then writes all words $\iota(v)$ for which there are words $u_2, u_3$ such that $u_0 u_2 \in \mathrm{dom}(\nu^{\mathrm{fs}})$, $\bigcap \nu^{\mathrm{fs}}(u_0 u_2) \neq \emptyset$, the machine $M$ on input $(u_0 u_2 1^\omega, w)$ writes $\iota(u_1) u_3$ in at most $|u_0 u_2|$ steps and $v \ll 11 u_3$. (In order to guarantee an infinite output, $N$ writes 11 from time to time.)
(S4) If (1) is false or the search in (2) is not successful then $N$ computes forever without writing. Let $r := f_N$ and $R := \mathrm{dom}(f_N)$. Then $R \subseteq \mathrm{dom}(\nu) \times \mathrm{dom}(\nu)$ and $R$ is r.e. We must prove correctness.

We show (12): Suppose $x = \delta(p) \in \nu(w)$. Then for some $u_1, q$, $f_M(p,w) = \iota(u_1)q$, hence for some prefix $u_0 \sqsubseteq p$ such that $w \ll u_0$ and $u_0 \in \Sigma^*11$ (since we my assume that $p$ has the subword 11 infinitely often), $M$ on input $(u_0 1^\omega, w)$ writes $\iota(u_1)$ in at most $|u_0|$ steps. Since $x \in \bigcap \nu^{\rm fs}(u_0)$ and $x \in \nu(u_1)$ by (14), $x \in \theta \circ g(u_0\iota(u_1))$, hence $x \in \nu(u)$ for some $u \ll g(u_0\iota(u_1))$. Therefore, there is some $u$ such that $x \in \nu(u)$ and the machine $N$ on input $(u,w)$ will find some words such that (S2) is true. Therefore $x \in \nu(u)$ for some $(u,w) \in R$, hence "$\supseteq$" is true in (12).

On the other hand, suppose $(u,w) \in R$ and $x \in \nu(u)$ for some $x$. Then on input $(u,w)$ the machine $N$ finds words $u_0, u_1$ such that the conditions in (S2) above are true. Since $u \ll g(u_0\iota(u_1))$ and $w \ll u_0$, $x \in \nu(u) \subseteq \bigcap \nu^{\rm fs}(u_0) \subseteq \nu(w)$. Therefore, "$\subseteq$" is true in (12).

For showing (13) suppose $(u,w) \in R$ and $x \in \nu(u)$ for some $x$ again. Then on input $(u,w)$ the machine $N$ finds words $u_0, u_1$ such that the conditions in (S2) above are true. Since $x \in \bigcap \nu^{\rm fs}(u_0)$, $x = \delta(u_0 p')$ for some $p' \in \Sigma^\omega$. Since $x \in \nu(w)$, $f_M(u_0 p', w) = \langle u_1, q\rangle = \iota(u_1)q$ for some $q \in \Sigma^\omega$ such that (14). Suppose $v \ll q$. Then for some $u_2, u_3$ such that $u_0 u_2 \in {\rm dom}(\bigcap \nu^{\rm fs})$, the machine $M$ on input $(u_0 u_2 1^\omega, w)$ writes $\iota(u_1)u_3$ in at most $|u_0 u_2|$ steps and $v \ll \iota(u_1)u_3$, therefore, $v \ll r(u,w)$. By (14),

$$\nu(w)^c \subseteq \theta(q) = \bigcup\{\nu(v) \mid v \ll q\} \subseteq \bigcup\{\nu(v) \mid v \in r(u,w)\} = \theta \circ r(u,w).$$

This proves $\psi^- \circ r(u,w) \subseteq \nu(w)$ in(13).

