Canonical Effective Subalgebras of Classical Algebras as Constructive Metric Completions

Andrej Bauer[1] and Jens Blanck[2]

[1] Faculty of Mathematics and Physics
University of Ljubljana, Slovenia
Andrej.Bauer@andrej.com

[2] Computer Science Department
Swansea University, UK
j.e.blanck@swansea.ac.uk

Abstract. We prove general theorems about unique existence of effective subalgebras of classical algebras. The theorems are consequences of standard facts about completions of metric spaces within the framework of constructive mathematics, suitably interpreted in realizability models. We work with general realizability models rather than with a particular model of computation. Consequently, all the results are applicable in various established schools of computability, such as type 1 and type 2 effectivity, domain representations, equilogical spaces, and others.

1 Introduction

Given an algebra, by which we mean a set with constants and operations, is there a largest subalgebra which carries a computable structure, and is the structure unique up to computable isomorphism? Without further assumptions the answer is in general negative. For example, within the context of Recursive Mathematics every computable subfield of reals may be properly extended to a subfield which is again computable, and this remains true even if we require the subfields to be effectively complete. However, as was proved by Moschovakis [1], by requiring also that the strict linear order be semidecidable, we are left with only one choice, namely the recursive reals. An analogous result for type 2 effectivity was established by Hertling [2].

We show how these results, as well as others, can be seen as standard facts about completions of metric spaces in the context of constructive mathematics, suitably interpreted in realizability models. We prove two main theorems which together give conditions under which an algebra $\mathcal{A}$, equipped with a complete metric d, has a unique effective subalgebra $\mathcal{B}$ that is effectively complete and for which the relation $d(x, y) < q$ is semidecidable in $x, y \in \mathcal{B}$ and $q \in \mathbb{Q}$.

Rather than choosing a specific model of computation, we work in a general realizability model. Thus our results apply to established schools of computable mathematics, such as type 1 and type 2 effectivity, domain representations, equilogical spaces, and others.

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The outline of the extended abstract is as follows. Sections 2–4 introduce the necessary background, namely realizability models, algebras, and premetric spaces. Section 5 states the main theorems, from which two specific important cases are inferred in Section 6. We conclude with a brief discussion of possible further directions of research.

2 Assemblies and Realizability

Among the different kinds of realizability the most suitable one for our purposes is *relative realizability*, because it subsumes type 1 and type 2 effectivity, domain representations, equilogical spaces, and other standard models of computation, see [3]. We review the basic definitions here and refer the readers to [4] for background material on realizability.

A *partial combinatory algebra (PCA)* is a set A with a partial application operation[3] $\cdot$ such that there exist elements $\mathsf{k}, \mathsf{s} \in A$ satisfying $\mathsf{k}\ x\ y = x$ and[4] $\mathsf{s}\ x\ y\ z \simeq (x\ z)\ (y\ z)$. A PCA is a general model of computation which supports encoding of pairs, natural numbers, recursion, partial recursive functions, etc. An *elementary sub-PCA* is a subset $B \subseteq A$ which is closed under application and contains k and s suitable for A. For the rest of the discussion we fix a PCA $\mathbb{A}$ and an elementary sub-PCA $\mathbb{A}^{\#} \subseteq \mathbb{A}$. The elements of $\mathbb{A}$ as "arbitrary" and those of $\mathbb{A}^{\#}$ as "effective" data or programs, although the exact meaning of these words depends on the particular choice of $\mathbb{A}$ and $\mathbb{A}^{\#}$.

An *assembly* $\mathbf{S} = (S, \Vdash_S)$ is a set S together with a *realizability relation* $\Vdash_S \subseteq \mathbb{A} \times S$, such that for every $x \in S$ there is at least one $\mathtt{x} \in \mathbb{A}$ for which $\mathtt{x} \Vdash_S x$. A *realized map* $f : \mathbf{S} \to \mathbf{T}$ between assemblies is a map $f : S \to T$ between the underlying sets which is *tracked* by some $\mathtt{f} \in \mathbb{A}^{\#}$, which means that whenever $\mathtt{x} \Vdash_S x$ then[5] $\mathtt{f}\ \mathtt{x}\downarrow$ and $\mathtt{f}\ \mathtt{x} \Vdash_T f(x)$. Note that we require maps to be realized by the elements of the *sub*algebra $\mathbb{A}^{\#}$. Assemblies and realized maps form a category Asm. An assembly $\mathbf{S}$ is *modest*, or a *modest set*, if each realizer realizes at most one element: for all $\mathtt{r} \in A$, $x, y \in S$, if $\mathtt{r} \Vdash_S x$ and $\mathtt{r} \Vdash_S y$ then $x = y$.

