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Curves That Must Be Retraced

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Abstract. We exhibit a polynomial time computable plane curve $\mathbf{\Gamma}$ that has finite length, does not intersect itself, and is smooth except at one endpoint, but has the following property. For every computable parametrization f of $\mathbf{\Gamma}$ and every positive integer m, there is some positive-length subcurve of $\mathbf{\Gamma}$ that f retraces at least m times. In contrast, every computable curve of finite length that does not intersect itself has a constant-speed (hence non-retracing) parametrization that is computable relative to the halting problem.

1 Introduction

A curve is a mathematical model of the path of a particle undergoing continuous motion. Specifically, in a Euclidean space $\mathbb{R}^n$, a curve is the range Γ of a continuous function $f : [a, b] \to \mathbb{R}^n$ for some $a < b$. The function f, called a *parametrization* of Γ, clearly contains more information than the pointset Γ, namely, the precise manner in which the particle "traces" the points $f(t) \in \Gamma$ as t, which is often considered a time parameter, varies from a to b. When the particle's motion is algorithmically governed, the parametrization must be computable (as a function on the reals; see below).

This paper shows that the geometry of a curve Γ may force every *computable* parametrization f of Γ to retrace various parts of its path (i.e., "go back and forth along Γ") many times, even when Γ is an efficiently computable, smooth, finite-length curve that does not intersect itself. In fact, our main theorem exhibits a plane curve $\mathbf{\Gamma} \subseteq \mathbb{R}^2$ with the following properties.

1. $\mathbf{\Gamma}$ is *simple*, i.e., it does not intersect itself.

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2. $\mathbf{\Gamma}$ is *rectifiable*, i.e., it has finite length.
3. $\mathbf{\Gamma}$ is *smooth except at one endpoint*, i.e., $\mathbf{\Gamma}$ has a tangent at every interior point and a 1-sided tangent at one endpoint, and these tangents vary continuously along $\mathbf{\Gamma}$.
4. $\mathbf{\Gamma}$ is *polynomial time computable* in the strong sense that there is a polynomial time computable position function $\vec{s} : [0,1] \to \mathbb{R}^2$ such that the velocity function $\vec{v} = \vec{s}'$ and the acceleration function $\vec{a} = \vec{v}'$ are polynomial time computable; the total distance traversed by $\vec{s}$ is finite; and $\vec{s}$ parametrizes $\mathbf{\Gamma}$, i.e., range$(\vec{s}) = \mathbf{\Gamma}$.
5. $\mathbf{\Gamma}$ *must be retraced* in the sense that every parametrization $f : [a,b] \to \mathbb{R}^2$ of $\mathbf{\Gamma}$ that is computable in *any* amount of time has the following property. For every positive integer m, there exist disjoint, closed subintervals $I_0, \ldots, I_m$ of $[a,b]$ such that the curve $\Gamma_0 = f(I_0)$ has positive length and $f(I_i) = \Gamma_0$ for all $1 \leq i \leq m$. (Hence f retraces Γ_0 at least m times.)

The terms "computable" and "polynomial time computable" in properties 4 and 5 above refer to the "bit-computability" model of computation on reals formulated in the 1950s by Grzegorczyk [9] and Lacombe [17], extended to feasible computability in the 1980s by Ko and Friedman [13] and Kreitz and Weihrauch [16], and exposited in the recent paper by Braverman and Cook [4] and the monographs [20,14,22,5]. As will be shown here, condition 4 also implies that the pointset $\mathbf{\Gamma}$ is polynomial time computable in the sense of Brattka and Weihrauch [2]. (See also [22,3,4].)

A fundamental and useful theorem of classical analysis states that every simple, rectifiable curve Γ has a *normalized constant-speed parametrization*, which is a one-to-one parametrization $f : [0,1] \to \mathbb{R}^n$ of Γ with the property that $f([0,t])$ has arclength tL for all $0 \leq t \leq 1$, where L is the length of Γ. (A simple, rectifiable curve Γ has exactly two such parametrizations, one in each direction, and standard terminology calls either of these *the* normalized constant-speed parametrization $f : [0,1] \to \mathbb{R}^n$ of Γ. The constant-speed parametrization is also called the *parametrization by arclength* when it is reformulated as a function $f : [0,L] \to \mathbb{R}^n$ that moves with constant speed 1 along Γ.) Since the constant-speed parametrization does not retrace any part of the curve, our main theorem implies that this classical theorem is not entirely constructive. Even when a simple, rectifiable curve has an efficiently computable parametrization, the constant-speed parametrization need not be computable.

