A Note on Closed Subsets in Quasi-zero-dimensional Qcb-spaces (Extended Abstract)

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Abstract. We introduce the notion of quasi-zero-dimensionality as a substitute for the notion of zero-dimensionality, motivated by the fact that the latter behaves badly in the realm of qcb-spaces. We prove that the category QZ of quasi-zero-dimensional qcb_0-spaces is cartesian closed. Prominent examples of spaces in QZ are the spaces in the sequential hierarchy of the Kleene-Kreisel continuous functionals. Moreover, we characterise some types of closed subsets of QZ-spaces in terms of their ability to allow extendability of continuous functions. These results are related to an open problem in Computable Analysis.

Keywords: Computable Analysis, Qcb-spaces, Extendability

1 Introduction

The category QCB of quotients of countably based spaces [15] has excellent closure properties. For example, it is cartesian closed, in contrast to the category Top of all topological spaces (see [1, 12]). This means that QCB allows us to form products and functions space with the usual transposing properties. Qcb-spaces are known to form exactly the class of topological spaces which can be handled appropriately by the representation based approach to Computable Analysis, the Type Two Model of Effectivity, TTE ([16]).

Unfortunately, exponentiation in QCB behaves badly in terms of preservation of classical topological notions like regularity, normality and zero-dimensionality. For example, the function space $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$ formed in QCB is neither zero-dimensional nor normal (see [13]) despite the fact that both the exponent $\mathbb{N}^{\mathbb{N}}$ and the basis $\mathbb{N}$ are even zero-dimensional Polish spaces. In [14] the notion of a quasi-normal qcb-space is introduced as a substitute for normality in the realm of qcb-spaces (see Section 2.7). This notion has the advantage of being preserved by exponentiation in QCB. Moreover, quasi-normal qcb-spaces admit a Tietze-Urysohn Extension Theorem for continuous real-valued functions defined on functionally closed subspaces.

In an analogous way, we introduce the notion of a quasi-zero-dimensional qcb-space (see Section 3). The category QZ of quasi-zero-dimensional qcb-spaces

Andrej Bauer, Peter Hertling, Ker-I Ko (Eds.)
6th Int'l Conf. on Computability and Complexity in Analysis, 2009, pp. 233-244
http://drops.dagstuhl.de/opus/volltexte/2009/2274

turns out to be an exponential ideal of QCB. In Section 4 we investigate extendability of continuous functions that have as codomain either a quasi-zero-dimensional qcb-space or the real numbers. We prove that a subspace X of a QZ-space Y admits continuous extendability of all continuous functions from X to $\mathbb{N}$ if, and only if, X is in closed in the zero-dimensional reflection of Y. Analogously, we characterise functionally closed subspaces of a quasi-normal qcb-space as those subspaces that admit continuous extendability of all continuous real-valued functions defined on them.

In Section 5 we discuss the relationship of our results with an open problem in Computable Analysis. The problem is whether two hierarchies of functionals over the reals coincide (see [2]).

Since this is an extended abstract, most proofs are omitted.

2 Preliminaries

We repeat some notions and basic facts about sequential spaces, qcb-spaces, pseudobases, and quasi-normal spaces. Moreover, we remind the reader of the definition of the completely regular reflection and of the zero-dimensional reflection of a sequential space.

2.1 Notations

We use sans-serif letters like X, Y etc. to denote topological spaces. We write $\mathcal{O}(\mathsf{X})$ for the topology of a topological space X and $\mathcal{A}(\mathsf{X})$ for the family of closed sets of X. In abuse of notation, we will denote the carrier set of a space X by the same symbol X.

We use the following symbols for relevant topological spaces: $\mathbb{R}$ for the space of real numbers endowed with the Euclidean topology, $\mathbb{I}$ for the unit interval $[0, 1]$ endowed with the Euclidean subspace topology, $\mathbb{N}$ for the discrete topological space of natural numbers $\{0, 1, 2, \dots\}$, $\mathbb{J}$ for the one-point compactification of $\mathbb{N}$ with carrier set $\mathbb{N} \cup \{\infty\}$, and the the sans-serif letter $\mathsf{2}$ for the two-point discrete space with points 0 and 1.

2.2 Sequential spaces, sequential coreflections

A subset A of a topological space X is called *sequentially closed*, if A contains any limit of any convergent sequence of points in A. Complements of sequentially closed sets are called *sequentially open*. For a given topology τ, we denote the topology of sequentially open sets by $seq(\tau)$. Spaces such that every sequentially open set is open are called *sequential*. The sequential coreflection (or sequentialisation) $seq(\mathsf{X})$ of X is the topological space that carries the topology $seq(\mathcal{O}(\mathsf{X}))$ consisting of all sequentially open sets of X. The operator seq is idempotent. Importantly, a function between two sequential spaces is topologically continuous if, and only if, it is sequentially continuous.