An assembly $\mathbf{S}$ is equivalent to a multi-valued representation $\delta_S : \mathbb{A} \to \mathcal{P}(S)$ via the correspondence $\mathtt{x} \Vdash_S x \iff x \in \delta_S(\mathtt{x})$. A modest set is equivalent to a single-valued representation. Traditional schools of computable mathematics typically use (single-valued) representations, for example:

- When $\mathbb{A} = \mathbb{A}^{\#} = \mathbb{N}$ is the first Kleene algebra, the modest sets are equivalent to type 1 representations, or *numbered sets*, which are used in the study of recursive mathematics. In this model "effective" means "computable by (type 1) Turing machine".

[3] We write $x\ y$ instead of $x \cdot y$, and associate application to the left, $x\ y\ z = (x\ y)\ z$.
[4] Kleene equality $a \simeq b$ means that if one side is defined then so is the other and they are equal.
[5] The expression $t\downarrow$ means "t is defined".

- When $\mathbb{A} = \mathbb{N}^{\mathbb{N}}$ is the second Kleene algebra and $\mathbb{A}^{\#}$ the subalgebra of total computable functions we get *type 2 representations*. In this case "effective" means "computable by type 2 Turing machine".
- The case $\mathbb{A} = \mathbb{A}^{\#} = \mathbb{N}^{\mathbb{N}}$ is the continuous version of type 2 effectivity in which "effective" means "continuously realized".
- When $\mathbb{A}$ is a universal Scott domain and $\mathbb{A}^{\#}$ its computable analogue, the modest assemblies are equivalent to domain representations and computable maps between them. Of course, "effective" is now interpreted in the sense of domain representations.
- With Scott's graph model $\mathbb{A} = \mathcal{P}\omega$ and its r.e. counterpart $\mathbb{A}^{\#} = \mathsf{RE}$ we obtain effective equilogical spaces [3].

Single-valued representations seem to be preferred to general assemblies, perhaps because from a programmer's perspective it makes little sense to use one realizer for representing several things, although lately multi-valued type 2 representations have turned out to be useful [5]. We use assemblies because they contain the category of sets, which allows us to consider classical and effective algebras in a single framework. Realizability toposes could be used instead, but assemblies are easier to describe and work with.

2.1 The realizability interpretation of first-order logic

Assemblies supports an interpretation of first-order intuitionistic logic in which a formula is deemed valid when there is an element $\mathtt{r} \in \mathbb{A}^{\#}$ witnessing it. The interpretation is given in terms of a *realizability relation* $\mathtt{r} \Vdash \phi$ which is read as "$\mathtt{r}$ realizers ϕ", and is defined inductively on the structure of the sentence ϕ:

- always $\mathtt{r} \Vdash \top$, and never $\mathtt{r} \Vdash \bot$,
- $\langle \mathtt{p}, \mathtt{q} \rangle \Vdash \phi \wedge \psi$ iff $\mathtt{p} \Vdash \phi$ and $\mathtt{q} \Vdash \psi$,[6]
- $\langle \overline{0}, \mathtt{r} \rangle \Vdash \phi \vee \psi$ iff $\mathtt{r} \Vdash \phi$, and $\langle \overline{1}, \mathtt{r} \rangle \Vdash \phi \vee \psi$ iff $\mathtt{r} \Vdash \psi$,[7]
- $\mathtt{r} \Vdash \phi \Rightarrow \psi$ iff for all $\mathtt{q} \in \mathbb{A}$, if $\mathtt{q} \Vdash \phi$ then $\mathtt{r}\,\mathtt{q}\downarrow$ and $\mathtt{r}\,\mathtt{q} \Vdash \psi$,
- $\mathtt{r} \Vdash \forall x \in \mathbf{S} \,.\, \phi(x)$ iff for all $\mathtt{a} \in \mathbb{A}$, $a \in S$, if $\mathtt{a} \Vdash_S a$ then $\mathtt{r}\,\mathtt{a}\downarrow$ and $\mathtt{r}\,\mathtt{a} \Vdash \phi(a)$,
- $\langle \mathtt{a}, \mathtt{r} \rangle \Vdash \exists x \in \mathbf{S} \,.\, \phi(x)$ iff for some $a \in S$, $\mathtt{a} \Vdash_S a$ and $\mathtt{r} \Vdash \phi(a)$,
- $\mathtt{r} \Vdash a = b$ iff $a = b$.