In addition to our main theorem, we prove that every simple, rectifiable curve Γ in $\mathbb{R}^n$ with a computable parametrization has the following two properties.

I. The length of Γ is lower semicomputable.
II. The constant-speed parametrization of Γ is computable relative to the length of Γ.

These two things are not hard to prove if the computable parametrization is one-to-one, (in fact, they follow from results of Müller and Zhao [19] in this case) but our results hold even when the computable parametrization retraces portions of the curve many times.

Taken together, I and II have the following two consequences.

1. The curve $\mathbf{\Gamma}$ of our main theorem has a finite length that is lower semi-computable but not computable. (The existence of polynomial-time computable curves with this property was first proven by Ko [15].)
2. Every simple, rectifiable curve Γ in $\mathbb{R}^n$ with a computable parametrization has a constant-speed parametrization that is Δ_2^0-computable, i.e., computable relative to the halting problem. Hence, the existence of a constant-speed parametrization, while not entirely constructive, is constructive relative to the halting problem.

2 Length, Computability, and Complexity of Curves

In this section we summarize basic terminology and facts about curves. As we use the terms here, a *curve* is the range Γ of a continuous function $f : [a, b] \to \mathbb{R}^n$ for some $a < b$. The function f is called a *parametrization* of Γ. Each curve clearly has infinitely many parametrizations.

A curve is *simple* if it has a parametrization that is one-to-one, i.e., the curve "does not intersect itself". The length of a simple curve Γ is defined as follows. Let $f : [a, b] \overset{1-1}{\to} \mathbb{R}^n$ be a one-to-one parametrization of Γ. For each *disection* $\vec{t}$ of $[a, b]$, i.e., each tuple $\vec{t} = (t_0, \ldots, t_m)$ with $a = t_0 < t_1 < \ldots < t_m = b$, define the *$f$-$\vec{t}$-approximate length* of Γ to be

$$\mathcal{L}_{\vec{t}}^{f}(\Gamma) = \sum_{i=0}^{m-1} |f(t_{i+1}) - f(t_i)|.$$

Then the *length* of Γ is

$$\mathcal{L}(\Gamma) = \sup_{\vec{t}} \mathcal{L}_{\vec{t}}^{f}(\Gamma),$$

where the supremum is taken over all dissections $\vec{t}$ of $[a, b]$. It is easy to show that $\mathcal{L}(\Gamma)$ does not depend on the choice of the one-to-one parametrization f, i.e. that the length is an intrinsic property of the pointset Γ.

In sections 4 and 5 of this paper we use a more general notion of length, namely, the 1-dimensional Hausdorff measure $\mathcal{H}^1(\Gamma)$, which is defined for *every* set $\Gamma \subseteq \mathbb{R}^n$. We refer the reader to [7] for the definition of $\mathcal{H}^1(\Gamma)$. It is well known that $\mathcal{H}^1(\Gamma) = \mathcal{L}(\Gamma)$ holds for every simple curve Γ.

A curve Γ is *rectifiable*, or *has finite length* if $\mathcal{L}(\Gamma) < \infty$. In sections 4 and 5 we use the notation $\mathcal{RC}$ for the set of all rectifiable simple curves.

Definition. Let $f : [a, b] \to \mathbb{R}^n$ be continuous.

1. For $m \in \mathbb{Z}^+$, f has *m-fold retracing* if there exist disjoint, closed subintervals $I_0, \ldots, I_m$ of $[a, b]$ such that the curve $\Gamma_0 = f(I_0)$ has positive length and $f(I_i) = \Gamma_0$ for all $1 \leq i \leq m$.
2. f is *non-retracing* if f does not have 1-fold retracing.
3. f has *bounded retracing* if there exists $m \in \mathbb{Z}^+$ such that f does not have m-fold retracing.
4. f has *unbounded retracing* if f does not have bounded retracing, i.e., if f has m-fold retracing for all $m \in \mathbb{Z}^+$.