Finally let $v$ be some word such that $\iota(v)$ is listed by the machine $N$ on input $(u,w)$, that is, $v \ll r(u,w)$. Then there are words $u_2, u_3$ such that $\bigcap \nu^{\rm fs}(u_0 u_2) \neq \emptyset$, the machine $M$ on input $(u_0 u_2 1^\omega, w)$ writes $\iota(u_1)u_3$ in at most $|u_0 u_2|$ steps and $v \ll 11u_3$. Since $\bigcap \nu^{\rm fs}(u_0 u_2) \neq \emptyset$ and $w \ll u_0$, there is some $p'$ such that $\delta(u_0 u_2 p') \in \nu(w)$ and $f_M(u_0 u_2 p', w) = \iota(u_1)u_3 q'$ for some $q'$. By (14) $\nu(u_1) \cap \theta(u_3 q') = \emptyset$. Since $\nu(u) \subseteq \nu(u_1)$ (by $u \ll g(u_0\iota(u_1))$ in (S2)) and $\nu(v) \subseteq \theta(u_3 q')$ (since $v \ll u_3$), $\nu(u) \cap \nu(v) = \emptyset$.

Since this is true for all $v \ll r(u,w)$, $\nu(u) \cap \theta \circ r(u,w) = \emptyset$, hence $\nu(u) \subseteq \psi^- \circ r(u,w)$.

Therefore, we have also proved (13). □

## 4 Counterexamples

A topological space is discrete iff every singleton $\{x\}$ is open iff every subset $B \subseteq X$ is open. A discrete space is $T_i$ for $i = 0, \ldots, 4$. Let "D" be the axiom stating that the topological space is discrete. Counterexamples show that the implications in Theorem 1.1 are proper. Since this is an extended abstract we include only two of them.

**Theorem 3.** *For computable topological spaces,*

$$\begin{array}{ll}
\mathrm{T}_0 \not\Longrightarrow \mathrm{WCT}_0 & \textit{by Example 1.2;}\\
\mathrm{T}_1 \not\Longrightarrow \mathrm{WCT}_0 & \textit{by Example 1.3;}\\
\mathrm{D} \not\Longrightarrow \mathrm{WCT}_0 & \textit{by Example 2;}\\
\mathrm{D}+\mathrm{WCT}_0 \not\Longrightarrow \mathrm{CT}_0 & \textit{by Example 3;}\\
\mathrm{D}+\mathrm{CT}_0 \not\Longrightarrow \mathrm{CT}_1 & \textit{by Example 4;}\\
\mathrm{D}+\mathrm{CT}_2 \not\Longrightarrow \mathrm{SCT}_2 & \textit{by Example 5;}\\
\mathrm{WCT}_3+\mathrm{CT}_2 \not\Longrightarrow \mathrm{SCT}_2 & \textit{by Example 5;}\\
\mathrm{T}_4+\mathrm{SCT}_2 \not\Longrightarrow \mathrm{WCT}_3 & \textit{by Example 7;}\\
\mathrm{SCT}_2 \not\Longrightarrow \mathrm{T}_3 & \textit{by Example 6;}\\
\mathrm{CT}_3 \not\Longrightarrow \mathrm{SCT}_3 & \textit{by Example 8.}
\end{array}$$

In the following examples let $(a_i)_{i\in\mathbb{N}}$, $(b_i)_{i\in\mathbb{N}}$, ..., $(e_i)_{i\in\mathbb{N}}$ be injective families with pairwise disjoint ranges and let $\{0,1,\ldots,7\}\subseteq\Sigma$.

*Example 2.* ($D$ but not $WCT_0$) Omitted. □

*Example 3.* ($D + WCT_0$ but not $CT_0$) Let $A\subseteq\mathbb{N}$ be some non-r.e. set. Let $X := \{a_i, b_i \mid i\in\mathbb{N}\}$ and let $\tau$ be the discrete topology on $X$. Below we will define sets $B, C, D\subseteq\mathbb{N}$ such that $\{A, B, C, D\}$ is a partition of $\mathbb{N}$. Define a notation $\nu$ of a basis $\beta$ of the topology as follows.

| | $0^i1$ | $0^i2$ | $0^i3$ | $0^i12$ | $0^i13$ | $0^i23$ |
|---|---|---|---|---|---|---|
| $i\in A\cup D$ | $\{a_i\}$ | $\{b_i\}$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $i\in B$ | $\{a_i\}$ | $\{a_i,b_i\}$ | $\{b_i\}$ | $\{a_i\}$ | $\emptyset$ | $\{b_i\}$ |
| $i\in C$ | $\{a_i,b_i\}$ | $\{b_i\}$ | $\{a_i\}$ | $\{b_i\}$ | $\{a_i\}$ | $\emptyset$ |

Since $\nu(0^ik)\cap\nu(0^im) = \nu(0^ikm)$, $\nu(u)\cap\nu(v) = \nu\circ g(u,v)$ for some computable function $g$. Therefore $\mathbf{X} := (X,\tau,\beta,\nu)$ is a computable topological space. Let $H := \{(0^ik, 0^jl) \mid i,j\in\mathbb{N};\ k,l\in\{1,2\};\ (i\neq j\vee k\neq l\}$. Then H satisfies (2) and (3) for the space $\mathbf{X}$. Therefore, $\mathbf{X}$ is a WCT$_0$-space.