A sentence ϕ is *valid*, written $\models \phi$, when there exists $\mathtt{r} \in A^{\#}$ such that $\mathtt{r} \Vdash \phi$. Note that $\mathtt{r}$ must be an element of the *sub*algebra $A^{\#}$. A formula with free variables is valid when its universal closure is valid. Intuitionistic logic is sound with respect to the realizability relation: if intuitionistic logic proves ϕ then ϕ is valid.

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[6] $\langle \mathtt{p}, \mathtt{q} \rangle$ is the encoding of the pair whose components are $\mathtt{p}$ and $\mathtt{q}$.
[7] $\overline{n}$ is the encoding of the natural number n.

2.2 The role of double negation

Negation $\neg\phi$ is defined as $\phi \Rightarrow \bot$. This gives us

$$\mathtt{r} \Vdash \neg\phi \text{ iff for all } \mathtt{q} \in \mathbb{A}, \text{ not } \mathtt{q} \Vdash \phi,$$
$$\mathtt{r} \Vdash \neg\neg\phi \text{ iff there is } \mathtt{q} \in \mathbb{A} \text{ such that } \mathtt{q} \Vdash \phi.$$

A realizer $\mathtt{r}$ of a doubly negated formula $\neg\neg\phi$ does not carry any information about the computational content of ϕ, because we may replace it with any other. Thus double negation is a way of erasing the constructive or computational meaning of a formula.

A formula which is equivalent to its double negation is called $\neg\neg$-*stable*. Since $\phi \Rightarrow \neg\neg\phi$ is always intuitionistically provable, only the direction $\neg\neg\phi \Rightarrow \phi$ is relevant. An important family of stable formulas are the *negative* ones, which are those built from $\bot$, $\top$, $=$, $\neg$, $\wedge$, $\Rightarrow$, $\forall$, and possibly other $\neg\neg$-stable primitive relations. The realizers of a $\neg\neg$-stable formula ϕ are computationally irrelevant in the sense that any information that can be computed with the help of a realizer $\mathtt{r} \models \phi$ can be computed without $\mathtt{r}$, the extreme case of which is that $\mathtt{r}$ itself can be computed from nothing, as long as it exists.

A mono $i : \mathbf{S} \rightarrowtail \mathbf{T}$ is $\neg\neg$-*stable* when $\models \forall\, x \in \mathbf{T} \,.\, (\neg\neg(x \in \mathbf{S}) \Rightarrow x \in \mathbf{S})$, where "$x \in \mathbf{S}$" is a shorthand for $\exists\, y \in \mathbf{S} \,.\, i(y) = x$. Up to isomorphism, such a mono is a restriction of $\mathbf{T}$ to a subset $S \subseteq T$, and the realizability relation $\Vdash_S$ is $\Vdash_T$ restricted to S. Thus the $\neg\neg$-stable monos of $\mathbf{T}$ correspond to subsets of T.

A mono $i : \mathbf{S} \rightarrowtail \mathbf{T}$ is $\neg\neg$-*dense* when $\models \forall\, y \in \mathbf{T} \,.\, \neg\neg\exists\, x \in \mathbf{S} \,.\, y = i(x)$. Such a mono is always isomorphic to a mono $i : \mathbf{S} \rightarrowtail \mathbf{T}$ such that $S = T$ and i is the identity map. Thus the $\neg\neg$-dense monos play in Asm the role of reductions between representations.