We now review the notions of computability and complexity of a real-valued function. An *oracle* for a real number t is any function $O_t : \mathbb{N} \to \mathbb{Q}$ with the property that $|O_t(s) - t| \leq 2^{-s}$ holds for all $s \in \mathbb{N}$. A function $f : [a, b] \to \mathbb{R}^n$ is *computable* if there is an oracle Turing machine M with the following property. For every $t \in [a, b]$ and every precision parameter $r \in \mathbb{N}$, if M is given r as input and *any* oracle O_t for t as its oracle, then M outputs a rational point $M^{O_t}(r) \in \mathbb{Q}^n$ such that $|M^{O_t}(r) - f(t)| \leq 2^{-r}$. A function $f : [a, b] \to \mathbb{R}^n$ is *computable in polynomial time* if there is an oracle machine M that does this in time polynomial in $r + l$, where l is the maximum length of the query responses provided by the oracle.

An *oracle* for a function $f : [a, b] \to \mathbb{R}^n$ is any function $\mathcal{O}_f : ([a, b] \cap \mathbb{Q}) \times \mathbb{N} \to \mathbb{Q}^n$ with the property that $|\mathcal{O}_f(q, r) - f(q)| \leq 2^{-r}$ holds for all $q \in [a, b] \cap \mathbb{Q}$ and $r \in \mathbb{N}$. A decision problem A is *Turing reducible* to a function $f : [a, b] \to \mathbb{R}^n$, and we write $A \leq_{\mathrm{T}} f$, if there is an oracle Turing machine M such that, for every oracle $\mathcal{O}_f$ for f, $M^{\mathcal{O}_f}$ decides A. It is easy to see that, if f is computable, then $A \leq_{\mathrm{T}} f$ if and only if A is decidable.

A curve is *computable* if it has a parametrization $f : [a, b] \to \mathbb{R}^n$, where $a, b \in \mathbb{Q}$ and f is computable. A curve is *computable in polynomial time* if it has a parametrization that is computable in polynomial time.

3 An Efficiently Computable Curve That Must Be Retraced

This section presents our main theorem, which is the existence of a smooth, rectifiable, simple plane curve $\mathbf{\Gamma}$ that is parametrizable in polynomial time but not computably parametrizable in any amount of time without unbounded retracing. Intuitively, our curve $\mathbf{\Gamma}$ has, for each $n \in \mathbb{N}$, a section of the form illustrated in Figure 3.1. The height $h(n)$ is positive, and the halting problem K is encoded into the width $w(n)$. Oversimplifying a bit, $w(n)$ is $2^{-(n+\tau(n))}$, where $\tau(n)$ is the number of steps executed by the nth Turing machine on input n. Thus $w(n)$ is 0 if $n \in K$, and $w(n)$ is so small as to be "indistinguishable" from 0 if $n \notin K$. The smallness of $w(n)$ implies that we can efficiently compute a parametrization that is retracing when $w(n)$ is 0. However, as we show in Lemma 3.12, a *nonretracing* parametrization must have a vertical component that distinguishes the case $w(n) = 0$ from the case $w(n) > 0$, and hence must solve the halting problem. It follows that no nonretracing parametrization is computable.

We now give a precise construction of the curve $\mathbf{\Gamma}$, followed by a brief discussion of how the construction achieves the intuition that we have just described. The rest of the section is devoted to proving that $\mathbf{\Gamma}$ has the desired properties.

Construction 3.1 *(1) For each $a, b \in \mathbb{R}$ with $a < b$, define the functions $\varphi_{a,b}, \xi_{a,b} : [a, b] \to \mathbb{R}$ by*