For more details about the theory of sequential spaces we refer to [3, 17].

2.3 Qcb-spaces

A qcb-space [15] is a topological *q*uotient of a *c*ountably-*b*ased topological space. Qcb_0-spaces, i.e. qcb-spaces that satisfy the T_0-property, are well-established to be exactly the class of sequential spaces which can be handled by the Type Two Model of Effectivity.

Qcb-spaces are hereditarily Lindelöf (i.e. any open cover of any subset has a countable subcover) and sequential. The category QCB of qcb-spaces as objects and of continuous functions as morphisms is cartesian closed. Moreover, QCB has all countable limits and all countable colimits. For two qcb-spaces A and B we denote by $\mathsf{A} \times \mathsf{B}$ their product, by $\mathsf{A} + \mathsf{B}$ their coproduct, and by B^{A} their function space formed in QCB.

More information can be found in [1, 11, 12, 15].

2.4 Pseudobases and pseudo-open decompositions

Given a topological space X, we say that a family $\mathcal{A}$ of subsets of X is a *pseudo-open decomposition* of a subset M, if $M = \bigcup \mathcal{A}$ holds and for every sequence $(x_n)_n$ that converges to some element $x_\infty \in M$ there is some set $B \in \mathcal{A}$ and some $n_0 \in \mathbb{N}$ such that $\{x_n, x_\infty \,|\, n \geq n_0\} \subseteq B \subseteq M$ holds. Clearly, a set has a pseudo-open decomposition if, and only if, it is sequentially open.

A *(sequential) pseudobase* for X is a family $\mathcal{B}$ of subsets such that every open set has a pseudo-open decomposition into members of $\mathcal{B}$. Any base of topological space is a pseudobase, but not vice versa. Pseudobases characterise qcb-spaces: a sequential space is a qcb-space if, and only if, it has a countable pseudobase. Any countably pseudobased space is hereditarily Lindelöf and its sequential coreflection is a qcb-space. In this paper we will only deal with spaces having a countable pseudobase. More information can be found in [4, 12, 15].

2.5 Completely regular reflections, functionally open sets

Let X be a sequential space. The *completely regular reflection* of X is defined to carry the topology that is induced by the base

$$\mathcal{B} := \left\{ h^{-1}(0,1] \,\middle|\, h \colon \mathsf{X} \to \mathbb{I} \text{ is continuous} \right\}.$$

We denote this topological space by $\mathcal{R}(\mathsf{X})$. It has the property that every real-valued function f on X is continuous w.r.t. the original topology $\mathcal{O}(\mathsf{X})$ if, and only if, f is continuous w.r.t. the topology $\mathcal{O}(\mathcal{R}(\mathsf{X}))$. If $\mathcal{R}(\mathsf{X})$ is a T_0-space then $\mathcal{R}(\mathsf{X})$ is a Tychonoff space.

A subset U of X is called *functionally open*, if there is a continuous function h from X to the unit interval $\mathbb{I} = [0,1]$ such that $h^{-1}\{0\} = \mathsf{X} \setminus A$. Complements of functionally open sets are called *functionally closed*. A common term for "functionally closed set" is *zero-set*, and for "functionally open set" is *cozero-set*. We denote the family of functionally open sets of X by $\mathcal{FO}(\mathsf{X})$ and the family of functionally closed sets by $\mathcal{FA}(\mathsf{X})$. If X is a hereditarily Lindelöf space then $\mathcal{FO}(\mathsf{X})$

forms the topology of the completely regular reflection $\mathcal{R}(\mathsf{X})$. Otherwise $\mathcal{FO}(\mathsf{X})$ need not be a topology.

Regularity, normality and perfect normality[1] are equivalent for hereditarily Lindelöf spaces, thus for countably pseudobased spaces and for qcb-spaces.

2.6 Zero-dimensional spaces, zero-dimensional reflections

A *zero-dimensional space* is a topological space that has a base consisting of *clopen* (= closed and open) sets. Any zero-dimensional T_0-space is regular. Zero-dimensional hereditarily Lindelöf spaces X are even *strongly zero-dimensional*, meaning that any pair of disjoint closed sets A, B can be separated by a clopen set C (i.e. $A \subseteq C \subseteq \mathsf{X} \setminus B$). Strongly zero-dimensional T_0-spaces are zero-dimensional and normal.