We show that $\mathbf{X}$ is not $SCT_0$.

Let $l, r\in\Sigma^*$ such that $\nu_{\mathbb{N}}(l) = 1$ and $\nu_{\mathbb{N}}(r) = 2$. We assume w.l.o.g. that $\nu_{\mathbb{N}}$ is injective. For $i\in\mathbb{N}$ let

$$S_i := \{\langle l, 0^i1\rangle,\ \langle r, 0^i3\rangle,\ \langle l, 0^i12\rangle,\ \langle r, 0^i23\rangle\},$$
$$T_i := \{\langle r, 0^i2\rangle,\ \langle l, 0^i3\rangle,\ \langle r, 0^i12\rangle,\ \langle l, 0^i13\rangle\}.$$

Suppose, the function $f :\subseteq \Sigma^\omega\times\Sigma^\omega\to\Sigma^*$ realizes the separation function $t_0^s$ for $\mathbf{X}$. If $\delta(p) = a_i$ and $\delta(q) = b_i$ then

$$f(p,q)\in\begin{cases} S_i \text{ if } & i\in B\\ T_i \text{ if } & i\in C\end{cases} \tag{15}$$

since $\nu(u)$ must be either $\{a_i\}$ or $\{b_i\}$ if $f(p,q) = \langle w, u\rangle$. Notice that $S_i\cap T_i = \emptyset$.

For all $i \in \mathbb{N}$ define $p_i, q_i \in \Sigma^\omega$ by $p_i := \iota(0^i1)\iota(0^i1)\iota(0^i1)\ldots$ and $q_i := \iota(0^i2)\iota(0^i2)\iota(0^i2)\ldots$. Let $F$ be the set of all computable functions $f :\subseteq \Sigma^\omega \times \Sigma^\omega \to \Sigma^*$ such that $f(p_i, q_i)$ exists for all $i \in A$. Consider $f \in F$. Then $f' : i \mapsto f(p_i, q_i)$ is computable such that $A \subseteq \mathrm{dom}(f')$. Since $A$ is not r.e. and $\mathrm{dom}(f')$ is r.e., $\mathrm{dom}(f') \setminus A$ is infinite. Since $F$ is countable, there is a bijective function $g : E \to F$ for some $E \subseteq \mathbb{N}$ such that $i \in \mathrm{dom}(g_i') \setminus A$ for all $i \in E$ ($g_i := g(i)$). Then $A \cap E = \emptyset$.

For each $i \in E$ we put $i$ to $B$ or $C$ in such a way that $g_i$ does not realize the separating function $t_0^s$ for $\mathrm{SCT}_0$.

$$B := \{i \in E \mid g_i(p_i, q_i) \notin S_i\},$$
$$C := \{i \in E \mid g_i(p_i, q_i) \in S_i\},$$

and $D := \mathbb{N} \setminus (A \cup B \cup C)$. Since $A \cap E = \emptyset$, $E = B \cup C$ and $B \cap C = \emptyset$, $\{A, B, C, D\}$ is a partition of $\mathbb{N}$.

Suppose some computable function $f$ realizes $t_0^s$. Since for $i \in A, \delta(p_i) = a_i$ and $\delta(q_i) = b_i$, $f(p_i, q_i)$ exists for all $i \in A$, hence $f = g_i$ for some $i \in E$.

If $i \in B$ then $g_i(p_i, q_i) \notin S_i$, hence by (15) the function $g_i$ does not realize $t_0^s$. If $i \in C$ then $g_i(p_i, q_i) \in S_i$, hence not in $T_i$ since $S_i \cap T_i = \emptyset$. By (15) the function $g_i$ does not realize $t_0^s$.