2.3 Semidecidable predicates

To illustrate how the realizability interpretation is used, and for later use, we explain how to treat semidecidable predicates in Asm. We say that a mono $i : \mathbf{S} \rightarrowtail \mathbf{T}$, seen as a predicate on $\mathbf{T}$, is *semidecidable* when

$$\models \forall\, x \in \mathbf{T} \,.\, \exists\, f \in \{0,1\}^{\mathbf{N}} \,.\, (x \in \mathbf{S} \iff \exists\, n \in \mathbf{N} \,.\, f(n) = 1)\,.$$

Here $\mathbf{N}$ is the modest set of natural numbers, cf. Section 3.2, and the exponential $\{0,1\}^{\mathbf{N}}$ is the modest set of those maps $\mathbb{N} \to \{0,1\}$ which are tracked by an element of $\mathbb{A}$. Markov Principle, which is valid in Asm, states that a formula of the form $\exists\, n \in \mathbf{N} \,.\, f(n) = 1$ is $\neg\neg$-stable. Therefore only $\neg\neg$-stable predicates can be semidecidable. We assume without loss of generality that $i : \mathbf{S} \rightarrowtail \mathbf{T}$ is $\neg\neg$-stable and that i is a subset inclusion. Validity of the above formula is then equivalent to there being $\mathtt{r} \in \mathbb{A}^{\#}$ which works as follows: if $\mathtt{x} \Vdash_T x$ then, for all $n \in \mathbb{N}$, $\mathtt{r}\ \mathtt{x}\ \overline{n}\!\downarrow$ and $\mathtt{r}\ \mathtt{x}\ \overline{n} \in \{\overline{0}, \overline{1}\}$, and furthermore, $x \in S$ if, and only if, $\mathtt{r}\ \mathtt{x}\ \overline{n} = \overline{1}$ for some $n \in \mathbb{N}$. The semidecidable predicates have the expected properties: decidable predicates are semidecidable, and the semidecidable predicates are closed under conjunctions and existential quantification over $\mathbf{N}$.

In type 1 effectivity our notion of semidecidability coincides with the usual one, while in type 2 effectivity the notion is known as "r.e. open subset". In a purely topological model, such as the continuous version of type 2 effectivity "semidecidable" means "topologically open". The interpretation in Set is trivial because there every subset is semidecidable (even decidable) thanks to the law of excluded middle.

3 Algebras

A *signature* Σ for an algebra is given by a list of *function symbols* $\mathsf{f}_1, \ldots, \mathsf{f}_k$. Each f_i has an *arity*, which is a non-negative integer. The set $\mathsf{Term}(\Sigma)$ of *terms over* Σ is built inductively from variables $x, y, z, \ldots$, and terms $\mathsf{f}(t_1, \ldots, t_n)$, where f is a function symbol with arity n and $t_1, \ldots, t_n$ are terms. We assume that a standard Gödel numbering $\ulcorner - \urcorner : \mathbb{N} \to \{\star\} + \mathsf{Term}(\Sigma)$ of terms is given.[8]

A Σ*-algebra* $\mathcal{A}$ in a category C with finite products is given by an object $|\mathcal{A}|$ called the *carrier* of $\mathcal{A}$, and for each function symbol f with arity n a morphism $\mathsf{f}^{\mathcal{A}} : |\mathcal{A}|^n \to |\mathcal{A}|$, called an *operation*. Each term $t \in \mathsf{Term}(\Sigma)$ whose free variables are among $x_1, \ldots, x_k$ determines a morphism $|\mathcal{A}|^k \to |\mathcal{A}|$: a variable x_i is the i-th projection, while a term $\mathsf{f}(t_1, \ldots, t_n)$ is the composition of $\mathsf{f}^{\mathcal{A}}$ with the the morphisms determined by $t_1, \ldots, t_n$. A *subalgebra* of $\mathcal{A}$ is a Σ-algebra $\mathcal{B}$ with a mono $\mathcal{B} \rightarrowtail \mathcal{A}$ such that the operations in $\mathcal{A}$ restrict to operations in $\mathcal{B}$. We write $\mathcal{B} \leq \mathcal{A}$ when $\mathcal{B}$ is a subalgebra of $\mathcal{A}$.