Let X be a sequential space. The *zero-dimensional reflection* of X is defined to be the space that carries the topology induced by the base

$$\mathcal{B} := \left\{ h^{-1}\{1\} \,\middle|\, h \colon \mathsf{X} \to 2 \text{ is continuous} \right\}.$$

We denote this space by $\mathcal{Z}(\mathsf{X})$. Clearly, $\mathcal{Z}(\mathsf{X})$ is zero-dimensional. If X is hereditarily Lindelöf then the zero-dimensional reflection $\mathcal{Z}(\mathsf{X})$ is hereditarily Lindelöf as well and thus strongly zero-dimensional (see [3]).

2.7 Quasi-normal spaces and the category QN

A *quasi-normal* space is defined to be the sequential coreflection of some normal space [14]. The category of quasi-normal qcb-spaces, which is denoted by QN, is cartesian closed and inherits finite products and exponentials from its super-category QCB. This is not the case for the category of normal qcb-spaces. Any continuous function $f \colon \mathsf{X} \to \mathbb{R}$ from a functionally closed subspace X of a space $\mathsf{Y} \in \mathsf{QN}$ can be extended to a continuous function $F \colon \mathsf{Y} \to \mathbb{R}$. Details can be found in [14].

3 Quasi-zero-dimensional Qcb-Spaces

In this section we introduce and investigate the notion of a quasi-zero-dimensional qcb-space.

The category QCB of qcb-spaces is known to be cartesian closed. However, forming function spaces in QCB does not preserve classical topological notions like regularity, normality and zero-dimensionality. For example, the function space $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$ formed in QCB is neither zero-dimensional nor normal (see [13]),

[1] A *normal* space is a T_1-space such that for a pair of disjoint closed sets (A, B) there exists a pair of disjoint open sets (U, V) such that $A \subseteq U$ and $B \subseteq V$. A perfectly normal space is a T_1-space in which every closed sets is functionally closed. Note that some authors omit the T_1-condition.

although both $\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}}$ are zero-dimensional and normal. Hence the final topology of the natural TTE-representation on $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$, which is equal to the topology of $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$, is not zero-dimensional. By contrast, the compact-open topology on $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$ is even strongly zero-dimensional.

This fact motivates the introduction of an appropriate substitute for the property of zero-dimensionality in the realm of qcb-spaces. We use the same idea as in [14], where the notion of quasi-normality is defined as a replacement for normality. The idea behind the following definition is the fact that finite products and function spaces in the category QCB are constructed as the sequential coreflection of their counterparts in classical topology, which enjoy the property of preserving zero-dimensionality.

Definition 1. A qcb-space X is called *quasi-zero-dimensional*, if X is the sequential coreflection of a zero-dimensional T_0-space.

So a qcb-space is quasi-zero-dimensional if, and only if, its convergence relation is induced by some zero-dimensional T_0-topology. Clearly, any quasi-zero-dimensional space is hereditarily disconnected. Simple examples of quasi-zero-dimensional spaces are zero-dimensional separable metrisable spaces, because they are equal to their own sequentialisation. By QZ we denote the full subcategory of QCB that are quasi-zero-dimensional spaces.

Recall that a quasi-normal space is defined to be the sequential coreflection of a normal space (see [14]). Since zero-dimensional hereditarily Lindelöf T_0-spaces are normal, we have:

Lemma 1. *Any QZ-space is a QN-space (and thus a Hausdorff space).*

3.1 Characterisation of quasi-zero-dimensionality

We will give now several characterisations of QZ-spaces. They are analogous to characterisations of quasi-normality given in [14, Section 3.2]. We begin with the following observation.

Lemma 2. *A qcb_0-space X is quasi-zero-dimensional if, and only if, it is the sequential coreflection of its zero-dimensional reflection $\mathcal{Z}(\mathsf{X})$.*

For the second characterisation, we define two families of (respectively, closed and open) subsets of a topological space X by

$$\mathcal{ZA}(\mathsf{X}) := \big\{h^{-1}\{\infty\} \,\big|\, h\colon \mathsf{X} \to \mathbb{J} \text{ continuous}\big\},\ \mathcal{ZO}(\mathsf{X}) := \big\{\mathsf{X} \setminus A \,\big|\, A \in \mathcal{ZA}(\mathsf{X})\big\}.$$

Here $\mathbb{J}$ denotes the one-point compactification of $\mathbb{N}$. Obviously, every set in $\mathcal{ZA}(\mathsf{X})$ is closed in the zero-dimensional reflection of X. We will sometimes use the term *$\mathcal{Z}$-closed* for the members of $\mathcal{ZA}(\mathsf{X})$ and *$\mathcal{Z}$-open* for members of $\mathcal{ZO}(\mathsf{X})$. In general, $\mathcal{ZO}(\mathsf{X})$ is not a topology, unless X is hereditarily Lindelöf.