$$\varphi_{a,b}(t) = \frac{b-a}{4} \sin \frac{2\pi(t-a)}{b-a}$$

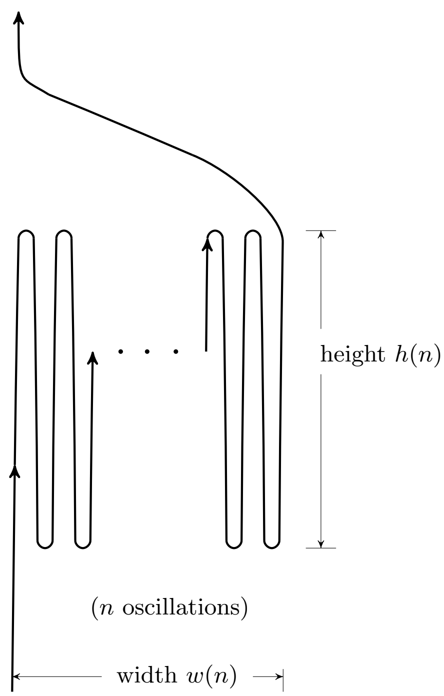

Fig. 3.1. Schematic view of the n^{th} section of $\mathbf{\Gamma}$

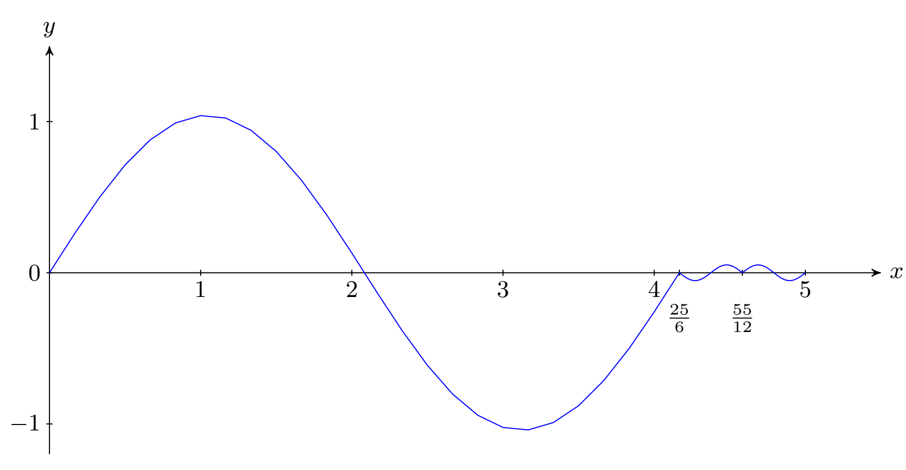

Fig. 3.2. $\psi_{0,5,1}$

and

$$\xi_{a,b}(t) = \begin{cases} -\varphi_{a,\frac{a+b}{2}}(t) & \text{if } a \le t \le \frac{a+b}{2} \\ \varphi_{\frac{a+b}{2},b}(t) & \text{if } \frac{a+b}{2} \le t \le b. \end{cases}$$

(2) *For each* $a, b \in \mathbb{R}$ *with* $a < b$ *and each positive integer* n*, define the function* $\psi_{a,b,n} : [a, b] \to \mathbb{R}$ *by*

$$\psi_{a,b,n}(t) = \begin{cases} \varphi_{a,d_0}(t) & \text{if } a \le t \le d_0 \\ \xi_{d_{i-1},d_i}(t) & \text{if } d_{i-1} \le t \le d_i, \end{cases}$$

where

$$d_i = \frac{a + 5b}{6} + i\frac{b-a}{6n}$$

for $0 \le i \le n$*. (See Figure 3.2.)*

(3) *Fix a standard enumeration* $M_1, M_2, \ldots$ *of (deterministic) Turing machines that take positive integer inputs. For each positive integer* n*, let* $\tau(n)$ *denote the number of steps executed by* M_n *on input* n*. It is well known that the* diagonal halting problem

$$K = \left\{ n \in \mathbb{Z}^+ \mid \tau(n) < \infty \right\}$$

is undecidable.

(4) *Define the horizontal and vertical acceleration functions* $a_x, a_y : [0, 1] \to \mathbb{R}$ *as follows. For each* $n \in \mathbb{N}$*, let*

$$t_n = \int_0^n e^{-x} dx = 1 - e^{-n},$$

noting that $t_0 = 0$ *and that* t_n *converges monotonically to* 1 *as* $n \to \infty$*. Also, for each* $n \in \mathbb{Z}^+$*, let*

