Effective Dispersion in Computable Metric Spaces

Zvonko Iljazović

University of Zagreb, Croatia

Abstract. We investigate the relationship between computable metric spaces (X, d, α) and (X, d, β), where (X, d) is a given metric space. In the case of Euclidean space, α and β are equivalent up to isometry, which does not hold in general. We introduce the notion of effectively dispersed metric space. This notion is essential in the proof of the main result of this paper: (X, d, α) is effectively totally bounded if and only if (X, d, β) is effectively totally bounded, i.e. the property that a computable metric space is effectively totally bounded (and in particular effectively compact) depends only on the underlying metric space.

1 Introduction

Let $k \in \mathbb{N}$, $k \geq 1$. We say that a function $f : \mathbb{N}^k \to \mathbb{Q}$ is **recursive** if there exist recursive functions $a, b, c : \mathbb{N}^k \to \mathbb{N}$ such that $f(x) = (-1)^{c(x)} \frac{a(x)}{b(x)+1}$, $\forall x \in \mathbb{N}^k$. A function $f : \mathbb{N}^k \to \mathbb{R}$ is said to be **recursive** if there exists a recursive function $F : \mathbb{N}^{k+1} \to \mathbb{Q}$ such that $|f(x) - F(x, i)| < 2^{-i}$, $\forall x \in \mathbb{N}^k$, $\forall i \in \mathbb{N}$.

A tuple (X, d, α) is said to be a **computable metric space** if (X, d) is a metric space and $\alpha : \mathbb{N} \to X$ is a sequence dense in (X, d) such that the function $\mathbb{N}^2 \to \mathbb{R}$, $(i, j) \mapsto d(\alpha(i), \alpha(j))$ is recursive. We say that α is an **effective separating sequence** in (X, d) (cf. [3]). If (X, d, α) is a computable metric space, then a sequence (x_i) in X is said to be **recursive** in (X, d, α) if there exists a recursive function $F : \mathbb{N}^2 \to \mathbb{N}$ such that $d(x_i, \alpha_{F(i,k)}) < 2^{-k}$, $\forall i, k \in \mathbb{N}$ and a point $a \in X$ is said to be **recursive** in (X, d, α) if the constant sequence $a, a, \dots$ is recursive. For example, if $q : \mathbb{N} \to \mathbb{Q}$ is a recursive surjection, then $(\mathbb{R}, d, q)$ is a computable metric space, where d is the Euclidean metric on $\mathbb{R}$. A sequence (x_i) is recursive in this computable metric space if and only if (x_i) is a recursive sequence of real numbers and $a \in \mathbb{R}$ is a recursive point in this space if and only if a is a recursive number.

Let (X, d) be a metric space and let $\mathcal{S}$ be a nonempty set whose elements are sequences in X. We say that $\mathcal{S}$ is a **computability structure** on (X, d) (cf. [3]) if the following three properties hold:

(i) if $(x_i), (y_j) \in \mathcal{S}$, then the function $\mathbb{N}^2 \to \mathbb{R}$, $(i, j) \mapsto d(x_i, y_j)$ is recursive;
(ii) if $(x_i)_{i \in \mathbb{N}} \in \mathcal{S}$, then $(x_{f(i)})_{i \in \mathbb{N}} \in \mathcal{S}$ for any recursive function $f : \mathbb{N} \to \mathbb{N}$;
(iii) if (y_i) is a sequence in X such that $d(y_i, x_{F(i,k)}) < 2^{-k}$, $\forall i, k \in \mathbb{N}$, where $F : \mathbb{N}^2 \to \mathbb{N}$ is a recursive function and $(x_i) \in \mathcal{S}$, then $(y_i) \in \mathcal{S}$.

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Let (X, d) be a metric space. If α is an effective separating sequence in (X, d), then the set $\mathcal{S}_\alpha$ of all recursive sequences in (X, d, α) is an example of a computability structure on (X, d). Suppose now that α and β are effective separating sequences in (X, d). We say that α is **equivalent** to β, $\alpha \sim \beta$, if α is a recursive sequence in (X, d, β). It follows easily that $\alpha \sim \beta$ if and only if $\mathcal{S}_\alpha = \mathcal{S}_\beta$.

A closed subset S of a computable metric space (X, d, α) is said to be *recursively enumerable* if $\{i \in \mathbb{N} \mid I_i \cap S \neq \emptyset\}$ is an r.e. set, where (I_i) is some effective enumeration of all open rational balls in (X, d, α), *co-recursively enumerable* if $X \setminus S = \cup_{i \in \mathbb{N}} I_{f(i)}$, where $f : \mathbb{N} \to \mathbb{N}$ is a recursive function and *recursive* if it is both r.e. and co-r.e. ([1]). It is not hard to see that if $\alpha \sim \beta$, then S is r.e. (co-r.e.) in (X, d, α) if and only if S is r.e. (co-r.e.) in (X, d, β) and consequently S is recursive in (X, d, α) if and only if S is recursive in (X, d, β). Hence the notions of recursive enumerability, co-recursive enumerability and recursiveness of a set are examples of notions which depend only on the induced computability structure and not on particular α which induces that structure.