Lemma 3. *Let X be a hereditarily Lindelöf space. Then $\mathcal{ZO}(\mathsf{X})$ is the family of all open sets of $\mathcal{Z}(\mathsf{X})$. Dually, $\mathcal{ZA}(\mathsf{X})$ is the family of all closed sets of $\mathcal{Z}(\mathsf{X})$.*

Obviously, every $\mathcal{Z}$-closed in a quasi-zero-dimensional space is functionally closed, because $\mathbb{J}$ is homeomorphic to the closed subspace $\{0, 2^{-n} \mid n \in \mathbb{N}\}$ of $\mathbb{I}$. It is not known for which QZ-spaces the converse is true as well.

Definition 2. We say that a qcb-space X has *Normann's property*, if $\mathsf{X} \in \mathsf{QZ}$ and every functionally closed set of X is closed in the zero-dimensional reflection $\mathcal{Z}(\mathsf{X})$ of X (i.e. $\mathcal{FA}(\mathsf{X}) = \mathcal{ZA}(\mathsf{X})$).

Clearly, zero-dimensional spaces have Normann's property.

Lemma 3 implies the following reformulation of Lemma 2.

Corollary 1. *A qcb_0-space X is quasi-zero-dimensional if, and only if, its convergence relation is induced by the topology $\mathcal{ZO}(\mathsf{X})$.*

We now work towards a characterisation of quasi-zero-dimensionality in terms of pseudobases. Recall that qcb-spaces are known to be those sequential spaces that have a countable pseudobase (see Section 2.4). We start with the following separation lemma for disjoint $\mathcal{Z}$-closed subsets.

Lemma 4. *Let X be a hereditarily Lindelöf space, and let A, B be disjoint closed subsets of $\mathcal{Z}(\mathsf{X})$. Then there is a continuous function $h \colon \mathsf{X} \to \mathbb{J}$ with $h^{-1}\{\infty\} = A$ and $B \subseteq h^{-1}\{0\}$.*

This lemma is instrumental in proving the following lemma about sequentially open sets that are $\mathcal{G}_\delta$-sets in the zero-dimensional reflection of a QZ-space.

Lemma 5. *Let X be a qcb-space equipped with a countable pseudobase consisting of sets in $\mathcal{ZA}(\mathsf{X})$. Then every sequentially open set $V \in \mathcal{O}(\mathsf{X})$ that is a $\mathcal{G}_\delta$-set in $\mathcal{Z}(\mathsf{X})$ is open in $\mathcal{Z}(\mathsf{X})$. Dually, every sequentially closed set $A \in \mathcal{A}(\mathsf{X})$ that is an $\mathcal{F}_\sigma$-set in $\mathcal{Z}(\mathsf{X})$ is closed in $\mathcal{Z}(\mathsf{X})$.*

Now we are ready to characterise quasi-zero-dimensional qcb-spaces in terms of properties of pseudobases.

Proposition 1. *A qcb_0-space X is quasi-zero-dimensional if, and only if, it has a countable pseudobase consisting of sets in $\mathcal{ZA}(\mathsf{X})$.*

Note that quasi-normal qcb-spaces are characterised via countable pseudobases consisting of functionally closed sets (see [14, Proposition 4]).

A continuous function $e \colon \mathsf{X} \to \mathsf{Y}$ between sequential spaces X, Y is said to *reflect convergent sequences*, if, for any sequence $(x_n)_n$ in X and any point $x_\infty \in \mathsf{X}$, $(x_n)_n$ converges to x_∞ in X whenever $(e(x_n))_n$ converges to $e(x_\infty)$ in Y.

Proposition 2. *A qcb-space X is quasi-zero-dimensional if, and only if, there are a qcb-space Z and a continuous injection $e \colon \mathsf{X} \to 2^{\mathsf{Z}}$ that reflects convergent sequences.*

3.2 Constructing quasi-zero-dimensional spaces

The category QZ of quasi-zero-dimensional qcb-spaces enjoys excellent closure properties. Like quasi-normality, quasi-zero-dimensionality is preserved by forming (i) countable products, (ii) subspaces, (iii) countable coproducts, and (iv) function spaces in the category QCB of qcb-spaces. So QZ inherits the cartesian-closed structure of QCB. In fact QZ is an exponential ideal of QCB.

Theorem 1. *The category* QZ *of quasi-zero-dimensional qcb-spaces is cartesian closed. Moreover, it has all countable limits, all countable colimits and is an exponential ideal of* QCB.

Proof. Similar to the proof of Theorem 6 in [14]. Alternatively, one can apply Proposition 2.