$$t_n^- = \frac{t_{n-1} + 4t_n}{5}, \quad t_n^+ = \frac{6t_n - t_{n-1}}{5},$$

noting that these are symmetric about t_n *and that* $t_n^+ \le t_{n+1}^-$*.*

(i) *For* $0 \le t \le 1$*, let*

$$a_x(t) = \begin{cases} -2^{-(n+\tau(n))}\xi_{t_n^-,t_n^+}(t) & \text{if } t_n^- \le t < t_n^+ \\ 0 & \text{if no such } n \text{ exists,} \end{cases}$$

where $2^{-\infty} = 0$*.*

(ii) *For* $0 \le t < 1$*, let*

$$a_y(t) = \psi_{t_{n-1},t_n,n}(t),$$

where n *is the unique positive integer such that* $t_{n-1} \le t < t_n$*.*

(iii) *Let* $a_y(1) = 0$*.*

(5) Define the horizontal and vertical velocity and position functions $v_x, v_y, s_x, s_y : [0,1] \to \mathbb{R}$ *by*

$$v_x(t) = \int_0^t a_x(\theta)d\theta, \quad v_y(t) = \int_0^t a_y(\theta)d\theta,$$
$$s_x(t) = \int_0^t v_x(\theta)d\theta, \quad s_y(t) = \int_0^t v_y(\theta)d\theta.$$

(6) Define the vector acceleration, velocity, and position functions $\vec{a}, \vec{v}, \vec{s} : [0,1] \to \mathbb{R}^2$ *by*

$$\vec{a}(t) = (a_x(t), a_y(t)),$$
$$\vec{v}(t) = (v_x(t), v_y(t)),$$
$$\vec{s}(t) = (s_x(t), s_y(t)).$$

(7) Let $\mathbf{\Gamma} = \text{range}(\vec{s})$.

Intuitively, a particle at rest at time $t = a$ and moving with acceleration given by the function $\varphi_{a,b}$ moves forward, with velocity increasing to a maximum at time $t = \frac{a+b}{2}$ and then decreasing back to 0 at time $t = b$. The vertical acceleration function a_y, together with the initial conditions $v_y(0) = s_y(0) = 0$ implied by (5), thus causes a particle to move generally upward (i.e., $s_y(t_0) < s_y(t_1) < \cdots$), coming to momentary rests at times $t_1, t_2, t_3, \ldots$. Between two consecutive such stopping times t_{n-1} and t_n, the particle's vertical acceleration is controlled by the function $\psi_{t_{n-1},t_n,n}$. This function causes the particle's vertical motion to do the following between times t_{n-1} and t_n.

(i) From time t_{n-1} to time $\frac{t_{n-1}+5t_n}{6}$, move upward from elevation $s_y(t_{n-1})$ to elevation $s_y(t_n)$.
(ii) From time $\frac{t_{n-1}+5t_n}{6}$ to time t_n, make n round trips to a lower elevation $s \in (s_y(t_{n-1}), s_y(t_n))$.

In the meantime, the horizontal acceleration function a_x, together with the initial conditions $v_x(0) = s_x(0) = 0$ implied by (5), ensure that the particle remains on or near the y-axis. The deviations from the y-axis are simply described: The particle moves to the right from time $\frac{t_{n-1}+4t_n}{5}$ through the completion of the n round trips described in (ii) above and then moves to the y-axis between times t_n and $\frac{6t_n - t_{n-1}}{5}$. The amount of lateral motion here is regulated by the coefficient $2^{-(n+\tau(n))}$. If $\tau(n) = \infty$, then there is no lateral motion, and the n round trips in (ii) are retracings of the particle's path. If $\tau(n) < \infty$, then these n round trips are "forward" motion along a curvy part of $\mathbf{\Gamma}$. In fact, $\mathbf{\Gamma}$ contains points of arbitrarily high curvature, but the particle's motion is kinematically realistic in the sense that the acceleration vector $\vec{a}(t)$ is polynomial time computable, hence continuous and bounded on the interval $[0,1]$. Figure 3.3 illustrates the path of the particle from time t_{n-1} to t_{n+1} with $n = 1$ and hypothetical (model dependent!) values $\tau(1) = 1$ and $\tau(2) = 2$.

The rest of this section is devoted to proving the following theorem concerning the curve $\mathbf{\Gamma}$.