From this contradiction we conclude that $\mathbf{X}$ is not $SCT_0$. By Theorem 1 $\mathbf{X}$ is not $CT_0$. □

*Example 4.* ($D$ and $CT_0$ but not $CT_1$) Omitted. □

*Example 5.* ($D$ and $CT_2$ but not $SCT_2$) Let $A \subseteq \mathbb{N}$ be an r.e. set with non-r.e. complement. Define a notation $\nu$ by
$\nu(0^i1) := \{a_i\}, \nu(0^i2) := \{a_i\}$ for $i \in A$,
$\nu(0^i1) := \{a_i\}, \nu(0^i2) := \{b_i\}$ for $i \notin A$
for all $i \in \mathbb{N}$. Then $\nu$ is a notation of a base $\beta$ of a topology (the discrete topology) $\tau$ on a subset $X \subseteq \mathbb{N}$ such that $\mathbf{X} = (X, \tau, \beta, \nu)$ is a computable topological space.

The space $\mathbf{X}$ is $T_i$ for $i = 0, \ldots, 4$ since it is discrete. It is $CT_2$ but not $SCT_2$: The set $H := \{(0^ik, 0^jl) \mid i, j \in \mathbb{N},\ k, l \in \{1, 2\}\}$ satisfies $\mathrm{CT}_2'$. Therefore, the space is $CT_2$. Suppose $\mathrm{SCT}_2$. Let $H$ be the r.e. set for $\mathrm{SCT}_2$. By (10), $i \notin A \Longrightarrow (0^i1, 0^i2) \in H$ and by (11), $i \in A \Longrightarrow (0^i1, 0^i2) \notin H$. Since $H$ is r.e., the complement of $A$ must be r.e. (contradiction). Notice that $x \neq y$ is not $(\delta, \delta)$-r.e., see Theorem 2.3. It can be shown easily that $\mathbf{X}$ is $WCT_3$. □

*Example 6.* ($SCT_2$ but not $T_3$) Omitted. □

*Example 7.* ($T_4$ and $SCT_2$ but not $WCT_3$) Omitted. □

*Example 8.* ($CT_3$ but not $SCT_3$) Define a notation $I$ of the open rational intervals by $I\langle u, v\rangle := (\nu_{\mathbb{Q}}(u); \nu_{\mathbb{Q}}(v)) \subseteq \mathbb{R}$. Let $\mathbb{R}_c \subseteq \mathbb{R}$ be the set of ($\rho$-) computable real numbers. There is a computable function $g : \Sigma^* \to \Sigma^*$ such that $\mathbb{R}_c \subseteq \bigcup_{i\in\mathbb{N}} I \circ g(0^i)$ and $\sum_{i\in\mathbb{N}} \mathrm{length}(I \circ g(0^i)) < 1$ [6, Theorem 4.2.8]. Let $z := \inf\{a \in \mathbb{Q} \mid [a; 1] \subseteq \bigcup_{i\in\mathbb{N}} I \circ g(0^i)\}$. Then $0 < z < 1$, $z$ is $\rho_>$-computable and not $\rho$-computable, hence not $\rho_<$-computable [6]. Furthermore for all $k$, $z \notin I \circ g(0^k)$.

Let $X := \mathbb{R}_c \cup \{z\}$. Define a notation $\nu$ of subsets of $X$ by $\nu(0v) := I(v) \cap X$ and $\nu(1v) := I(v) \cap (-\infty; z) \cap X$ $(v \in \mathrm{dom}(I))$.Then $\beta := \mathrm{range}(\nu)$ is a base of a topology $\tau$ such that $\mathbf{X} := (X, \tau, \beta, \nu)$ is a computable topological space. Notice that for $x < z$, $z \in \mathrm{cls}_X((x; z) \cap X)$. Let $\delta$ be the inner representation for the points of $\mathbf{X}$.

Proposition 1: The multi-function $h : x \rightrightarrows a$ mapping each $x \in X$ such that $x < z$ to some $a \in \mathbb{Q}$ such that $x < a < z$ is $(\delta, \nu_{\mathbb{Q}})$-computable.