If C and D are categories with finite products and $F : \mathsf{C} \to \mathsf{D}$ a functor which preserves finite products then a Σ-algebra $\mathcal{A}$ in C is mapped by F to a Σ-algebra $F(\mathcal{A})$ in $\mathcal{D}$, where $|F(\mathcal{A})| = F(|\mathcal{A}|)$ and $\mathsf{f}^{F(\mathcal{A})} = F(\mathsf{f}^{\mathcal{A}})$. The mapping preserves valid equations in $\mathcal{A}$, and also reflects them if F is faithful.

A *(first-order) formula* ϕ *over* Σ is a formula in first-order logic with terms over Σ. If $\mathcal{A}$ is a Σ-algebra in C, where C is either Set or Asm, then we may interpret such a ϕ as a statement about $\mathcal{A}$: the terms are interpreted according to $\mathcal{A}$, while the logic is interpreted either in the standard set-theoretic way, as given by Tarski, or using the realizability interpretation from Section 2.1. We write $\mathcal{A} \models_{\mathsf{C}} \phi$ when ϕ is valid when so interpreted. We refer to interpretations in Set as "classical" and those in Asm as "effective". More generally the adjectives "classical" and "effective" are used distinguish between the two settings. For example, a "classical algebra" is an algebra in Set, while an "effective algebra" is one in Asm. Similarly, a (classical) space is "classically complete" if the formula expressing completeness is valid in Set, and an (effective) space is "effectively complete" if the same formula is valid in Asm. Note however that the exact interpretation of "effective" depends on the choice of the underlying computational model.

[8] The special value $\ulcorner n \urcorner = \star$ signifies that n is not a valid Gödel code. This is not necessary for enumeration of all terms, but we do need it when we consider enumerations of closed terms, of which there may be none.

3.1 Subalgebras generated by subassemblies

Suppose $\mathcal{A}$ is classical Σ-algebra, and consider a subset $C \subseteq |\mathcal{A}|$ of the carrier. Then there exists the least subalgebra $\mathcal{I} \leq \mathcal{A}$ such that $C \subseteq |\mathcal{I}|$, namely the intersection of all subalgebras that contain C. We say that $\mathcal{I}$ is *generated by* C and denote it by $\langle C \rangle_{\mathcal{A}}$.

Now let $\mathcal{A}$ be an effective Σ-algebra and $\mathbf{C} \rightarrowtail |\mathcal{A}|$ a subassembly of $|\mathcal{A}|$. There exists the least effective subalgebra $\langle \mathbf{C} \rangle_{\mathcal{A}} \leq \mathcal{A}$ containing $\mathbf{C}$ as a subassembly. One way of proving this is to work in the internal language of the realizability topos $\mathsf{RT}(\mathbb{A}, \mathbb{A}^{\#})$, where $\langle \mathbf{C} \rangle_{\mathcal{A}}$ is the intersection of all subalgebras of $\mathcal{A}$ that contain the assembly $\mathbf{C}$, just like in Set. A special case is the *initial subalgebra* $\langle \emptyset \rangle_{\mathcal{A}}$ which is generated by the empty subassembly. It is always modest, even if $\mathcal{A}$ is not, and is effectively enumerated by a realized map $e : \mathbf{N} \to \{\star\} + \langle \emptyset \rangle_{\mathcal{A}}$ which is essentially the composition of the Gödel numbering of the closed terms over Σ with their interpretation in $\mathcal{A}$.

3.2 Algebras characterized by their universal properties

When a classical algebra is characterized up to isomorphism by a universal property, we may use the property to identify the corresponding effective algebra. It turns out that we usually get the generally accepted "correct" computability structure:

