Computability of Homology for Compact Absolute Neighbourhood Retracts

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Abstract. In this note we discuss the information needed to compute the homology groups of a topological space. We argue that the natural class of spaces to consider are the compact absolute neighbourhood retracts, since for these spaces the homology groups are finite. We show that we need to specify both a function which defines a retraction from a neighbourhood of the space in the Hilbert cube to the space itself, and a sufficiently fine over-approximation of the set. However, neither the retraction itself, nor a description of an approximation of the set in the Hausdorff metric, is sufficient to compute the homology groups. We express the conditions in the language of computable analysis, which is a powerful framework for studying computability in topology and geometry, and use cubical homology to perform the computations.

Keywords: computability, homology, compact absolute neighbourhood retract

1 Introduction

Homology theory is one of the cornerstones of algebraic topology. The first homology theory, simplicial homology, was developed to provide invariants of a topological space (expressed as a simplicial complex) which could be more easily computed than the homotopy invariants. Other homology theories, most notably singular homology, were developed which extended the simplicial homology to arbitrary topological spaces, topological pairs and continuous functions. For an introduction to homology theory, see [ES52], [Mun84], [Mas91] or [Spa81]. However, while the simplicial homology can be easily computed by purely algebraic means, it is not clear precisely what information is needed about a space in order to compute its homology groups using a digital computer. The purpose of this article is to discuss the computability of homology for general metric spaces.

As is standard in computability theory, we use Turing machines as the underlying computational model. We consider different representations of the input

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sets and/or functions in terms of symbols over some alphabet. Since the class of compact subsets of a (infinite) separable metric space has continuum cardinality, we need to represent these sets by streams of data, yielding successively better approximations to the set.

Since homology groups are well defined (by the Eilenburg-Steenrod axioms [ES45]) and finite for the class of compact absolute neighbourhood retracts, we restrict attention to these spaces. A natural way of describing a compact absolute neighbourhood retract is to specify a neighbourhood retraction onto the set. However, we shall see that this information itself is not quite sufficient to compute the homology; we also need to give a bounding set for the set which is a subset of the domain of the retraction.

The original approach to homology theory via simplicial complexes is well-suited to the computation of the homology of a topological space when an explicit construction of the space is known. However, it is less-well suited for the computation of the homology of an arbitrary continuous function, unless a homotopic simplicial map can easily be constructed. Further, the relative simplicity of interval methods for rigorous evaluation of continuous functions suggests the development of a homology theory based on cubical complexes. The first algorithms for the computation of cubical homology were developed in [KMŚ98,KMW99]; see [KMM04] for a self-contained exposition. More advanced algorithms have since been developed [MMP05,MPŻ08,MB09]. The computational homology package CHomP [KMP] contains implementations of the computation of the homology of simplicial and cubical complexes by Kalies, Mrozek and Pilarczyk.

The main results of this paper are that the homology of a general compact separable metric space X cannot be computable from a name of X as a compact set, and neither can the homology of a compact absolute neighbourhood retract X be computed from a name of a neighbourhood retraction $r : U \longrightarrow X$. However, the homology can be computed given both pieces of data; this is equivalent to a name of r and a single bound on X.

2 Preliminaries

In this section we review the main concepts and results from the theory of retracts, homology theory, computational cubical homology and computable analysis that we require.

2.1 Theory of retracts

Let E be a metrisable space. Recall that if $X \subset E$, then a function $r : E \longrightarrow X$ is a *retraction* if $r|_X = \mathrm{id}_X$. If U is an open neighbourhood of X in E and $r : U \longrightarrow X$ is such that $r|_X = \mathrm{id}_X$, then r is a *neighbourhood retraction.* We say X is a *neighbourhood retract* if there exists a neighbourhood retraction $r : U \longrightarrow X$. If $X \subset E$, we denote the embedding of X in E as $i : X \longrightarrow E$; note that $r|_X = r \circ i$. We say that $r : U \longrightarrow X$ is a *weak (neighbourhood) retraction* if $r|_X \sim \mathrm{id}|_X$, i.e. r is homotopic to the identity of X.