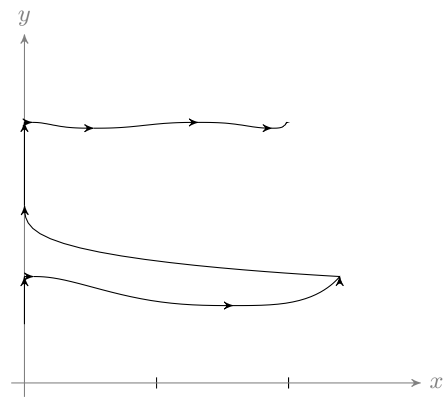

Fig. 3.3. Example of $\vec{s}(t)$ from t_0 to t_2

Theorem 3.2. *(main theorem). Let $\vec{a}, \vec{v}, \vec{s}$, and $\mathbf{\Gamma}$ be as in Construction 3.1.*

1. *The functions $\vec{a}, \vec{v}$, and $\vec{s}$ are Lipschitz and computable in polynomial time, hence continuous and bounded.*
2. *The total length, including retracings, of the parametrization $\vec{s}$ of $\mathbf{\Gamma}$ is finite and computable in polynomial time.*
3. *The curve $\mathbf{\Gamma}$ is simple, rectifiable, and smooth except at one endpoint.*
4. *Every computable parametrization $f : [a, b] \to \mathbb{R}^2$ of $\mathbf{\Gamma}$ has* unbounded retracing.

For the remainder of this section, we use the notation of Construction 3.1.

The following two observations facilitate our analysis of the curve $\mathbf{\Gamma}$. The proofs are routine calculations.

Observation 3.3 *For all $n \in \mathbb{Z}^+$, if we write*

$$d_i^{(n)} = \frac{t_{n-1} + 5t_n}{6} + i\frac{t_n - t_{n-1}}{6n}$$

and

$$e_i^{(n)} = d_i^{(n)} + \frac{t_n - t_{n-1}}{12n}$$

for all $0 \le i < n$, then

$$t_{n-1} < t_n^- < d_0^{(n)} < e_0^{(n)} < d_1^{(n)} < e_1^{(n)} < \cdots < d_{n-1}^{(n)} < e_{n-1}^{(n)} < t_n < t_n^+ < t_{n+1}^-.$$

Observation 3.4 *For all $a, b \in \mathbb{R}$ with $a < b$,*

$$\int_a^b \int_a^t \varphi_{a,b}(\theta) d\theta dt = \frac{(b-a)^3}{8\pi}.$$

We now proceed with a quantitative analysis of the geometry of $\mathbf{\Gamma}$. We begin with the horizontal component of $\vec{s}$.

Lemma 3.5 *1. For all* $t \in [0,1] - \bigcup_{n \in K}(t_n^-, t_n^+)$, $v_x(t) = s_x(t) = 0$.
2. For all $n \in K$ *and* $t \in (t_n^-, t_n)$, $v_x(t) > 0$.
3. For all $n \in K$ *and* $t \in (t_n, t_n^+)$, $v_x(t) < 0$.
4. For all $n \in \mathbb{Z}^+$, $s_x(t_n) = \frac{(e-1)^3}{1000\pi e^{3n}} 2^{-(n+\tau(n))}$.
5. $s_x(1) = 0$.

The following lemma analyzes the vertical component of $\vec{s}$. We use the notation of Observation 3.3, with the additional proviso that $d_n^{(n)} = t_n$.