- The natural numbers $\mathbb{N}$ are the initial commutative semiring with unit. In Asm this is the modest set $\mathbf{N} = (\mathbb{N}, \Vdash_{\mathbb{N}})$ where $\overline{n} \Vdash_{\mathbb{N}} n$ for each $n \in \mathbb{N}$.
- The initial commutative ring in Set are the integers $\mathbb{Z}$, while in Asm it is the modest set $\mathbf{Z} = (\mathbb{Z}, \Vdash_{\mathbb{Z}})$ where, for each $m, n \in \mathbb{N}$ and $k \in \mathbb{Z}$, $\langle \overline{m}, \overline{n} \rangle \Vdash_{\mathbb{Z}} k$ when $k = m - n$.
- The field of fractions over the integers in Set are the rationals $\mathbb{Q}$. In Asm it is the modest set $\mathbf{Q} = (\mathbb{Q}, \Vdash_{\mathbb{Q}})$ where, for all $k, m, n \in \mathbb{N}$ and $q \in \mathbb{Q}$, $\langle \overline{k}, \overline{m}, \overline{n} \rangle \Vdash_{\mathbb{Q}} q$ when $q = (k - m)/n$.
- The reals $\mathbb{R}$ are the Cauchy-complete archimedean ordered field. The counterpart in assemblies is the modest set $\mathbf{R} = (R, \Vdash_{\mathbb{R}})$ where $\mathtt{x} \Vdash_{\mathbb{R}} x$ when $\mathtt{x} \in \mathbb{A}$ represents a fast Cauchy sequence[9] of rational numbers converging to x, and $R = \{x \in \mathbb{R} \mid \exists\, \mathtt{x} \in A \,.\, \mathtt{x} \Vdash_{\mathbb{R}} x\}$. Depending on the PCA $\mathbb{A}$ the set R could consist just of the computable reals, or all reals, or all reals computable with respect to an oracle, etc.

Unfortunately, such universal characterizations are not always available.

Apart from first-order formulas over a signature Σ we shall also consider more general first-order formulas which additionally refer to the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, and the rationals $\mathbb{Q}$. We call them *extended formulas over the signature* Σ. When they are interpreted in Set, the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ receive their usual meaning, whereas in Asm we interpret them as the corresponding assemblies $\mathbf{N}$, $\mathbf{Z}$, and $\mathbf{Q}$, as described above. An extended formula may *not* refer

[9] A sequence $(a_n)_n$ is fast Cauchy if $|a_m - a_n| \leq 2^{-\min(m,n)}$ for all $m, n \in \mathbb{N}$.

directly to the real numbers because Propositions 1 and 2 fail for formulas that refer to the reals. An extended formula over Σ which is also negative is called *extended negative formula over* Σ.

3.3 Transfer of algebras from sets to assemblies

Every set S may be represented as a *constant assembly* $\nabla S = (S, \Vdash_{\nabla S})$ where $\mathtt{r} \Vdash_S x$ holds for all $\mathtt{r} \in A$ and $x \in S$. In other words, in ∇S every realizer realizes every element. Every function $f : S \to T$ between sets S and T is realized as a map $\nabla f : \nabla S \to \nabla T$, for example by the realizer $\mathsf{s}\,\mathsf{k}\,\mathsf{k}$. This gives us a full and faithful embedding $\nabla : \mathsf{Set} \to \mathsf{Asm}$.

The functor ∇ preserves finite limits, and finite products in particular. Therefore, ∇ maps a Σ-algebra $\mathcal{A}$ in Set to a Σ-algebra $\nabla\mathcal{A}$ in Asm. The mapping preserves and reflects equations because ∇ is faithful. Even more, it preserves all negative formulas:

Proposition 1. *Let $\mathcal{A}$ be a Σ-algebra in Set and ϕ an extended negative formula over Σ. Then $\mathcal{A} \models_{\mathsf{Set}} \phi$ if, and only if, $\nabla\mathcal{A} \models_{\mathsf{Asm}} \phi$.*

A $\neg\neg$-*dense* subalgebra $\mathcal{B} \leq \mathcal{A}$ in Asm is a subalgebra for which the mono $|\mathcal{B}| \rightarrowtail |\mathcal{A}|$ is $\neg\neg$-dense. We may assume that $|\mathcal{B}| = |\mathcal{A}|$ and that the mono $|\mathcal{B}| \rightarrowtail |\mathcal{A}|$ is the identity map.