Recall that the *Hilbert cube* is the countably infinite product space $[-1,+1]^\infty$. We can give a metric by

$$d(x,y) = \left(\sum_{k=1}^{\infty} \left(\frac{x_k - y_k}{k} \right)^2 \right)^{1/2}.$$

The relative interior of the Hilbert cube is the subset $(-1,+1)^\infty$, which is not locally-compact.

A countable base for the Hilbert cube is given by open sets of the form

$$I_1 \times I_2 \times \cdots I_k \times [-1,+1] \times \cdots$$

with each I_j of the form (a_j,b_j), $(a_j,+1]$, $[-1,b_j)$ or $[-1,+1]$ for $a_j,b_j \in \mathbb{Q}$. The closures of these sets have the form

$$[a_1,b_1] \times \cdots \times [a_k,b_k] \times [-1,+1] \times \cdots$$

with $a_i,b_i \in \mathbb{Q}$ and $-1 \leq a_i < b_i \leq +1$ for $i = 1,\ldots,k$. A countable base for the relative interior of the Hilbert cube is given by the sets

$$(a_1,b_1) \times \cdots \times (a_k,b_k) \times (-1,+1) \times \cdots$$

with $a_i,b_i \in \mathbb{Q}$ and $-1 \leq a_i < b_i \leq +1$ for $i = 1,\ldots,k$.

A space X is an *absolute neighbourhood retract* if, whenever it embeds as a closed subset of a normal space Y, there is an open neighbourhood U of X in Y and a retraction $r : U \longrightarrow X$. It can be shown that a separable metric space is an absolute neighbourhood retract if, and only if, it embeds as a neighbourhood retract in the Hilbert cube. We can therefore consider absolute neighbourhood retracts as subsets of the Hilbert cube. A space is a *Euclidean neighbourhood retract* if it embeds as a neighbourhood retract in Euclidean space $\mathbb{R}^d$ for some d.

2.2 Homology Theory

Recall that a *topological pair* is a pair (X,A) where X is a topological space and A is a subset of X. By a slight abuse of notation, we will sometimes write X for the pair $(X,\emptyset)$. A *map of pairs* $f : (X,A) \longrightarrow (Y,B)$ is a continuous function $f : X \longrightarrow Y$ such that $f(A) \subset B$.

Recall that a *graded abelian group* G is a sequence $(G_q)_{q=0}^{\infty}$ of abelian groups. A homomorphism ϕ between graded abelian groups G and H is a sequence of group homomorphisms $\phi_q : G_q \longrightarrow H_q$. A graded abelian group G is *finite* if each G_q is finite, and $G_q = \{e\}$ for all but finitely many q.

Recall that a (finite or infinite) sequence of (graded) group homomorphisms

$$\cdots \longrightarrow G_k \xrightarrow{\phi_k} G_{k+1} \xrightarrow{\phi_{k+1}} G_{k+2} \longrightarrow \cdots$$

is *exact* if $\mathrm{im}(\phi_k) = \ker(\phi_{k+1})$ for all k.

There are a large number of homology theories, each with different properties. However, they all satisfy the following axioms.

Axioms 1 (Eilenberg-Steenrod) *A homology theory consists of a covariant functor H_* from (a full subcategory of) the category of topological pairs to the category of graded abelian groups, and a natural transformation ∂_* of degree -1 from $H_*(X,A)$ to $H_*(A)$ satisfying Axioms (i) to (iv) below.*

In other words, $H_q(X,A)$ is an abelian group for $q = 0,1,\ldots$, if $f : (X,A) \longrightarrow (Y,B)$ then $H_q(f) : H_q(X,A) \longrightarrow H_q(Y,B)$, $H_*(g \circ f) = H_*(g) \circ H_*(f)$ and $\delta_q : H_q(X,A) \longrightarrow H_{q-1}(A)$.