If α and β are effective separating sequences in a metric space (X, d), then α and β need not be equivalent. For example, if $c \in \mathbb{R}$ is a nonrecursive number and (α_i) a recursive sequence of real numbers dense in $(\mathbb{R}, d)$, where d is the Euclidean metric, then $(\alpha_i + c)$ is an effective separating sequence in $(\mathbb{R}, d)$, c is a recursive point in $(\mathbb{R}, d, (\alpha_i + c))$ and c is not recursive in $(\mathbb{R}, d, (\alpha_i))$. Hence (α_i) and $(\alpha_i + c)$ are not equivalent.

Let $(X, d, (\alpha_i))$ be a computable metric space and f an isometry of (X, d). By an isometry of (X, d) we mean a surjective map $f : X \to X$ such that $d(f(x), f(y)) = d(x, y)$, $\forall x, y \in X$. Then $(X, d, (f(\alpha_i)))$ is also a computable metric space and in general the sequences (α_i) and $(f(\alpha_i))$ are not equivalent by the previous example. Note that f "maps" the computability structure induced by (α_i) on the computability structure induced by $(f(\alpha_i))$, i.e.

$$\mathcal{S}_{(f(\alpha_i))} = \{(f(x_i)) \mid (x_i) \in \mathcal{S}_{(\alpha_i)}\}.$$

In particular, if A is the set of all recursive points in $(X, d, (\alpha_i))$ and B the set of all recursive points in $(X, d, (f(\alpha_i)))$, then $f(A) = B$.

We say that effective separating sequences (α_i) and (β_i) in a metric space (X, d) are **equivalent up to isometry** if $(\alpha_i) \sim (f(\beta_i))$ for some isometry f of (X, d). It is easy to see that this relation is an equivalence relation on the set of all effective separating sequences in (X, d).

If (X, d, α) is a computable metric space, then clearly the metric space (X, d) is totally bounded if and only if for each $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $X = \cup_{0 \leq i \leq m} B(\alpha_i, 2^{-k})$. Here $B(x, r)$ for $x \in X$ and $r > 0$ denotes the open ball of radius r centered at x. We say that a computable metric space (X, d, α) is **effectively totally bounded** if there exists a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that

$$X = \bigcup_{i=0}^{f(k)} B(\alpha_i, 2^{-k}),$$

$\forall k \in \mathbb{N}$ ([3]).

Example 1. If S is a recursive nonempty compact subset of $\mathbb{R}^n$, then there exists a recursive sequence (x_i) in S and a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that $S \subseteq \cup_{0 \leq i \leq f(k)} B(x_i, 2^{-k})$, $\forall k \in \mathbb{N}$ ([4]) and therefore $(S, d, (x_i))$ is an effectively totally bounded computable metric space, where d is the Euclidean metric on S.

Example 2. Let $\omega : \mathbb{N} \to \mathbb{Q}$ be a recursive sequence which converges to a nonrecursive number $\gamma \in \mathbb{R}$ and such that $\omega(0) = 0$, $\omega(i) < \omega(i+1)$, $\forall i \in \mathbb{N}$. It is easy to construct a recursive sequence of rational numbers α which is dense in $[0, \gamma]$. Then the tuple $([0, \gamma], d, \alpha)$ is a computable metric space, where d is the Euclidean metric on $[0, \gamma]$. Suppose that $([0, \gamma], d, \alpha)$ is effectively totally bounded. Then $[0, \gamma] = \cup_{0 \leq i \leq f(k)} B(\alpha_i, 2^{-k})$, $\forall k \in \mathbb{N}$, for some recursive function $f : \mathbb{N} \to \mathbb{N}$. If $h : \mathbb{N} \to \mathbb{Q}$ is defined by $h(k) = \max\{\alpha_i \mid 0 \leq i \leq f(k)\}$, $k \in \mathbb{N}$, then h is a recursive function and $|\gamma - h(k)| < 2^{-k}$, $\forall k \in \mathbb{N}$ which contradicts the fact that γ is a nonrecursive number. Hence the computable metric space $([0, \gamma], d, \alpha)$ is not effectively totally bounded, although the metric space $([0, \gamma], d)$ is totally bounded.

It is not hard to check that if α and β are equivalent effective separating sequences in a metric space (X, d), then (X, d, α) is effectively totally bounded if and only if (X, d, β) is effectively totally bounded. Furthermore, if f is an isometry of (X, d) and (α_i) an effective separating sequence, then $(X, d, (\alpha_i))$ is effectively totally bounded if and only if $(X, d, (f(\alpha_i)))$ is effectively totally bounded. This follows immediately from the fact that $f(B(x, r)) = B(f(x), r)$, $\forall x \in X$, $\forall r > 0$. Therefore, if α and β are effective separating sequences equivalent up to isometry, then (X, d, α) is effectively totally bounded if and only if (X, d, β) is effectively totally bounded.