Proof 1: If $x < z$ and $x \in I \circ g(0^k)$, then $\sup I \circ g(0^k) < z$, since $z \not\leq \inf I \circ g(0^k)$ (since $x < z$), $z \notin I \circ g(0^k)$ and $z \neq \sup I \circ g(0^k)$ (since $z \notin \mathbb{Q}$). There is a machine $M$ that on input $p$ searches for some $k \in \mathbb{N}$ such that $0g(0^k) \ll p$ and writes some $u$ such that $\nu_{\mathbb{Q}}(u) = \sup I \circ g(0^k)$. Let $\delta(p) = x < z$. Since $x \in \mathbb{R}_c$, there is some $k$ such that $x \in I \circ g(0^k)$, hence $0g(0^k) \ll p$. We obtain $\nu_{\mathbb{Q}} \circ f_M(p) < z$. Therefore, the multi-function $h$ is $(\delta, \nu_{\mathbb{Q}})$-computable.

Proposition 2: The multi-function $f : (x, U) \rightrightarrows V$ mapping each $(x, (a; b)) \in X \times \mathrm{range}(I)$ such that $x \in (a; b)$ to some $(c; d) \in \mathrm{range}(I)$ such that $x \in (c; d) \subseteq [c; d] \subseteq (a; b)$ is $(\delta, I, I)$-computable.

Proof 2: Every $\delta$-name of $x$ lists arbitrarily short rational intervals containing $x$. Search for a sufficiently short interval $(c; d)$.

We show that $t_3'$ from Definition 3 is computable. Suppose $x \in W \in \beta$. If $W = \nu(0w) = I(w) \cap X$ for some $w$ then $W' := I(w)$. If $W = \nu(1w) = I(w) \cap (-\infty; z) \cap X$ for some $w$ then by means of $h$ find some $e \in \mathbb{Q}$ such that $x < e < z$ and let $W' := I(w) \cap (-\infty; e)$. Then $x \in W' \cap X \subseteq W$. By means of $f$ from $x$ and $(a; b) := W'$ find $(c; d) \in \mathrm{range}(I)$ such that $x \in (c; d) \subseteq [c; d] \subseteq (a; b)$. Then $x \in (c; d) \cap X \subseteq [c; d] \cap x \subseteq W$.

From $a, b, c$ and $d$ some $u$ and $q$ can be computed such that $\nu(u) = (c; d) \cap X$ and $\psi^-(q) = [c; d] \cap X$. Then $x \in \nu(u) \subseteq \psi^-(q) \subseteq W$. Therefore, $t_3'$ is $(\delta, \nu, [\nu, \psi^-])$-computable.

Suppose, $\mathbf{X}$ is $SCT_3$. Let $R$ be the r.e. set for $\mathrm{SCT}_3$ from Definition 3. There is some $w$ such that $\nu(w) = (0; z) \cap X$. Suppose $(u, w) \in R$. Then $\nu(u) \subseteq \nu(w)$, hence for some $a, b \in \mathbb{Q}$ such that $a < b < z$, $\nu(u) = (a; b) \cap X$ or $\nu(u) = (a; z) \cap X$. If $\nu(u) = (a; z) \cap X$, then $z \in \mathrm{cls}_X(\nu(u))$, but $\mathrm{cls}_X(\nu(u)) \subseteq \nu(w) = (0; z)$ by $\mathrm{SCT}_3$, hence $z \in \nu(w) = (0; z)$ (contradiction). Therefore, $\sup \nu(u) = (a; b)$ for some rational numbers $a, b$ such that $a < b < z$.

The function $U \mapsto \sup U$ for all $U = (a; x) \in \beta$ such that $x < z$ is $(\nu, \nu_{\mathbb{Q}})$-computable. Since $R$ is r.e., the number $y := \sup\{\sup \nu(u) \mid (u, w) \in R\}$ is $\rho_<$-computable such that $y \leq z$. Since $(0; z) = \nu(w) = \bigcup_{(u,w) \in R} \nu(u)$, for every $x < z$ there is some $(u, w) \in R$ such that $x < \sup \nu(u)$. Therefore, $y = z$, hence $z$ is $\rho_<$-computable. Contradiction. Therefore, $\mathbf{X}$ is not $SCT_3$. Notice that $U \neq \emptyset$ is not $\nu$-r.e. □