1. Homotopy: *If $f_0, f_1 : (X,A) \longrightarrow (Y,B)$ are homotopic, then*
$$H_*(f_0) = H_*(f_1) : H_*(X,A) \longrightarrow H_*(Y,B).$$
2. Exactness: *Each pair (X, A) induces a long exact sequence in homology, via the inclusions $i : A \longrightarrow X$ and $j : X \longrightarrow (X,A)$ by*
$$\cdots \longrightarrow H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X,A) \xrightarrow{\partial_*} H_{q-1}(A) \longrightarrow \cdots .$$
3. Excision: *If (X,A) is a pair and U is a subset of X such that $\bar{U} \subset A^\circ$, then the inclusion map $i : (X \setminus U, A \setminus U) \longrightarrow (X,A)$ induces an isomorphism in homology*
$$i_* : H_*(X \setminus U, A \setminus U) \approx H_*(X,A).$$
4. Dimension: *If P is a one-point space, then*
$$H_q(P) \equiv \begin{cases} 0 \text{ if } q \neq 0 \\ \mathbb{Z} \text{ if } q = 0. \end{cases}$$

It is well-known that the homology of a compact absolute neighbourhood retract is uniquely determined by the axioms. For the homology of simplicial set is determined by the axioms, and can be effectively computed from the axioms (though the computation is usually performed in practice using the simplicial homology theory). Additionally, any compact absolute neighbourhood retract is *dominated* by a finite simplicial complex, allowing computation of the homology. That any compact absolute neighbourhood retract has the homotopy type of a finite simplicial complex was a long-standing open conjecture, finally proved by West [Wes77]. We shall use a similar technique to relate the homology of a compact absolute neighbourhood retract to that of a finite cubical complex.

Recall that a single-valued function $f : X \longrightarrow Y$ is a *selection* of a multi-valued function $F : X \rightrightarrows Y$ if $f(x) \in F(x)$ for all $x \in X$. It is not difficult to show that if $F : X \rightrightarrows Y$ is convex-valued and f_0, f_1 are two continuous selections of F, then f_0 and f_1 are homotopic. We say that $F : (X,A) \rightrightarrows (Y,B)$ is a multivalued map of pairs if $F : X \rightrightarrows Y$ and $F(a) \subset B$ for all $a \in A$. Hence if $F : (X,A) \rightrightarrows (Y,B)$ is a multivalued map of pairs with convex values, then any continuous selections $f_0, f_1 : (X,A) \longrightarrow (Y,B)$ are homotopic, and so have the same homology. We can therefore speak of the homology of a multivalued map.

2.3 Computational Homology

The computational homology approach begins with the computation of the homology of *cubical sets* which are essentially finite unions of cubes in Euclidean space.

The following definition is modified from [KMM04, Definitions 2.1,3,9].

Definition 2 (Cubical Set). *An* elementary interval *is a closed interval* $I \subset \mathbb{R}$ *of the form* $I = [k, k]$ *or* $I = [k, k+1]$ *for some* $k \in \mathbb{Z}$*. An* elementary cube $Q \subset \mathbb{R}^d$ *is a finite product of elementary intervals* $Q = I_1 \times I_2 \times \cdots \times I_d$*. An* elementary cubical chain *is a formal sum of oriented elementary cubes. The* boundary ∂Q *of an elementary cube* Q *is the formal sum of the elementary cubes of dimension* $\dim(Q) - 1$ *with the natural orientation.*

An elementary cubical complex $\mathcal{Q}$ *is a set of elementary cubes* Q *such that if* $Q \in \mathcal{Q}$*, then any elementary cube which is a subset of* Q *is also an element of* $\mathcal{Q}$*.*