There exist totally bounded metric spaces with effective separating sequences nonequivalent up to isometry (Section 2). Nevertheless, the equivalence

$$(X, d, \alpha) \text{ effectively totally bounded } \Leftrightarrow (X, d, \beta) \text{ effectively totally bounded} \tag{1}$$

holds in general and that is the main result of this paper which will be proved in Section 3 where we introduce the notion of effectively dispersed metric space. In Section 2 we also prove that each two effective separating sequence in Euclidean space $\mathbb{R}^n$ are equivalent up to isometry.

2 Isometries and computability structures

Let $n \geq 1$ and let d be the Euclidean metric on $\mathbb{R}^n$. The main step in proving that every two effective separating sequences in $(\mathbb{R}^n, d)$ are equivalent up to isometry is the following proposition.

Proposition 1. *Let $a_0, \ldots, a_n$ be recursive points in $\mathbb{R}^n$ which are geometrically independent (i.e. $a_1 - a_0, \ldots, a_n - a_0$ are linearly independent vectors) and let (x_i) be a sequence in $\mathbb{R}^n$ such that $(d(x_i, a_k))_{i \in \mathbb{N}}$ is a recursive sequence of real numbers for each $k \in \{0, \ldots, n\}$. Then (x_i) is a recursive sequence.*

Proposition 1 is essentially a consequence of the fact that we can compute each component of x_i by certain formula which involves addition, subtraction, multiplication and division of numbers $d(x_i, a_0), \dots, d(x_i, a_n)$ and components of the points $a_0, \dots, a_n$. It follows from Proposition 1 that for geometrically independent recursive points $a_0, \dots, a_n$ in $\mathbb{R}^n$ and $x \in \mathbb{N}$ the following implication holds:

$$\text{the numbers } d(x, a_0), \dots, d(x, a_n) \text{ are recursive } \Rightarrow \text{ the point } x \text{ is recursive.} \tag{2}$$

However, in a general computable metric space it is not possible to find $n \in \mathbf{N}$ and recursive points $a_0, \dots, a_n$ such that the implication (2) holds. This shows the following example.

Example 3. Let p be the metric on $\mathbb{R}^2$ given by $p((x_1, y_1), (x_2, y_2)) = \max\{|x_2 - x_1|, |y_2 - y_1|\}$. If (α_i) is a recursive dense sequence in $\mathbb{R}^2$, then $(\mathbb{R}^2, p, (\alpha_i))$ is a computable metric space and the induced computability structure coincides with the usual computability structure on $\mathbb{R}^2$. Suppose $(x_0, y_0), \dots, (x_k, y_k)$ are any recursive points in $\mathbb{R}^2$. Let $M > 0$ be some upper bound of the set $\{|x_0|, |y_0|, \dots, |x_k|, |y_k|\}$. Let $a, b \in \mathbb{R}$ be such that $a > 3M$, $|b| < M$ and such that a is a recursive, and b a nonrecursive number. Then $p((a, b), (x_0, y_0)), \dots$ $p((a, b), (x_k, y_k))$ are recursive numbers, but (a, b) is a nonrecursive point.

The following corollary is an immediate consequence of Proposition 1.

Corollary 1. *Suppose $(\mathbb{R}^n, d, \alpha)$ is a computable metric space, $f : \mathbb{R}^n \to \mathbb{R}^n$ an isometry and $a_0, \dots, a_n$ recursive points in $(\mathbb{R}^n, d, \alpha)$ which are geometrically independent and such that $f(a_0), \dots, f(a_n)$ are recursive points in $\mathbb{R}^n$ in the usual sense. Then $f \circ \alpha$ is a recursive sequence in the usual sense.*

The next step in proving that every two effective separating sequences in $(\mathbb{R}^n, d)$ are equivalent up to isometry is the following lemma.

Lemma 1. *Let $a_0, \dots, a_n$ be geometrically independent points in $\mathbb{R}^n$ such that $d(a_i, a_j)$ is a recursive number for all $i, j \in \{0, \dots, n\}$. Then there exists an isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(a_0), \dots, f(a_n)$ are recursive points.*

The idea in the proof of Lemma 1 is to find an isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(a_0) = (0, \dots, 0)$, $f(a_i) \in \{(t_1, \dots, t_i, 0, \dots, 0) \mid t_1, \dots, t_i \in \mathbb{R},\ t_i \neq 0\}$, $\forall i \in \{1, \dots, n\}$ and then to show that the points $f(a_0), \dots, f(a_n)$ are recursive.