Obviously, all zero-dimensional metric spaces are in QZ. Theorem 1 implies that all Kleene-Kreisel spaces [5] of the form $\mathbb{N}^{\mathsf{Z}}$ belong to QZ. Furthermore, for all $k \in \mathbb{N}$ the space $\mathsf{N}\langle k \rangle$ of Kleene-Kreisel continuous functional of order k (see [7, 8, 10]) is a quasi-zero-dimensional space. The hierarchy $(\mathsf{N}\langle k \rangle)_k$ is recursively defined by the formula $\mathsf{N}\langle 0 \rangle := \mathbb{N}$ and $\mathsf{N}\langle k+1 \rangle := \mathbb{N}^{\mathsf{N}\langle k \rangle}$. On the other hand, the Euclidean space $\mathbb{R}$ is not quasi-zero-dimensional by being connected.

Remark 1. One can show that there is a cartesian closed embedding of QZ into the cartesian closed category $\mathsf{k_2 0dim}$ considered by G. Lukács in [9]. This category is itself equivalent to a full reflective sub-ccc of the category of Hausdorff k-spaces.

4 Extendability of continuous functions

In this section we investigate extendability of continuous functions defined on subspaces of quasi-zero-dimensional spaces. Moreover, we classify subspaces in terms of their ability to admit extendability of continuous functions.

4.1 A transitivity property for $\mathcal{Z}$-closed sets

It is well-known that the subspace operator on topological spaces has the following transitivity property: Any closed subset of a closed subspace is closed in the original space, whereas the analogous statement for functionally closed sets is false in general (see [3, 2.1.B]).

In [14], it is shown that functionally closed sets in quasi-normal qcb-spaces do have the transitivity property. Recall that functionally closed sets of a QN-space Y are exactly the closed sets of the completely regular reflection of Y.

In Proposition 1 and Lemma 3 we have seen that $\mathcal{Z}$-closed sets play a similar role for QZ-spaces as functionally closed sets do for QN-spaces. Validity of the transitivity property for $\mathcal{Z}$-closed sets is related to extendability of continuous functions with zero-dimensional codomains as follows: Let X be a $\mathcal{Z}$-closed subspace of a sequential space Y. If any continuous function from X to $\mathbb{J}$ (the one-point compactification of $\mathbb{N}$) is extendable onto Y, then any closed subset A of $\mathcal{Z}(\mathsf{X})$ is also closed in $\mathcal{Z}(\mathsf{Y})$: Choose continuous functions $f\colon \mathsf{X} \to \mathbb{J}$ and $g\colon \mathsf{Y} \to \mathbb{J}$ with $f^{-1}\{\infty\} = A$, $g^{-1}\{\infty\} = \mathsf{X}$ and extend f to a continuous function $F\colon \mathsf{Y} \to \mathbb{J}$. Then the function $h\colon \mathsf{Y} \to \mathbb{J}$ defined by $h(y) := \min\{F(y), g(y)\}$ is a continuous function witnessing that A is closed in $\mathcal{ZA}(\mathsf{Y})$.

Fortunately, the transitivity property for $\mathcal{Z}$-closed sets is valid in the realm of QZ-spaces (see Proposition 3). So the zero-dimensional reflection of any $\mathcal{Z}$-closed subspace is a subspace of the zero-dimensional reflection of its QZ-superspace.

Proposition 3. *Let* $\mathsf{Y} \in \mathsf{QZ}$*. Let* X *be a subspace of* Y *with* $\mathsf{X} \in \mathcal{ZA}(\mathsf{Y})$*. Then every set that is closed in* $\mathcal{Z}(\mathsf{X})$ *is closed in* $\mathcal{Z}(\mathsf{Y})$*. Moreover,* $\mathcal{Z}(\mathsf{X})$ *is a topological subspace of* $\mathcal{Z}(\mathsf{Y})$*.*

4.2 Extendability of continuous functions into QZ-spaces

In this section we work towards showing that, for any zero-dimensional Polish space B, any continuous B-valued function defined on a $\mathcal{Z}$-closed subspace of a QZ-space is continuously extendable. We start by showing that clopens of $\mathcal{Z}$-closed subspaces extend to clopens of the whole space, provided that the latter is in QZ.

Lemma 6. *Let* $\mathsf{Y} \in \mathsf{QZ}$*, and let* X *be a subspace of* Y *with* $\mathsf{X} \in \mathcal{ZA}(\mathsf{Y})$*. Then for every set* D *that is clopen in* X *there is a clopen* C *in* Y *with* $D = C \cap \mathsf{X}$*.*

Proof. By Proposition 3, both D and $\mathsf{X} \setminus D$ are closed sets in $\mathcal{Z}(\mathsf{Y})$. By strong zero-dimensionality of $\mathcal{Z}(\mathsf{Y})$, there is a clopen set C in Y with $D \subseteq C \subseteq \mathsf{X} \setminus D$. Clearly, $C \cap \mathsf{X} = D$. □

Lemma 6 can be reformulated by stating that any continuous function from a $\mathcal{Z}$-closed subset into the two-point discrete space $\mathsf{2}$ has a continuous extension.