Lemma 3.6 *1. For all* $n \in \mathbb{Z}^+$ *and* $t \in (t_{n-1}, d_0^{(n)})$, $v_y(t) > 0$.
2. For all $n \in \mathbb{Z}^+$, $0 \leq i < n$, *and* $t \in (d_i^{(n)}, e_i^{(n)})$, $v_y(t) < 0$.
3. For all $n \in \mathbb{Z}^+$, $0 \leq i < n$, *and* $t \in (e_i^{(n)}, d_{i+1}^{(n)})$, $v_y(t) > 0$.
4. For all $n \in \mathbb{Z}^+$, $0 \leq i < n$, *and* $t \in \{e_i^{(n)}, d_i^{(n)}, t_n\}$, $v_y(t) = 0$.
5. For all $n \in \mathbb{Z}^+$ *and* $0 \leq i \leq n$, $s_y(d_i^{(n)}) = s_y(d_0^{(n)})$.
6. For all $n \in \mathbb{Z}^+$ *and* $0 \leq i < n$, $s_y(e_i^{(n)}) = s_y(e_0^{(n)})$.
7. For all $n \in \mathbb{N}$, $s_y(t_n) = \frac{5^3(e-1)^3}{6^3 \cdot 8\pi} \sum_{i=1}^{n} \frac{1}{e^{3i}}$.
8. For all $n \in \mathbb{Z}^+$, $s_y(e_0^{(n)}) = s_y(t_n) - \frac{(e-1)^3}{12^3 n^3 8\pi e^{3n}}$.
9. $s_y(1) = \frac{5^3(e-1)^3}{6^3 \cdot 8\pi(e^3-1)}$.

By Lemmas 3.5 and 3.6, we see that $\vec{s}$ parametrizes a curve from $\vec{s}(0) = (0,0)$ to $\vec{s}(1) = (0, \frac{5^3(e-1)^3}{6^3 8\pi(e^3-1)})$.

It is clear from Observation 3.3 and Lemmas 3.5 and 3.6 that the curve Γ does not intersect itself. We thus have the following.

Corollary 3.7 $\mathbf{\Gamma}$ *is a simple curve from* $\vec{s}(0) = (0,0)$ *to* $\vec{s}(1) = (0, \frac{5^3(e-1)^3}{6^3 8\pi(e^3-1)})$.

Lemma 3.8 *The functions* $\vec{a}, \vec{v}$, *and* $\vec{s}$ *are Lipschitz, hence continuous, on* $[0,1]$.

Since every Lipschitz parametrization has finite total length [1], and since the length of a curve cannot exceed the total length of any of its parametrizations, we immediately have the following.

Corollary 3.9 *The total length, including retracings, of the parametrization* $\vec{s}$ *is finite. Hence the curve* $\mathbf{\Gamma}$ *is rectifiable.*

Lemma 3.10 *The curve* $\mathbf{\Gamma}$ *is smooth except at the endpoint* $\vec{s}(1)$.

Lemma 3.11 *The functions* $\vec{a}, \vec{v}$, *and* $\vec{s}$ *are computable in polynomial time. The total length including retracings, of* $\vec{s}$ *is computable in polynomial time.*

Definition. A *modulus of uniform continuity* for a function $f : [a,b] \to \mathbb{R}^n$ is a function $h : \mathbb{N} \times \mathbb{N}$ such that, for all $s, t \in [a,b]$ and $r \in \mathbb{N}$,

$$|s-t| \leq 2^{-h(r)} \implies |f(s) - f(t)| \leq 2^{-r}.$$

It is well known (e.g., see [14]) that every computable function $f : [a,b] \to \mathbb{R}^n$ has a modulus of uniform continuity that is computable.

Lemma 3.12 *Let $f : [a,b] \to \mathbb{R}^2$ be a parametrization of $\mathbf{\Gamma}$. If f has bounded retracing and a computable modulus of uniform continuity, then $\mathrm{K} \leq_{\mathrm{T}} f_y$, where f_y is the vertical component of f.*

4 Lower Semicomputability of Length

In this section we prove that every computable curve Γ has a lower semicomputable length. Our proof is somewhat involved, because our result holds even if every computable parametrization of Γ is retracing.

Construction 4.1 *Let $f : [0,1] \to \mathbb{R}^n$ be a computable function. Given an oracle Turing machine M that computes f and a computable modulus $m : \mathbb{N} \to \mathbb{N}$ of the uniform continuity of f, the (M,m)-cautious polygonal approximator of* range(f) *is the function $\pi_{M,m} : \mathbb{N} \to \{$polygonal paths$\}$ computed by the following algorithm.*

input $r \in \mathbb{N}$;
$S := \{\}$; *// S may be a multi-set*
for i:=0 **to** $2^{m(r)}$ **do**
  $a_i := i2^{-m(r)}$;
  use M to compute x_i with
    $|x_i - f(a_i)| \leq 2^{-(r+m(r)+1)}$;
  add x_i to S;
output a longest path inside a minimum spanning tree of S.