Proposition 2. *Let (α_i) be an effective separating sequence in $\mathbb{R}^n$. Then there exists an isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $(f(\alpha_i))$ is a recursive sequence in $\mathbb{R}^n$.*

Proof. Let $i_0, \dots, i_n \in \mathbb{N}$ be such that $\alpha_{i_0}, \dots, \alpha_{i_n}$ are geometrically independent points. By Lemma 1 there exists an isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(a_{i_0}), \dots, f(a_{i_n})$ are recursive points. The claim of the theorem now follows from Corollary 1. □

Note the following: if (x_i) and (y_i) are recursive dense sequences in $\mathbb{R}^n$, then (x_i) and (y_i) are equivalent as effective separating sequences. This and Proposition 2 imply the following.

Theorem 1. *If α and β are effective separating sequences in $(\mathbb{R}^n, d)$, then α and β are equivalent up to isometry.*

Euclidean space $\mathbb{R}^n$ is not totally bounded, but each open (or closed) ball in $\mathbb{R}^n$ is totally bounded. We say that a computable metric space (X, d, α) **can be exhausted effectively by totally bounded balls** if there exists $\tilde{x} \in X$ and a recursive function $F : \mathbb{N}^2 \to \mathbb{N}$ such that

$$B(\tilde{x}, m) \subseteq \bigcup_{i=0}^{F(k,m)} B(\alpha_i, 2^{-k}),$$

$\forall k, m \in \mathbb{N}$. It is clear that if such a function F exists for one $\tilde{x} \in X$, then it exists for each $\tilde{x} \in X$. It is obvious that each effectively totally bounded computable metric space can be exhausted effectively by totally bounded balls. Furthermore, if α is some recursive dense sequence in $\mathbb{R}^n$, then $(\mathbb{R}^n, d, \alpha)$ can be exhausted effectively by totally bounded balls. It is easy to conclude from this and Theorem 1 that any computable metric space of the form $(\mathbb{R}^n, d, \alpha)$ can be exhausted effectively by totally bounded balls.

In the contrast to the fact that the equivalence (1) holds in general, which will be proved later, the equivalence

(X, d, α) can be exhausted effectively by totally bounded balls

$$\Updownarrow$$

(X, d, β) can be exhausted effectively by totally bounded balls

does not hold in general, as the following example shows.

Example 4. Let the number γ be as in Example 2. It is easy to construct a recursive sequence of rational numbers α' which is dense in $\langle -\infty, \gamma]$. Let d be the Euclidean metric on $\langle -\infty, 0]$ and let (x_i) be some recursive sequence of real numbers which is dense in $\langle -\infty, 0]$. Then the computable metric space $(\langle -\infty, 0], d, (x_i))$ can be exhausted effectively by totally bounded balls. On the other hand, if $\alpha : \mathbb{N} \to \langle -\infty, 0]$ is defined by $\alpha(i) = \alpha'(i) - \gamma$, then α is an effective separating sequence in $(\langle -\infty, 0], d)$ and the computable metric space $(\langle -\infty, 0], d, \alpha)$ cannot be exhausted effectively by totally bounded balls which can be deduced from the fact that 0 is not a recursive point in this space.

The previous example also shows that effective separating sequences in a metric space (X, d) need not be equivalent up to isometry. The following two examples show that effective separating sequences in (X, d) need not be equivalent up to isometry even when (X, d) is totally bounded.

Example 5. Let $([0,\gamma],d,\alpha)$ be the computable metric space of Example 2. Let $\alpha' : \mathbb{N} \to \mathbb{R}$ be defined by $\alpha'(2i) = \frac{\alpha(i)}{2}$, $\alpha'(2i+1) = -\frac{\alpha(i)}{2}$, $i \in \mathbb{N}$ and let $\alpha'' : \mathbb{N} \to [0,\gamma]$ be defined by $\alpha''(i) = \alpha'(i) + \frac{\gamma}{2}$. Then α'' is an effective separating sequence in $([0,\gamma],d)$. Since the point $\frac{\gamma}{2}$ is recursive in $([0,\gamma],d,\alpha'')$, but not in $([0,\gamma],d,\alpha)$, and since $\frac{\gamma}{2}$ is a fixed point of each isometry of $([0,\gamma],d)$ (namely the only isometries are the identity and the map $t \mapsto \gamma - t$, $t \in [0,\gamma]$), we conclude that effective separating sequences α and α'' are not equivalent.

Example 6. Let S be the unit circle in $\mathbb{R}^2$ and let d be the Euclidean metric on S. Since S is a recursive set, there exists a recursive sequence (x_i) in S such that $(S,d,(x_i))$ is an effectively totally bounded computable metric space (Example 1). Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation with the center $(0,0)$ such that $f(1,0)$ is a nonrecursive point. Then $(f(x_i))$ is an effective separating sequence in (S,d) nonequivalent to (x_i). Let $A = \{x_i \mid i \in \mathbb{N}\} \cup \{f(x_i) \mid i \in \mathbb{N}\}$, let $T = \{(x,y) \in S \mid x \leq 0 \text{ or } (x,y) \in A\}$ and let d' be the Euclidean metric on T. Then (x_i) and (y_i) are effective separating sequences in (T,d') and it follows easily that they are not equivalent up to isometry in this metric space.