We now investigate the full subcategory ZEXT of QCB consisting of those quasi-zero-dimenional qcb-spaces $\mathsf{Z} \in \mathsf{QZ}$ that have the following property: For all spaces $\mathsf{Y} \in \mathsf{QZ}$, for all $\mathcal{Z}$-closed subspaces X of Y and for all continuous functions $f\colon \mathsf{X} \to \mathsf{Z}$ there exists a continuous function $F\colon \mathsf{Y} \to \mathsf{Z}$ extending f. Lemma 6 states that $\mathsf{2}$ is an object of ZEXT.

Given two qcb-spaces Y, B, we say that a subspace X of Y *admits a continuous extension operator for* B, if there exists a continuous function $E\colon \mathsf{B}^{\mathsf{X}} \to \mathsf{B}^{\mathsf{Y}}$ satisfying $E(f)(x) = f(x)$ for all $x \in \mathsf{X}$ and all continuous functions $f\colon \mathsf{X} \to \mathsf{B}$. Cartesian closedness of QZ (see Theorem 1) yields the following characterisation of the objects in ZEXT.

Proposition 4. *A space* $\mathsf{Z} \in \mathsf{QZ}$ *is an object of* ZEXT *if, and only if, any* $\mathcal{Z}$*-closed subspace* X *of any space* $\mathsf{Y} \in \mathsf{QZ}$ *admits a continuous extension operator* $E\colon \mathsf{Z}^{\mathsf{X}} \to \mathsf{Z}^{\mathsf{Y}}$ *for* Z*.*

The category ZEXT enjoys excellent closure properties.

Proposition 5.
1. *If* $\mathsf{A}, \mathsf{B} \in \mathsf{ZEXT}$*, then* $\mathsf{A} \times \mathsf{B} \in \mathsf{ZEXT}$*.*
2. *If* $\mathsf{B} \in \mathsf{ZEXT}$ *and* $\mathsf{A} \in \mathsf{QCB}$*, then* $\mathsf{B}^{\mathsf{A}} \in \mathsf{ZEXT}$*.*
3. *If* $\mathsf{A}, \mathsf{B} \in \mathsf{ZEXT}$*, then* $\mathsf{A} + \mathsf{B} \in \mathsf{ZEXT}$*.*
4. *If* $\mathsf{B} \in \mathsf{ZEXT}$ *and* A *is a* QCB*-retract of* B*, then* $\mathsf{A} \in \mathsf{ZEXT}$*.*
5. *If* $\mathsf{B} \in \mathsf{ZEXT}$ *and* A *is a* $\mathcal{Z}$*-open subspace of* B*, then* $\mathsf{A} \in \mathsf{ZEXT}$*.*

In the category of sequential spaces and hence in QCB the discrete space $\mathbb{N}$ is homeomorphic to the function space $2^{2^{\mathbb{N}}}$ by [2, Proposition 3]. Moreover, by [6, Theorem 7.8] every zero-dimensional Polish space is homeomorphic to a closed subset of the Baire space $\mathbb{N}^{\mathbb{N}}$. In turn, this closed subspace is a retract of $\mathbb{N}^{\mathbb{N}}$ by [2, Proposition 2]. We obtain by Proposition 5 and Lemma 6:

Example 1. The following spaces are objects of ZEXT:
(a) the discrete space $\mathbb{N}$,
(b) the Baire space $\mathbb{N}^{\mathbb{N}}$,
(c) any zero-dimensional Polish space,
(d) the one-point compactification $\mathbb{J}$ of $\mathbb{N}$,
(e) for any $k \in \mathbb{N}$ the space $\mathsf{N}\langle k \rangle$ of all Kleene-Kreisel continuous functionals of order k (see Section 3.2).

4.3 Subspaces that admit continuous extendability

Now we study under which conditions a subspace admits continuous extendability of continuous functions. We start with the following simple observation.

Lemma 7. *Let* $\mathsf{Y} \in \mathsf{QZ}$ *and let* X *be a subspace of* Y *such that for every subset* $D \subseteq \mathsf{X}$ *that is clopen in* X *there exists a clopen* C *in* Y *with* $D = C \cap \mathsf{X}$*. Then* X *is sequentially closed.*

We have already seen that the property of X being closed in $\mathcal{ZA}(\mathsf{Y})$ is sufficient to guarantee extendability of all continuous $\mathbb{N}$-valued functions defined on X. We show that this condition is also necessary.