A cubical complex *is a set* $\mathcal{X}$ *of the form* $\mathcal{X} = \{s_l(Q) \mid Q \in \mathcal{Q}\}$ *where* $\mathcal{Q}$ *is an elementary cubical complex and* $s_l(x) = x/2^l$ *is a scaling transformation. A cubical complex* $\mathcal{X}'$ *is a* refinement *of* $\mathcal{X}$ *if* $Q = \bigcup\{Q' \in \mathcal{X}' | Q' \subset Q\}$ *for all* $Q \in \mathcal{X}$*.*

The support $|\mathcal{X}|$ *of a cubical complex* $\mathcal{X}$ *is the union of all elementary cubes of* X*. A set* X *is* cubical *if there is a cubical complex* $\mathcal{X}$ *such that* $X = |\mathcal{X}|$*.*

Definition 3 (Cubical Map). *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be cubical complexes. A cubical function is a multivalued function* $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ *such that* $\mathcal{F}(Q_1 \cap Q_2) = \mathcal{F}(Q_1) \cap \mathcal{F}(Q_2)$*. A cubical function is* convex *if* $|\mathcal{F}(Q)|$ *is convex for all* $Q \in \mathcal{X}$*.*

The support $|\mathcal{F}|$ *of a cubical function* $\mathcal{F}$ *is the lower-semicontinuous multivalued map* $|\mathcal{F}| : |\mathcal{X}| \rightrightarrows |\mathcal{Y}|$ *defined by* $|\mathcal{F}|(x) = |\mathcal{F}(Q)|$ *for* $x \in \operatorname{rel\,int}(Q)$*. We say that a multivalued map* $F : X \longrightarrow Y$ *is* cubical *if there are cubical complexes* $\mathcal{X}$ *and* $\mathcal{Y}$ *with* $X = |\mathcal{X}|$*,* $Y = |\mathcal{Y}|$*, and a cubical function* $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ *such that* $F = |\mathcal{F}|$*.*

The following theorem asserts that cubical homology is effectively computable.

Theorem 4.

1. *Let* $(\mathcal{X}, \mathcal{A})$ *be cubical complexes. Then the cubical homology* $H_*(|\mathcal{X}|, |\mathcal{A}|)$ *is effectively computable given* $\mathcal{X}$ *and* $\mathcal{A}$*.*
2. *Let* $\mathcal{F} : (\mathcal{X}, \mathcal{A}) \longrightarrow (\mathcal{Y}, \mathcal{B})$ *be a convex cubical function. Then the cubical homology* $H_*(|\mathcal{F}|)$ *is effectively computable given* $\mathcal{F}$*.*

Note that the cubical homology theory is essentially a combinatorial theory (for cubical complexes and convex cubical functions) which induces a topological theory on the supports. It is possible to show that the homology of a cubical set (or map) does not depend on the cubical complex (or function) used for the representation. However, the cubical theory is only defined on the special classes of cubical sets and cubical maps. To extend the theory to arbitrary sets and maps, we need to reduce to the cubical theory. The main results of this paper involve showing that these reductions can be performed effectively.

2.4 Computability Theory

In this section we give an overview of computability in analysis, following the type-two effectivity theory of [Wei00].

Let Σ be a finite alphabet, such as the binary digits $\{0,1\}$ or the ASCII character set. By Σ^* we mean the set of finite words on Σ, and by Σ^ω the set of infinite sequences. We say a function $\eta :\subset \Sigma^{*/\omega} \times \cdots \times \Sigma^{*/\omega} \longrightarrow \Sigma^{*/\omega}$ is *computable* if it can be evaluated by a Turing machine. The set of computable functions is closed under composition.

We will sometimes need a computable *tupling* operation $\tau\ (\Sigma^*)^\omega \longrightarrow \Sigma^\omega$, denoted $(w_1, w_2, \ldots) \mapsto \langle w_1, w_2, \ldots \rangle$.

Let M be a set. A *notation* of M is a partial surjective function $\nu :\subset \Sigma^* \longrightarrow M$. A *representation* of M is a partial surjective function $\delta :\subset \Sigma^\omega \longrightarrow M$. A δ-*name* of $x \in M$ is an element $p \in \Sigma^\omega$ such that $\delta(p) = x$.