3 Effective dispersion

Let (X,d) be a metric space. A nonempty subset S of X is said to be $r-$**dense** in (X,d), where $r \in \mathbb{R}$, $r > 0$, if $X = \cup_{s \in S} B(s,r)$. Note that a set S is dense in (X,d) if and only if S is $r-$dense in (X,d) for all $r > 0$. We say that a finite sequence $x_0,\dots,x_n$ of points in X is $r-$dense in (X,d) if the set $\{x_0,\dots,x_n\}$ is $r-$dense in (X,d). Hence (X,d) is totally bounded if and only if for each $\varepsilon > 0$ there exists a finite sequence of points in X which is $\varepsilon-$dense in (X,d).

Let $s \in \mathbb{R}$. A nonempty subset S of X is said to be $s-$**dispersed** in (X,d) if $d(x,y) > s$, $\forall x,y \in S$, $x \neq y$. A finite sequence $x_0,\dots,x_n$ of points in X is said to be $s-$**dispersed** in (X,d) if $d(x_i,x_j) > s$, $\forall i,j \in \{0,\dots,n\}$, $i \neq j$. Note that if $x_0,\dots,x_n$ is an s-dispersed finite sequence, then $\{x_0,\dots,x_n\}$ is an $s-$dispersed set, while converse does not hold in general.

Proposition 3. *Let (X,d) be a totally bounded metric space and let $s > 0$. Then the set $A = \{k \in \mathbb{N} \mid$ there exists a finite sequence $x_1,\dots,x_k$ which is $s-$dispersed in $(X,d)\}$ is finite.*

Proof. Let $y_0,\dots,y_p$ be an $\frac{s}{2}-$dense finite sequence in (X,d). Suppose that a finite sequence $x_1,\dots,x_k$ is $s-$dispersed. For each $i \in \{1,\dots,k\}$ let $j_i \in \{0,\dots,p\}$ be such that $x_i \in B(y_{j_i},\frac{s}{2})$. If $i,i' \in \{1,\dots,k\}$, $i \neq i'$, then $j_i \neq j_{i'}$ since $d(x_i,x_{i'}) > s$. Therefore we have an injection $\{1,\dots,k\} \to \{0,\dots,p\}$, hence $k < p$. This shows that A is finite. □

Let (X,d) be a totally bounded metric space. If $S \subseteq X$, $S \neq \emptyset$, and $s > 0$, then, by Proposition 3, the set $\{k \in \mathbb{N} \mid$ there exists a finite sequence $x_1,\dots,x_k$ of points in S which is $s-$dispersed in $(X,d)\}$ is finite. We denote the maximum of this set by $\rho(S,s)$. If $x_0,\dots,x_n$ is a finite sequence in X, then we will write $\rho(x_0,\dots,x_n;s)$ instead of $\rho(\{x_0,\dots,x_n\},s)$.

Example 7. With the Euclidean metric on $[0,3]$ we have $\rho([0,1],s)=1$ if $s\geq 1$, $\rho([0,1],s)=2$ if $s\in\left[\frac{1}{2},1\right\rangle$ and $\rho(0,1,3;s)=\begin{cases}1,\ 3\leq s,\\ 2,\ 1\leq s<3,\\ 3,\ 0<s<1.\end{cases}$

Suppose (X,d) is a totally bounded metric space, $s>0$ and $n=\rho(X,\frac{s}{2})$. Then there exists a finite sequence $x_0,\dots,x_{n-1}$ which is $\frac{s}{2}-$dispersed in (X,d) and such that the finite sequence $a,x_0,\dots,x_{n-1}$ is not $\frac{s}{2}-$dispersed for each $a\in X$. Therefore for each $a\in X$ there exists $i\in\{0,\dots,n-1\}$ such that $d(a,x_i)<s$. Hence the finite sequence $x_0,\dots,x_{n-1}$ is $s-$dense.

Now, let α and β be effective separating sequences in (X,d) such that the computable metric space (X,d,α) is effectively totally bounded. In order to prove that (X,d,β) is also effectively totally bounded, it would be enough to prove that for each $k\in\mathbb{N}$ we can effectively find the number $\rho(X,2^{-k})$. Namely, in that case for any $k\in\mathbb{N}$ we can effectively find $i_1,\dots,i_n\in\mathbb{N}$ such that the finite sequence $\beta_{i_1},\dots,\beta_{i_n}$ is $2^{-(k+1)}-$dispersed, where $n=\rho(X,2^{-(k+1)})$ and then this finite sequence of points (and consequently the finite sequence $\beta_0,\dots,\beta_{\max\{i_1,\dots,i_n\}}$) must be $2^{-k}-$dense. However, the number $\rho(X,2^{-k})$ cannot be found effectively in general, as the following example shows.