Lemma 8. *Let* X *be a subspace of a* $\mathsf{Y} \in \mathsf{QZ}$*. If every continuous function* $h\colon \mathsf{X} \to \mathbb{N}$ *can be extended to a continuous function* $H\colon \mathsf{Y} \to \mathbb{N}$*, then* $\mathsf{X} \in \mathcal{ZA}(\mathsf{Y})$*.*

We obtain as an easy consequence:

Corollary 2. *Let* A *be a retract of a space* $\mathsf{Y} \in \mathsf{QZ}$*. Then* A *is homeomorphic to a* $\mathcal{Z}$*-closed subspace of* Y*.*

Lemma 8 generalises to all non-compact QZ-spaces replacing $\mathbb{N}$ as codomain space.

Proposition 6. *Let* $\mathsf{Z} \in \mathsf{QZ}$ *such that* Z *is not compact. Let* X *be a subspace of a space* $\mathsf{Y} \in \mathsf{QZ}$ *such that every continuous function* $f\colon \mathsf{X} \to \mathsf{Z}$ *can be extended to a continuous function* $F\colon \mathsf{Y} \to \mathsf{Z}$*. Then* $\mathsf{X} \in \mathcal{ZA}(\mathsf{Y})$*.*

Proposition 6 is a consequence of Lemma 8 and the following equivalence.

Lemma 9. *A space* $\mathsf{X} \in \mathsf{QZ}$ *is not compact if, and only if,* $\mathbb{N}$ *is a retract of* X.

We do not know whether Lemma 8 is valid for the two-point discrete space $\mathsf{2}$ replacing $\mathbb{N}$. However, the (possibly) stronger condition on a subspace X to admit a continuous extension operator for the continuous functions with codomain $\mathsf{2}$ is enough to ensure that X is $\mathcal{Z}$-closed.

Proposition 7. *Let* $\mathsf{Y} \in \mathsf{QZ}$. *Let* X *be a subspace of* Y *that admits a continuous extension operator* $E\colon \mathsf{2}^{\mathsf{X}} \to \mathsf{2}^{\mathsf{Y}}$. *Then* $\mathsf{X} \in \mathcal{ZA}(\mathsf{Y})$.

With the help of Propositions 2, 5 and 7 one can prove:

Proposition 8. *A qcb-space* X *is an object of* ZEXT *if, and only if, there is a qcb-space* Z *such that* X *is a retract of* $\mathsf{2}^{\mathsf{Z}}$.

We summerise some of the above results in a characterisation theorem for sets that are closed in the zero-dimensional reflection.

Theorem 2 (Characterisation of $\mathcal{Z}$-closed subsets). *Let* Y *be a quasi-zero-dimensional qcb-space, and let* X *be a subspace of* Y. *Then the following statements are equivalent:*

(a) The set X *is closed in the zero-dimensional reflection* $\mathcal{Z}(\mathsf{Y})$ *of* Y.
(b) The set is $\mathcal{Z}$*-closed (i.e.* $\mathsf{X} \in \mathcal{ZA}(\mathsf{Y})$*).*
(c) The subspace X *admits a continuous extension operator* $E\colon \mathsf{2}^{\mathsf{X}} \to \mathsf{2}^{\mathsf{Y}}$.
(d) The subspace X *admits a continuous extension operator* $E\colon \mathbb{N}^{\mathsf{X}} \to \mathbb{N}^{\mathsf{Y}}$.
(e) Any continuous function $f\colon \mathsf{X} \to \mathbb{N}$ *can be extended to a continuous function* $F\colon \mathsf{Y} \to \mathbb{N}$.
(f) There is a non-compact quasi-zero-dimensional qcb-space Z *such that any continuous function* $f\colon \mathsf{X} \to \mathsf{Z}$ *can be extended to a continuous function* $F\colon \mathsf{Y} \to \mathsf{Z}$.

4.4 Characterisation of functionally closed subsets

In this section we present a characterisation of all functionally closed subsets of quasi-normal spaces that is similar to Theorem 2.

In [14] it is shown that real-valued functions defined on a functionally closed subspace can be extended to the whole space, provided the latter is a quasi-normal qcb-space. We remark that cartesian closedness of QN implies the following uniform versions of this extendability result.

Proposition 9. *Let* X *be a functionally closed subspace of a space* $\mathsf{Y} \in \mathsf{QN}$. *Then* X *admits continuous extension operators* $E_{\mathbb{I}}\colon \mathbb{I}^{\mathsf{X}} \to \mathbb{I}^{\mathsf{Y}}$ *and* $E_{\mathbb{R}}\colon \mathbb{R}^{\mathsf{X}} \to \mathbb{R}^{\mathsf{Y}}$.