Definition. Let (X,d) be a metric space. Let $\Gamma \subseteq X$ and $\epsilon > 0$. Let

$$\Gamma(\epsilon) = \left\{ p \in X \;\middle|\; \inf_{p' \in \Gamma} d(p,p') \leq \epsilon \right\}$$

be the *Minkowski sausage* of Γ with radius ϵ.

Let $d_{\mathrm{H}} : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}$ be such that for all $\Gamma_1, \Gamma_2 \in \mathcal{P}(X)$

$$d_{\mathrm{H}}(\Gamma_1, \Gamma_2) = \inf \{ \epsilon \mid \Gamma_1 \subseteq \Gamma_2(\epsilon) \text{ and } \Gamma_2 \subseteq \Gamma_1(\epsilon) \}.$$

Note that d_{H} is the *Hausdorff distance* function.

Let $\mathcal{K}(X)$ be the set of nonempty compact subsets of X. Then $(\mathcal{K}(X), d_{\mathrm{H}})$ is a metric space [6].

Theorem 4.2. *(Frink [8], Michael [18]). Let (X, d) be a compact metric space. Then $(\mathcal{K}(X), d_{\mathrm{H}})$ is a compact metric space.*

Definition. Let $\mathcal{RC}$ be the set of all simple rectifiable curves in $\mathbb{R}^n$.

Theorem 4.3. *([21] page 55). Let $\Gamma \in \mathcal{RC}$. Let $\{\Gamma_n\}_{n\in\mathbb{N}} \subseteq \mathcal{RC}$ be a sequence of rectifiable curves such that $\lim_{n\to\infty} d_{\mathrm{H}}(\Gamma_n, \Gamma) = 0$. Then $\mathcal{H}^1(\Gamma) \leq \liminf_{n\to\infty} \mathcal{H}^1(\Gamma_n)$.*

This theorem has the following consequence.

Theorem 4.4. *Let $\Gamma \in \mathcal{RC}$. For all $\epsilon > 0$, there exists $\delta > 0$ such that for all $\Gamma' \in \mathcal{RC}$, if $d_{\mathrm{H}}(\Gamma, \Gamma') < \delta$, then $\mathcal{H}^1(\Gamma') > \mathcal{H}^1(\Gamma) - \epsilon$.*

Theorem 4.5. *Let $\Gamma \in \mathcal{RC}$ such that $\Gamma = \gamma([0, 1])$, where γ is a continuous function. (Note that γ may not be one-one.) Let $S(a) = \{\gamma(a_i) \mid a_i \in a\}$ for all dissection a. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of dissections of Γ such that*

$$\lim_{n\to\infty} \mathrm{mesh}(a_n) = 0.$$

Then

$$\lim_{n\to\infty} \mathcal{H}^1(LMST(a_n)) = \mathcal{H}^1(\Gamma),$$

where $LMST(a)$ is the longest path inside the Minimum Euclidean Spanning Tree of $S(a)$.

This result implies that when the sampling density is high, the number of leaves in the minimum spanning tree is asymptotically smaller than the total number of nodes.

We now have the machinery to prove the main result of this section.

Theorem 4.6. *Let $\gamma : [0, 1] \to \mathbb{R}^n$ be computable such that $\Gamma = \gamma([0, 1]) \in \mathcal{RC}$. Then $\mathcal{H}^1(\Gamma)$ is lower semicomputable.*

5 Δ_2^0-Computability of the Constant-Speed Parametrization

In this section we prove that every computable curve Γ has a constant speed parametrization that is Δ_2^0-computable.

Theorem 5.1. *Let $\Gamma = \gamma^*([0, 1]) \in \mathcal{RC}$. ($\gamma^*$ may not be one-one.) Let $l = \mathcal{H}^1(\Gamma)$ and O_l be an oracle such that for all $n \in \mathbb{N}$, $|O_l(n) - l| \leq 2^{-n}$. Let f be a computation of γ^* with modulus m. Let γ be the constant speed parametrization of Γ. Then γ is computable with oracle O_l.*

Corollary 5.2 *Let Γ be a curve with the property described in property 5 of Theorem 3.2. Then the length of Γ – $\mathcal{H}^1(\Gamma)$ is not computable.*

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