Example 8. Let (λ_i) be a recursive sequence of real numbers such that $\lambda_i\geq 0$, $\forall i\in\mathbb{N}$ and such that the set $\{i\in\mathbb{N}\mid\lambda_i=0\}$ is not recursive ([2]). We may assume $\lambda_i<4^{-i}$, $\forall i\in\mathbb{N}$. Let $t_i=4^{-i}+\lambda_i$, $i\in\mathbb{N}$, $X=\{t_i\mid i\in\mathbb{N}\}\cup\{0\}$ and let d be the Euclidean metric on X. Then $(X,d,(t_i))$ is an effectively totally bounded computable metric space. Let $i\in\mathbb{N}$. It is straightforward to check that $\rho(X,4^{-i})=i+1$ if $\lambda_i=0$ and $\rho(X,4^{-i})=i+2$ if $\lambda_i>0$. Therefore the function $\mathbb{N}\to\mathbb{N}$, $i\mapsto\rho(X,2^{-i})$ is not recursive.

Although $\rho(X,2^{-k})$ cannot be found effectively in general, we are going to prove that for $k\in\mathbb{N}$ we can effectively find numbers $a_k\in\langle 0,2^{-k}\rangle\cap\mathbb{Q}$ and $\rho(X,a_k)$ and this will imply that (X,d,β) if effectively totally bounded.

Suppose (X,d) is a totally bounded metric space, $S\subseteq X$, $S\neq\emptyset$ and $s>0$. It is immediate from the definition of the number $\rho(S,s)$ that there exists $r>0$ such that $\rho(S,s)=\rho(S,s+2r)$. Here, of course, r depends on S and s. In the following lemma we claim that s and r can be chosen so that $\rho(S,s)=\rho(S,s+2r)$ holds whenever S is in certain family of subsets of X.

Lemma 2. *Let (X,d) be a totally bounded metric space and let $s_0>0$. Then there exists $r_0>0$ such that for each $r\in\langle 0,r_0\rangle$ and each finite sequence $x_0,\dots,x_p$ which is $r-$dense in (X,d) there exists $s\in[s_0,s_0+r\rangle\cap\mathbb{Q}$ and $m_1,\dots,m_n\in\{0,\dots,p\}$ such that the finite sequence $x_{m_1},\dots,x_{m_n}$ is $(s+2r)-$dispersed, $d(x_i,x_j)\neq s$, $\forall i,j\in\{0,\dots,p\}$ and $\rho(x_0,\dots,x_p;s)=n$.*

Proof. Let $n=\rho(X,s_0)$ and $y_1,\dots,y_n$ be a finite sequence which is s_0-dispersed in (X,d). Since $d(y_i,y_j)>s_0$, $\forall i,j\in\{1,\dots,n\}$, $i\neq j$, there exists $r_0>0$ such that

$$d(y_i,y_j)>s_0+5r_0,\ \forall i,j\in\{1,\dots,n\},\ i\neq j.$$

Let $r \in \langle 0, r_0 \rangle$ and let $x_0, \dots, x_p$ be an $r-$dense sequence in (X, d). For $i \in \{1, \dots, n\}$ let $m_i \in \{0, \dots, p\}$ be such that $y_i \in B(x_{m_i}, r)$. If $i, j \in \{1, \dots, n\}$, $i \neq j$, then

$$d(y_i, y_j) \leq d(y_i, x_{m_i}) + d(x_{m_i}, x_{m_j}) + d(x_{m_j}, y_j) < 2r + d(x_{m_i}, x_{m_j})$$

and therefore

$$d(x_{m_i}, x_{m_j}) > s_0 + 3r, \tag{3}$$

$\forall i, j \in \{1, \dots, n\}$, $i \neq j$. Let $s \in [s_0, s_0 + r\rangle \cap \mathbb{Q}$ be such that $d(x_i, x_j) \neq s$, $\forall i, j \in \{0, \dots, p\}$. From (3) we get that $d(x_{m_i}, x_{m_j}) > s + 2r$, $\forall i, j \in \{1, \dots, n\}$, $i \neq j$, hence the finite sequence $x_{m_1}, \dots, x_{m_n}$ is $s + 2r-$dispersed. This implies $\rho(x_0, \dots, x_p; s) \geq n$. On the other hand

$$\rho(x_0, \dots, x_p; s) \leq \rho(X, s) \leq \rho(X, s_0) = n.$$

Therefore $\rho(x_0, \dots, x_p; s) = n$. □

The next lemma provides conditions under which equality $\rho(X, s+2r) = \mathrm{card}(T)$ holds, where $T \subseteq X$ and $s, r > 0$.