Now we investigate under which condition a subspace admits continuous extendability of continuous real-valued functions. We begin with the following simple observation which is analogous to Lemma 7.

Lemma 10. *Let* $\mathsf{Y} \in \mathsf{QN}$*. Let* X *be a* QCB*-subspace of* Y *such that every continuous function* $f\colon \mathsf{X} \to \mathbb{I}$ *can be extended to a continuous function* $F\colon \mathsf{Y} \to \mathbb{I}$*. Then* X *is sequentially closed.*

The fact that every qcb-space is hereditarily Lindelöf implies the following observation.

Lemma 11. *Let* $\mathsf{Y} \in \mathsf{QN}$*. Let* X *be a* QCB*-subspace of* Y *such that every continuous function* $f\colon \mathsf{X} \to \mathbb{R}$ *can be extended to a continuous function* $F\colon \mathsf{Y} \to \mathbb{R}$*. Then* $\mathsf{X} \in \mathcal{FA}(\mathsf{Y})$*.*

We obtain the following corollary which parallels Corollary 2.

Corollary 3. *Let* A *be a retract of a space* $\mathsf{Y} \in \mathsf{QN}$*. Then* A *is homeomorphic to a functionally closed subspace of* Y*.*

We do not know whether non-uniform extendability of all continuous functions on X into the unit interval $\mathbb{I} = [0,1]$ implies that X is functionally closed. However, if X admits a continuous extension operator for $\mathbb{I}$ as codomain, then X must be functionally closed (cf. Proposition 7).

Proposition 10. *Let* Y *be a quasi-normal qcb-space. Let* X *be a* QCB*-subspace of* Y *that admits a continuous extension operator* $E\colon \mathbb{I}^{\mathsf{X}} \to \mathbb{I}^{\mathsf{Y}}$*. Then* $\mathsf{X} \in \mathcal{FA}(\mathsf{Y})$*.*

We summerise the above results in a characterisation theorem for subsets of quasi-normal qcb-spaces that are closed in the completely regular reflection.

Theorem 3 (Characterisation of functionally closed subsets). *Let* Y *be a quasi-normal qcb-space, and let* X *be a* QCB*-subspace of* Y*. Then the following statements are equivalent:*

(a) The set X *is functionally closed in* Y *(i.e.* $\mathsf{X} \in \mathcal{FA}(\mathsf{Y})$*).*
(b) The set X *is closed the completely regular reflection* $\mathcal{R}(\mathsf{Y})$ *of* Y*.*
(c) The subspace X *admits a continuous extension operator* $E\colon \mathbb{I}^{\mathsf{X}} \to \mathbb{I}^{\mathsf{Y}}$*.*
(d) The subspace X *admits a continuous extension operator* $E\colon \mathbb{R}^{\mathsf{X}} \to \mathbb{R}^{\mathsf{Y}}$*.*
(e) Any continuous function $f\colon \mathsf{X} \to \mathbb{R}$ *can be extended to a continuous function* $F\colon \mathsf{Y} \to \mathbb{R}$*.*

5 Discussion

We have seen that the category QZ of quasi-zero-dimensional qcb-spaces and the category QN of quasi-normal qcb-spaces enjoy several similarities, for example they are exponential ideals of QCB. Both classes of topological spaces possess a distinguished family of closed subsets ($\mathcal{Z}$-closed subsets in the case of QZ and functionally closed subsets in the case of QN) with following property: either class is characterised by the existence of a countable pseudobase consisting of sets in the respective family of closed subsets. Functionally closed subspaces of QN-spaces are characterised as those subspaces that admit continuous extendability of real-valued functions, while $\mathcal{Z}$-closed subsets are exactly the class of sets which allow to extend continuous functions that have a Kleene-Kreisel space of the form $\mathbb{N}^{\mathsf{Z}}$ as codomain.

It is not known whether every QZ-space Y has *Normann's property* (i.e. every functionally closed set in Y is $\mathcal{Z}$-closed). This question is related to an open problem in Computable Analysis, namely whether or not two natural hierarchies of continuous functionals over the reals (called the intensional hierarchy and the extensional hierarchy, see [2]) coincide. D. Normann [10] proved that the two hierarchies agree if, and only if, for all $k \geq 2$ the space $\mathsf{N}\langle k \rangle$ of Kleene-Kreisel continuous functionals of type k (see Section 3.2) has Normann's property.

Acknowledgement

I would like to thank the anonymous referees for their valuable suggestions.

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