If $\delta_0, \ldots, \delta_k$ are representations of $M_0, \ldots, M_k$ respectively, then a function $f : M_1 \times \cdots \times M_k \longrightarrow M_0$ is *computable* if there is a computable function $\eta :\subset \Sigma^\omega \times \cdots \times \Sigma^\omega \longrightarrow \Sigma^\omega$ such that $f(\delta_1(p_1), \ldots, \delta_k(p_k)) = \delta_0(\eta(p_1, \ldots, p_k))$ whenever the left-hand side is defined. If the representations of $M_0, \ldots, M_k$ being used are clear from the context, we simply say that f is (effectively) computable.

If M is a topological space, we are interested in representations which are compatible with the topological structure. A *computable topological space* is a tuple (M, τ, σ, ν) where σ is a sub-base for a T_0 topology τ on M, and ν is a notation of τ. The *standard representation* of (M, τ, σ, ν) is the representation δ of M defined by

$$\delta\langle w_1, w_2, w_3, \ldots \rangle = x \iff \{\nu(w_i) \mid i \in \mathbb{N}\} = \{I \in \sigma \mid x \in I\}.$$

In other words, p encodes a list of all sub-basic sets I containing x. By the T_0 hypothesis on (M, τ), this p encodes a unique element of M.

If M is the Hilbert cube and ν is an encoding of the standard basis set β, then the standard arithmetical operations $+$, $-$, $\times$ and $\div$ are computable with respect to the standard representation.

Given a locally-compact Hausdorff space X and a base β for X with notation ν we can construct representations for open and compact subsets of X as follows:

1. A $\theta_<$-name of an open subset U of X encodes a list of all $I \in \beta$ such that $\bar{I} \subset U$.
2. A $\kappa_>$-name of a compact subset C of X encodes a list of all tuples $(J_1, \ldots, J_k) \in \beta^*$ such that $C \subset \bigcup_{i=1}^k J_i$.
3. A κ-name of a compact subset C of X encodes a list of all tuples $(J_1, \ldots, J_k) \in \beta^*$ such that $C \subset \bigcup_{i=1}^k J_i$ and $J_i \cap C \neq \emptyset$ for all $i = 1, \ldots, k$.

These are standard representations with respect to the Scott topology on open sets, and the (upper) Vietoris topology on compact sets. We can also construct representations for continuous functions:.

4. Let U be an open subset of X, and $f : U \longrightarrow Y$ a continuous function. A γ-name of f encodes a list of all pairs $(I, J) \in \beta_X \times \beta_Y$ such that $\bar{I} \subset U$ and $f(\bar{I}) \subset J$.

Note that a γ-name of f implicitly contains a $\theta_<$-name of $\mathrm{dom}(f)$. This is a standard representation with respect to the compact-open topology on continuous functions.

3 Computability of Homology Groups

In this section we present the main results on computability and uncomputability of homology groups. We first show that homology is uncomputable with respect to certain representations of the space, and then find conditions under which homology is computable.

3.1 Uncomputability of homology

We now show that the homology cannot be computed from a κ-name of X, nor from a γ-name of a neighbourhood retract $r : U \longrightarrow X$ alone. These results are strong, in the sense that there is no space for which the homology can be computed from the given data.

Theorem 5. *Let X be a compact absolute neighbourhood retract. The homology function H_* is discontinuous at X in the Vietoris topology, and hence is uncomputable.*

Proof. It suffices to construct a sequence of compact absolute neighbourhood retracts X_n such that $X_n \to X$ in the Vietoris topology, but $H_*(X_n)$ does not converge to $H_*(X)$. Let x_i be a sequence of points such that each $x_i \notin X$ but $\lim_{n\to\infty} x_i = x_\infty \in X$. Let $X_n = X \cup \bigcup_{i=n+1}^{2n} x_i$. Then each X_n is an absolute neighbourhood retract and $X_n \to X$ in the Vietoris topology on compact sets, but $H_0(X_n) \approx H_0(X) \oplus \mathbb{Z}^n$, so the homology does not converge.