Lemma 3. *Let (X, d) be a totally bounded metric space, $r, s > 0$ and let S be an $r-$dense subset of X such that there exists a finite nonempty subset T of S which is $s+2r$ dispersed and such that $\rho(S, s) = \mathrm{card}(T)$. Then $\rho(X, s+2r) = \mathrm{card}(T)$.*

Proof. Certainly $\rho(X, s + 2r) \geq \mathrm{card}(T)$. On the other hand, let $x_1, \dots, x_n$ be an $(s + 2r)-$dispersed sequence in (X, d). For each $i \in \{1, \dots, n\}$ let $y_i \in S$ be such that $d(x_i, y_i) < r$. For all $i, j \in \{1, \dots, n\}$, $i \neq j$, we have

$$s + 2r < d(x_i, x_j) \leq d(x_i, y_i) + d(y_i, y_j) + d(y_j, x_j) < d(y_i, y_j) + 2r$$

which implies $s < d(y_i, y_j)$. Hence $y_1, \dots, y_n$ is an $s-$dispersed sequence and therefore $\rho(S, s) \geq n$, i.e. $\mathrm{card}(T) \geq n$. We conclude that $\mathrm{card}(T) \geq \rho(X, s + 2r)$ and it follows $\rho(X, s + 2r) = \mathrm{card}(T)$. □

Lemma 3, together with Lemma 2, gives the idea how to compute the number $\rho(X, s+2r)$, $s, r > 0$. The next step is to include effectiveness into consideration. We first state the following lemma.

Lemma 4. *Let $F : \mathbb{N}^4 \to \mathbb{R}$ be a recursive function. Let S be the set of all $(k, n, l, p) \in \mathbb{N}^4$ such that $F(i, j, n, l) \neq 0$, $\forall i, j \in \{0, \dots, k\}$ and such that*

$$\mathrm{card}\{(i, j) \in \{0, \dots, k\} \times \{0, \dots, k\} \mid F(i, j, n, l) > 0\} = p.$$

Then S is a recursively enumerable set.

Let $\sigma : \mathbb{N}^2 \to \mathbb{N}$ and $\eta : \mathbb{N} \to \mathbb{N}$ be some fixed recursive functions with the following property: $\{(\sigma(j, 0), \dots, \sigma(j, \eta(j))) \mid j \in \mathbb{N}\}$ is the set of all finite sequences in $\mathbb{N}$, i.e. the set $\{(a_0, \dots, a_n) \mid n \in \mathbb{N},\ a_0, \dots, a_n \in \mathbb{N}\}$. Such functions, for instance, can be defined using the Cantor pairing function. We are going to use the following notation: $(j)_i$ instead of $\sigma(j,i)$ and $\overline{j}$ instead of $\eta(j)$. Hence

$$\{((j)_0, \dots, (j)_{\overline{j}}) \mid j \in \mathbb{N}\}$$

is the set of all finite sequences in $\mathbb{N}$.

Suppose (X,d) is a metric space, (γ_i) a sequence in X such that the function $\mathbb{N}^2 \to \mathbb{R}$, $(i,j) \mapsto d(\gamma_i, \gamma_j)$ is recursive and (s_n) a recursive sequence of real numbers. Then the function $\mathbb{N}^2 \to \mathbb{N}$, $(k,n) \mapsto \rho(\gamma_0, \dots, \gamma_k; s_n)$ need not be recursive and we see this similarly as in Example 8. However, we have the following lemma.

Lemma 5. *Let (X,d) be a metric space, (γ_i) a sequence in X such that the function $\mathbb{N}^2 \to \mathbb{R}$, $(i,j) \mapsto d(\gamma_i, \gamma_j)$ is recursive and (s_n) a recursive sequence of real numbers.*

(i) The set

$$S = \{(k,n,p) \in \mathbb{N}^3 \mid d(\gamma_i, \gamma_j) \neq s_n,\ \forall i,j \in \{0, \dots, k\},\ \rho(\gamma_0, \dots, \gamma_k; s_n) = p\}$$

is recursively enumerable.

(ii) The set

$$T = \{(l,n) \in \mathbb{N}^2 \mid \textit{the finite sequence } \gamma_{(l)_0}, \dots, \gamma_{(l)_{\overline{l}}} \textit{ is } s_n - \textit{dispersed}\}$$

is recursively enumerable.

Proof. (i) We apply Lemma 4 to the function $F : \mathbb{N}^4 \to \mathbb{R}$ defined by

$$F(i,j,n,l) = d(\gamma_i, \gamma_j) - s_n,$$

$i,j,n,l \in \mathbb{N}$ and we get that the set

$$\{(k,n,l,p) \in \mathbb{N}^4 \mid d(\gamma_i, \gamma_j) \neq s_n,\ \forall i,j \in \{0, \dots, k\},\ \rho(\gamma_0, \dots, \gamma_k; s_n) = p\}$$

is r.e. which implies that S is r.e.