Proposition 2. *Let $\mathcal{A}$ be an effective Σ-algebra and $\mathcal{B} \leq \mathcal{A}$ a $\neg\neg$-dense subalgebra. Then $\mathcal{A}$ and $\mathcal{B}$ satisfy the same extended negative formulas over Σ.*

The proofs of both propositions are standard exercises in performing an induction over the structure of ϕ. The deeper reason for their truth is the fact that sets are precisely the sheaves for the double negation topology on the realizability topos $\mathsf{RT}(\mathbb{A}, \mathbb{A}^\#)$.

4 Premetric spaces

A *metric algebra* is a Σ-algebra $\mathcal{A}$ whose carrier is a metric space and the operations are continuous maps. A metric algebra is *complete* if its carrier is a complete metric space. We face a difficulty when we try to transfer metric algebras from sets to assemblies: ∇ maps a metric $d : S \times S \to \mathbb{R}$ to the realized map $\nabla d : \nabla S \times \nabla S \to \nabla\mathbb{R}$, which is not a metric anymore because its codomain $\nabla\mathbb{R}$ is not the object $\mathbf{R}$ of real numbers in Asm. To overcome the problem we use a formulation of metric spaces which does not directly refer to real numbers, is classically equivalent to the usual metric spaces,[10] and is constructively acceptable. Such a notion, namely *premetric spaces*, was defined by Fred Richman [6]. We use a slight variation:

[10] We allow infinite distances but that is inessential.

Definition 1. *A* premetric space (X, d) *is a set* X *with a ternary relation* $d \subseteq X \times X \times \mathbb{Q}$ *satisfying the following conditions, where we write* $d(x, y) \leq q$ *instead of* $(x, y, q) \in d$*:*

1. *if* $q < 0$ *then not* $d(x, y) \leq q$*,*
2. $d(x, y) \leq 0$ *if, and only if,* $x = y$*,*
3. *if* $d(x, y) \leq q$ *then* $d(y, x) \leq q$*,*
4. *if* $d(x, y) \leq q$ *and* $d(y, z) \leq r$ *then* $d(x, z) \leq q + r$*,*
5. $d(x, y) \leq q$ *if, and only if,* $d(x, y) \leq r$ *for all* $r > q$*.*

Richman's definition also requires that for all $x, y \in X$ there is a rational $q \geq 0$ such that $d(x, y) \leq q$. We omit the requirement because we do not need it, and because it is the only axiom which is not a negative formula.

Every metric space (M, d) is a premetric space (M, d') with $d' = \{(x, y, q) \in X \times X \times \mathbb{Q} \mid d(x, y) \leq q\}$. Classically, the converse holds if we allow infinite distances[11] because the metric d may be recovered from the premetric d' as $d(x, y) = \inf \{q \in \mathbb{Q} \mid d'(x, y) \leq q\}$. Constructively however the infimum need not exist.

The basic theory of premetric spaces parallels that of metric spaces. The notions of completeness, continuity, density, etc., are all easily expressed in terms of the premetric. In fact, the whole theory is constructively valid (even without choice), as was shown by Richman [6]. Despite our allowing infinite distances, the following theorem still holds constructively, and is therefore valid both in Set and Asm.

Proposition 3. *Let* X *be a premetric space and* $e : X \to Y$ *its* completion*, i.e., an isometry with a dense image into a complete premetric space* Y*. Then every locally uniformly continuous*[12] $f : X \to Z$ *to a complete premetric space* Z *has a unique locally uniformly continuous extension* $\overline{f} : Y \to Z$ *along* e*.*

An easy consequence of the theorem is that any two completions of a premetric space are isometrically isomorphic.

When a premetric space (X, d) is transferred from Set to Asm by ∇, the relation $d \subseteq X \times X \times \mathbb{Q}$ is mapped to the mono $\nabla d \rightarrowtail \nabla X \times \nabla X \times \nabla \mathbb{Q}$, which is $\neg\neg$-stable. The axioms for premetric structure are extended negative formulas, so by Proposition 1 they are preserved. This proves the following proposition:

Proposition 4. *If* (X, d) *is a classical premetric space then* $(\nabla X, \nabla d)$ *is an effective premetric space. Furthermore,* (X, d) *and* $(\nabla X, \nabla d)$ *satisfy the same extended negative formulas.*

Moreover, ∇ preserves the completeness property, which follows easily from the observation that the exponential assembly $(\nabla X)^{\mathbb{N}}$ is isomorphic to $\nabla(X^{\mathbb{N}})$:

Proposition 5. *A classical premetric space* (X, d) *is classically complete if, and only if,* $(\nabla X, \nabla d)$ *is effectively complete.*

[11] With Richman's extra axiom the correspondence between metric and premetric spaces is exact, classically.