Theorem 6. *Let $X \subset \mathbb{R}^\infty$ be a compact absolute neighbourhood retract. The homology of X cannot be computed from a γ-name of a neighbourhood retraction $r : U \to \mathbb{R}^\infty$ with $r(U) = X$.*

Proof. Let $p \in \Sigma^\omega$ be a γ-name of r encoding a sequence (I_k, J_k) of basic open sets such that $r(\bar{I}_k) \subset J_k$. Let U' be an open ball $\mathrm{cl}(U') \cap \mathrm{cl}(U) = \emptyset$, let $x' \in U'$ and $r' : U' \longrightarrow \{x'\}$. Let p' be a γ-name of r' encoding a sequence (I'_k, J'_k) such that $r'(\bar{I}'_k) \subset J'_k$.

Take $\hat{U} = U \cup U'$, $\hat{X} = X \cup X'$ and define $\hat{r} : \hat{U} \longrightarrow \hat{X}$ by $\hat{r}(a) = r(a)$ if $a \in U$ and $\hat{r}(a) = r'(a)$ if $a \in U'$. Then $\hat{r}$ is a retraction from $\hat{U}$ to $\hat{X}$. We can construct names of $\hat{r}$ by taking an arbitrarily long prefix of a name of $\tilde{r}$, and then splicing in a name of r'. For $n \in \mathbb{N}$, define $(\hat{I}_{n,k}, \hat{J}_{n,k}) = (I_k, J_k)$ for $k \leq n$, and $(\hat{I}_{n+2j-1}, \hat{J}_{n+2j-1}) = (I_{n+j}, J_{n+j})$, $(\hat{I}_{n+2j}, \hat{J}_{n+2j}) = (I_{n+j}, J_{n+j})$ for $j \in \mathbb{N}$. Let $\hat{p}_n$ be the encoding of the sequence $(\hat{I}_{n,i}, \hat{J}_{n,i})$. Then for all $n \in \mathbb{N}$, $\hat{p}_n$ is an encoding of $\hat{r}$, but $\lim_{n\to\infty} \hat{p}_n = p$, which is a name of r. This means that given the name p of r, at no point can we deduce p is a name of r and not $\hat{r}$, and so at no point can we deduce $H_0(X)$.

We note that while the first result is due to an argument that the homology is discontinuous, for the second we needed to consider the details of the representation. This suggests that the homology can "almost" be computed from a name of a neighbourhood retraction. In the next section we shall see that this is indeed the case.

3.2 Homology of Euclidean neighbourhood retracts

To give an idea of the general method, we first prove effective computability of $H_*(X)$ for a Euclidean neighbourhood retract X.

Theorem 7. *Let X be a compact Euclidean neighbourhood retract. Then $H_*(X)$ can be effectively computed from a γ-name of a retraction $r : U \longrightarrow X$ with U an open subset of $\mathbb{R}^d$, and from a $\kappa_>$-name of X as a compact subset of $\mathbb{R}^d$.*

Proof (Proof (Sketch)). From the $\kappa_>$-name of X and a $\theta_<$-name of U we can effectively compute a cubical set C such that $X \subset C^\circ$ and $C \subset U$. Since $r(C) = X \subset C^\circ$, every point x of C has a basic open neighbourhood I such that $f(\bar{I}) \subset J$ with $J \subset C$. From a γ-name of r, we can therefore compute a convex-valued cubical map $R : C \rightrightarrows C$ such that $r(x) \in R(x)$ for all x. Since C is a cubical set and R is a cubical map, $H_*(C)$ and $H_*(R)$ can be computed using Theorem 4.