(ii) Let $F : \mathbb{N}^4 \to \mathbb{R}$ be given by $F(i,j,n,l) = d(\gamma_{(l)_i}, \gamma_{(l)_j}) - s_n$. Let T' be the set associated to F as in Lemma 4, hence $T' = \{(k,n,l,p) \in \mathbb{N}^4 \mid d(\gamma_{(l)_i}, \gamma_{(l)_j}) \neq s_n,\ \forall i,j \in \{0, \dots, k\}$ and $\rho(\gamma_{(l)_0}, \dots, \gamma_{(l)_k}; s_n) = p\}$. Then for all $l,n \in \mathbb{N}$ we have $(l,n) \in T$ if and only if

$$d(\gamma_{(l)_i}, \gamma_{(l)_j}) \neq s_n, \forall i,j \in \{0, \dots, \bar{l}\},\ \rho(\gamma_{(l)_0}, \dots, \gamma_{(l)_{\bar{l}}}; s_n) = \bar{l} + 1$$

and this holds if and only if $(\bar{l}, n, l, \bar{l}+1) \in T'$. Therefore T is r.e. □

A totally bounded metric space (X,d) is said to be **effectively dispersed** if there exists a recursive function $a : \mathbb{N} \to \mathbb{Q}$ such that $a(i) \in \langle 0, 2^{-i} \rangle$, $\forall i \in \mathbb{N}$ and such that the function $\mathbb{N} \to \mathbb{N}$, $i \mapsto \rho(X, a(i))$ is recursive.

Theorem 2. *Let (X,d,α) be an effectively totally bounded computable metric space. Then (X,d) is effectively dispersed.*

Proof (Sketch). Let $f : \mathbb{N} \to \mathbb{N}$ be a recursive function such that $\alpha_0, \dots, \alpha_{f(k)}$ is a 2^{-k}−dense sequence for each $k \in \mathbb{N}$. Let $q : \mathbb{N} \to \mathbb{Q}$ be some fixed recursive function whose image is $\mathbb{Q} \cap \langle 0, \infty\rangle$.

Let $i \in \mathbb{N}$. Let s_0 be some positive number such that $s_0 < 2^{-i}$. By Lemma 2 there exist $k, n, l \in \mathbb{N}$ such that $s_0 + 3 \cdot 2^{-k} < 2^{-i}$, $q_n \in [s_0, s_0 + 2^{-k}\rangle$ and such that the following holds:

$$\alpha_{(l)_0}, \dots, \alpha_{(l)_{\bar{l}}} \text{ is } (q_n + 2 \cdot 2^{-k}) - \text{dispersed finite sequence,} \tag{4}$$

$$d(\alpha_i, \alpha_j) \neq q_n, \forall i, j \in \{0, \dots, f(k)\},\ \rho(\alpha_0, \dots, \alpha_{f(k)}; q_n) = \bar{l} + 1, \tag{5}$$

and

$$\{(l)_0, \dots, (l)_{\bar{l}}\} \subseteq \{0, \dots, f(k)\}. \tag{6}$$

Since (4) and (5) are r.e. relations (Lemma 5) and (6) is recursive, we can express n, k and l recursively in i. The claim of the theorem follows from

$$\rho(X, q_n + 2 \cdot 2^{-k}) = \bar{l} + 1,$$

and this equality can be deduced from Lemma 3. □

Theorem 3. *Let (X, d, α) be a computable metric space such that (X, d) is effectively dispersed. Then (X, d, α) is effectively totally bounded.*

The idea in the proof of Theorem 3 is that for a given $i \in \mathbb{N}$ we effectively find $i_1, \dots, i_n \in \mathbb{N}$ such that the finite sequence $\alpha_{i_1}, \dots, \alpha_{i_n}$ is s−dispersed, where $s \in \langle 0, 2^{-(i+1)}\rangle \cap \mathbb{Q}$ and $n = \rho(X, s)$. Then the finite sequence of points $\alpha_{i_1}, \dots, \alpha_{i_n}$ must be 2^{-i}−dense which shows that (X, d, α) is effectively totally bounded.

Let (X, d, α) be a computable metric space. Theorem 2 and Theorem 3 give the following equivalence:

$$(X, d, \alpha) \text{ is effectively totally bounded } \Leftrightarrow (X, d) \text{ is effectively dispersed .}$$

Corollary 2. *Let α and β be effective separating sequences in a metric space (X, d). Then (X, d, α) is effectively totally bounded if and only if (X, d, β) is effectively totally bounded.*

A computable metric space (X, d, α) is said to be **effectively compact** if (X, d, α) is effectively totally bounded and (X, d) is compact (cf. [3]). If α and β are effective separating sequences in a metric space (X, d), then, by Corollary 2, (X, d, α) is effectively compact if and only if (X, d, β) is effectively compact.

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