Further results can be obtained in combination with the positive results from Theorem 1. Figure 1 visualizes the interplay between the computable versions of $T_i$ for $i = 0, 1, 2, 3$ from Definitions 2 and 3 we have proved. "$A \longrightarrow B$" means $A \Longrightarrow B$, "$A \not\longrightarrow B$" means that we have constructed a computable topological space for which $A \wedge \neg B$, and $A \overset{C}{\not\longrightarrow} B$" means that we have constructed a computable topological space for which $(A \wedge C) \wedge \neg B$. Remember that $\mathrm{SCT}_0 \iff \mathrm{CT}_0 \iff \mathrm{CT}'_0$, $\mathrm{CT}_1 \iff \mathrm{CT}'_1 \iff \mathrm{CT}_2 \iff \mathrm{CT}'_2$ and $\mathrm{CT}_3 \iff \mathrm{CT}'_3$.

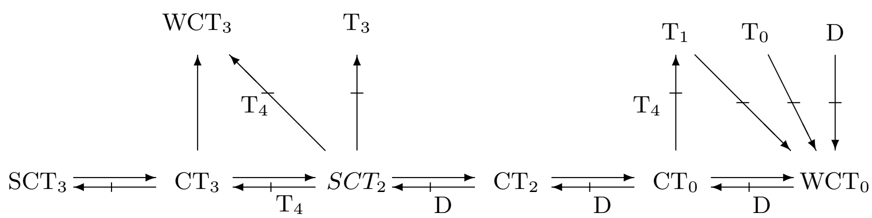


**Fig. 1.** The relation between computable $T_0$-, $T_1$-, $T_2$- and $T_3$-separation.

## 5 Further Results

For a computable topological space $\mathbf{X} = (X, \tau, \beta, \nu)$ and $B \subseteq X$ the subspace $\mathbf{X}_B = (B, \tau_B, \beta_B, \nu_B)$ of $\mathbf{X}$ to $B$ is the computable topological space defined by $\mathrm{dom}(\nu_B) := \mathrm{dom}(\nu)$, $\nu_B(w) := \nu(w) \cap B$. The separation axioms from Definitions 2 and 3 are invariant under restriction to subspaces.

**Theorem 4.** *If a computable topological space satisfies some separation axiom from Definitions 2 and 3 then each subspace satisfies this axiom.*

**Proof:** Straightforward. □

The product of two $T_i$-spaces is a $T_i$-space for $i = 0, 1, 2, 3$. This is no longer true for some of the computable separation axioms. By definition for the product $\mathbf{X}_1 \times \mathbf{X}_2 = \overline{\mathbf{X}} = (X_1 \times X_2, \overline{\tau}, \overline{\beta}, \overline{\nu})$ of two computable topological spaces $\mathbf{X}_1 = (X_1, \tau_1, \beta_1, \nu_1)$ and $\mathbf{X}_2 = (X_2, \tau_2, \beta_2, \nu_2)$, $\overline{\nu}\langle u_1, u_2 \rangle = \nu_1(u_1) \times \nu_2(u_2)$.

*Example 9.* The space $\mathbf{X}$ from Example 5 is $CT_2$ but not $SCT_2$. Let $\mathbf{R}$ be the computable real line from Example 1.1. We show that the product $\mathbf{X} \times \mathbf{R}$ is not $WCT_0$. Suppose, $\mathbf{X} \times \mathbf{R}$ is $WCT_0$. Since every base element of $\mathbf{X} \times \mathbf{R}$ has the form $\nu(u) \times (a; b)$ ($a, b \in \mathbb{Q}$, $a < b$) no singleton $\{(x, y)\}$ ($x \in X$, $y \in \mathbb{R}$) is open. By Theorem 2.1, $\mathbf{X} \times \mathbf{R}$ is $SCT_2$. By Theorem 1 the relation $(x, x') \neq (y, y')$ is $([\delta, \rho], [\delta, \rho])$-r.e. where $\delta$ is the inner representation of the points of $\mathbf{X}$. There is a machine $M$ that halts on input $(\langle p, p' \rangle, \langle q, p' \rangle)$ for $p, q \in \mathrm{dom}(\delta)$ and $p' \in \mathrm{dom}(\rho)$ iff $\delta(p) \neq \delta(q)$. There is a computable element $p' \in \mathrm{dom}(\rho)$. Therefore, there is a machine $N$ that halts on input $(p, q)$ iff $\delta(p) \neq \delta(q)$, hence $x \neq y$ is $(\delta, \delta)$-r.e. By Theorem 1, $\mathbf{X}$ must be $SCT_2$. But $\mathbf{X}$ is not $SCT_2$.