Let $i : X \longrightarrow C$ be the embedding of X in C, and $p : C \longrightarrow X$ be the restriction of r to C. Then $p \circ i = \mathrm{id}_X$, so $H_*(p \circ i) = \mathrm{id}_{H_*(X)}$. Hence $H_*(p)$ is surjective and $H_*(i)$ is injective. Since $i \circ p = r|_C$, the cubical map R is an over-approximation to $i \circ p$, so $H_*(i \circ p) = H_*(R)$. Then $H_*(X) = H_*(p)(H_*(C)) \approx H_*(i)(H_*(p)(H_*(C))) = H_*(i \circ p)(H_*(C)) = H_*(R)(H_*(C))$, so can be effectively computed.

The presentation of $H_*(X)$ is as a subgroup of $H_*(C)$ for which we have an explicit presentation. The subgroup is the image of $H_*(C)$ under the homomorphism $H_*(R)$. Notice that $H_*(R)$ is a projection on $H_*(C)$, since $H_*(R) = H_*(i \circ p) = H_*(i \circ (p \circ i) \circ p) = H_*(i \circ p \circ i \circ p) = H_*(i \circ p)^2 = H_*(R)^2$.

3.3 Computation of homology for compact absolute neighbourhood retracts

Lemma 8. *Let (X, A) be a pair of compact absolute neighbourhood retracts embedded in the Hilbert cube. Then given $\kappa_>$-names of X and A, and γ-names of $r_X : U_X \longrightarrow X$ and $r_A : U_A \longrightarrow A$, it is possible to effectively compute a pair $(\hat{X}, \hat{A})$ of cubical sets, and maps of pairs $i : (X, A) \longrightarrow (\hat{X}, \hat{A})$ and $p : (\hat{X}, \hat{A}) \longrightarrow (X, A)$ such that $p \circ i \sim \mathrm{id}_{X,A}$.*

Proof. By the effective Urysohn lemma [Wei01], we can construct a function $\phi : U_X \longrightarrow [0, 1]$ such that $\phi(x) = 1$ on a small neighbourhood of A, and $\phi(x) = 0$ outside U_A. We define $q : U_X \longrightarrow X$ by $q(x) = r_X(\phi(x) r_A(x) + (1 - \phi(x))x)$. It is straightforward to verify that q maps a small neighbourhood V_A of A in U_X

to A, that $q|_X$ is homotopic to the identity, and that we can compute a γ-name of q.

Since $X \subset U_X$, and using the topology of the Hilbert cube, we can effectively compute a cubical subset $\hat{X}$ of $\mathbb{R}^d$ such that $X \subset \hat{X}^\circ \times (-1,+1)^\infty$ and $\hat{X} \times [-1,+1]^\infty \subset U_X$. Further, we can ensure that $\hat{X}$ has a cubical subset $\hat{A}$ such that $A \subset \hat{A}^\circ \times (-1,+1)^\infty$ and $\hat{A} \times [-1,+1]^\infty \subset V_A$.

We take $i : (X,A) \longrightarrow (\hat{X},\hat{A})$ as $i(x) = \pi(x)$, which is clearly computable, and $p : (\hat{X},\hat{A}) \longrightarrow (X,A)$ by $p(x) = q(x,0,\ldots)$. Since q is homotopic to the identity on (X,A), we find $p \circ i \sim \mathrm{id}_{X,A}$ by the homotopy extension theorem.

Lemma 9. *Let $(\hat{X},\hat{A})$ and $(\hat{Y},\hat{B})$ be cubical sets, and $f : (\hat{X},\hat{A}) \longrightarrow (\hat{Y}^\circ,\hat{B}^\circ)$. Then given a γ-name of f, it is possible to effectively compute a convex cubical map $F : (\hat{X},\hat{A}) \longrightarrow (\hat{Y},\hat{B})$ such that f is a selector of F.*

Proof. Given a γ-name of f, we list all pairs (I,J) such that $f(\bar{I}) \subset J$, that $\bar{J} \subset \hat{Y}^\circ$ and $\bar{J} \subset \hat{B}^\circ$ if $I \cap A \neq \emptyset$. We eventually obtain an open cover of $\hat{X}$ by such sets I. By refining $\hat{X}$ if necessary, we can assume that each cell Q of $\hat{X}$ lies in some I with corresponding J. We define $\mathcal{F}(Q) = \{Q' \in \mathcal{K}(\hat{Y}) \mid J \cap Q' \neq \emptyset\}$. It is easy to verify that $|\mathcal{F}|$ is the required convex cubical map.

We can now compute the homology of an arbitrary topological pair.

Theorem 10. *Let (X,A) be a pair of compact absolute neighbourhood retracts embedded in the Hilbert cube. Then the homology $H_*(X,A)$ can be effectively computed from $\kappa_>$-names of X and A, and γ-names of r_X and r_A.*

Proof. Let $i : (X,A) \longrightarrow (\hat{X},\hat{A})$ and $p : (\hat{X},\hat{A}) \longrightarrow (X,A)$ be as given by Lemma 8, so that $p \circ i \sim \mathrm{id}_{X,A}$. Then $H_*(p \circ i) = \mathrm{id}_{H_*(X,A)}$ so $H_*(p)$ is surjective, and $H_*(i)$ is injective, and hence $H_*(X,A) \approx H_*(i \circ p)(H_*(\hat{X},\hat{A}))$. By Lemma 9 we can effectively compute a cubical map $P : (\hat{X},\hat{A}) \rightrightarrows (\hat{X},\hat{A})$ such that $i \circ p$ is a selection of P. Since $i \circ p$ is a selection of P, $H_*(i \circ p) = H_*(P)$. The result follows since we can compute the homology of $H_*(\hat{X},\hat{A})$ and $H_*(P)$ by Theorem 4.

We now consider the computation of the homology of a map of pairs.

Theorem 11. *Let (X,A) and (Y,B) be compact absolute neighbourhood retracts, equipped with the information needed to compute the homology. Let $f : (X,A) \longrightarrow (Y,B)$ be a map of pairs. Then the homology $H_*(f)$ can be computed from a γ-name of f.*

Proof. From Lemma 8, the approximate projection $p : (\hat{X},\hat{A}) \longrightarrow (X,A)$ can be effectively computed, as can the approximate embedding $i : (Y,B) \longrightarrow (\hat{Y},\hat{B})$. Then $i_{Y,B} \circ f \circ p_{X,A} : (\hat{X},\hat{A}) \longrightarrow (\hat{Y},\hat{B})$ can be effectively computed. We can therefore compute a cubical map $\hat{f} : (\hat{X},\hat{A}) \longrightarrow (\hat{Y},\hat{B})$ which is an over-approximation to $i_{Y,B} \circ f \circ p_{X,A}$. The homology of f is then given by $H_*(f) \approx H_*(p_{Y,B}) \circ H_*(\hat{f})$, since $H_*(p_{Y,B})$ is a projection of $H_*(\hat{Y},\hat{B})$ onto $H_*(Y,B)$ considered as a subgroup of $H_*(\hat{Y},\hat{B})$.

4 Conclusions

In this paper, we have considered the information required to compute the homology groups of compact absolute neighbourhood retracts. We have shown that the homology can be computed given a bound for the set, and the name of a neighbourhood retract from a subset of the Hilbert cube to the space. The derivations use standard homotopy arguments to reduce the problem to a problem of computing cubical homology.

An interesting question for further research is whether the requirements that X be a compact absolute neighbourhood retract can be weakened. If X is not compact, then the homology groups are not finite, but X still has the homotopy type of a (now infinite) simplicial complex. If X is not an absolute neighbourhood retract, then it cannot be embedded in Euclidean space or the Hilbert cube as a neighbourhood retract, and so a different representation of X is required.

An alternative approach would be to reduce the problem to the problem of computing simplicial homology. However, since existing numerical approaches work better with interval arithmetic and cubical sets, the cubical approach is closer to existing implementations.

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