**Theorem 5.** *1. The $SCT_2$-, $WCT_3$-, $CT_3$- and $SCT_3$-spaces are closed under product.*
*2. The $WCT_0$-, $CT_0$- and $CT_2$-spaces are not closed under product.*

**Proof:** 1. Suppose, $\mathbf{X}_1$ and $\mathbf{X}_2$ are $SCT_2$. By Theorem 1, $x_i \neq y_i$ is $(\delta_i, \delta_i)$-r.e. for $i = 1, 2$, hence $(x_1, x_2) \neq (y_1, y_2)$ is $([\delta_1, \delta_2], [\delta_1, \delta_2])$-r.e., hence again by Theorem 1, $\mathbf{X}_1 \times \mathbf{X}_2$ is $SCT_2$.

Suppose, $\mathbf{X}_1$ and $\mathbf{X}_2$ are $WCT_3$. Let $(x_1, x_2) \in W_1 \times W_2$. From $x_i$ and $W_i$ we can find $U_i \in \beta_i$ such that $x_i \in U_i \subseteq \overline{U}_i \subseteq W_i$ (for $i = 1, 2$). Then $(x_1, x_2) \in U_1 \times U_2 \subseteq \overline{U_1 \times U_2} = \overline{U}_1 \times \overline{U}_2 \subseteq W_1 \times W_2$.

Suppose, $\mathbf{X}_1$ and $\mathbf{X}_2$ are $CT_3'$. We consider computability w.r.t. $\nu_i$, $\delta_i$, $\psi_i^-$, $\overline{\nu}$, $\overline{\delta}$ and $\overline{\psi}^-$. Suppose $(x_1, x_2) \in (W_1, W_2) \in \beta_1 \times \beta_2$. From $((x_1, x_2), (W_1, W_2))$ we can compute $x_1$, $x_2$, $W_1$ and $W_2$. Using $t_3'$ for $\mathbf{X_1}$ and $\mathbf{X_2}$ we can compute $(U_i, B_i)$ such hat $U_i \in \beta_i$ $B_i \subseteq X_i$ is closed and $x_i \in U_i \subseteq B_i \subseteq W_i$ $(i = 1, 2)$. Observe that $(x_1, x_2) \in U_1 \times U_2 \subseteq B_1 \times B_2 \subseteq W_1 \times W_2$. Form $(U_1, B_1)$ and $(U_2, B_2)$ we can compute $((u_1, u_2), (B_1, B_2))$.

Suppose, $\mathbf{X}_1$ and $\mathbf{X}_2$ are $SCT_3$. For $\mathbf{X_i}$ $(i = 1, 2)$ let $R_i$ be the r.e. set and let $r_i$ be the computable function for $SCT_3$ from Definition 3. There is a computable function $h$ such that $\psi_1^-(p_1) \times \psi_2^-(p_2) = \overline{\psi}^- \langle p_1, p_2 \rangle$. Let

$$\overline{R} := \{(\langle u_1, u_1 \rangle, \langle w_1, w_2 \rangle) \mid (u_1, w_1) \in R_1 \wedge (u_2, w_2) \in R_2\}\,,$$
$$\overline{r}(\langle u_1, u_1 \rangle, \langle w_1, w_2 \rangle) := h(r_1(u_1, w_1), r_2(u_2, w_2))\,.$$

A straightforward calculation shows that $\overline{R}$ is the r.e. set and $\overline{r}$ be the computable function for $SCT_3$ from Definition 3 for the product $\mathbf{X}_1 \times \mathbf{X}_2$.

2. In Example 9, the spaces $\mathbf{X}$ and $\mathbf{R}$ are $CT_2$, $CT_0$ and $WCT_0$. Their product $\mathbf{X} \times \mathbf{R}$, however, is not $WCT_0$, hence not $CT_0$ and not $CT_2$. □

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