[12] A map is locally uniformly continuous if it is uniformly continuous on every closed ball.

4.1 Complete subalgebras

When $\mathcal{A}$ is a classical complete premetric Σ-algebra we may ask whether every subalgebra $\mathcal{B} \leq \mathcal{A}$ is contained in the least *complete* subalgebra $\overline{\mathcal{B}} \leq \mathcal{A}$. The premetric closure $\overline{|\mathcal{B}|}$ in $|\mathcal{A}|$ is an obvious candidate. For it to be a subalgebra, each operation $\mathsf{f}^{\mathcal{B}} : |\mathcal{B}|^n \to |\mathcal{B}|$ must extend to a map $\mathsf{f}^{\overline{\mathcal{B}}} : \overline{|\mathcal{B}|}^n \to \overline{|\mathcal{B}|}$, which it does by Theorem 3 as long as the operations on $\mathcal{B}$ are locally uniformly continuous. We have proved the following proposition.

Proposition 6. *Let $\mathcal{A}$ be a classical complete Σ-algebra. The closure $\overline{|\mathcal{B}|}$ of the carrier of a subalgebra $\mathcal{B} \leq \mathcal{A}$ is the least complete subalgebra of $\mathcal{A}$ containing $\mathcal{B}$, provided the operations on $\mathcal{B}$ are locally uniformly continuous.*

The argument which proved Proposition 6 is constructively valid. Its interpretation in Asm gives the following effective version.

Proposition 7. *Let $\mathcal{A}$ be an effective*[13] *complete Σ-algebra. The effective closure $\overline{|\mathcal{B}|}$ of the carrier of a subalgebra $\mathcal{B} \leq \mathcal{A}$ is the least effective complete subalgebra of $\mathcal{A}$ containing $\mathcal{B}$, provided the operations on $\mathcal{B}$ are effectively locally uniformly continuous.*

We remark that the complete subalgebra $\overline{\mathcal{B}}$ generated by $\mathcal{B}$ is modest if $\mathcal{B}$ is modest, even if $\mathcal{A}$ is not.

5 Main Theorems

The results of the previous sections give us a method for finding canonical effective subalgebras of classical algebras. Let $\mathcal{A}$ be a classical premetric Σ-algebra. In general there will be many effective subalgebras $\mathcal{B} \leq \nabla\mathcal{A}$, each carving out a different piece of $\mathcal{A}$ with its own effective structure. Our first theorem gives conditions which severely cut down the number of possibilities. Define the relation $d(x, y) < q$ for $x, y \in |\mathcal{A}|$ and $q \in \mathbb{Q}$ by $d(x, y) < q \iff \exists\, r \in \mathbb{Q} \,.\, d(x, y) \leq r \wedge r < q$.

Theorem 1. *Suppose $\mathcal{A}$ is a classical premetric Σ-algebra in which the initial subalgebra $\langle\emptyset\rangle_{\mathcal{A}}$ is classically dense. Up to isomorphism, there is at most one effectively complete subalgebra $\mathcal{B} \leq \nabla\mathcal{A}$ on which the relation $d(x, y) < q$ is semidecidable.*

We omit the proof, and just note that $\mathcal{B}$, if it exists, is the effective completion of the initial subalgebra $\langle\emptyset\rangle_{\nabla\mathcal{A}}$.

When the initial subalgebra $\langle\emptyset\rangle_{\mathcal{A}}$ is not dense, Theorem 1 cannot be applied. Quite often this can be fixed with a judicious addition of new constants and operations. For example, the initial subring of the ring $\mathcal{C}[0, 1]$ of continuous real functions on the closed unit interval is the ring of integers (embedded as

[13] To be precise, we are talking about an "effectively complete effectively premetric effective Σ-algebra".

constant functions), which is not dense. If we adjoin the identity function and the constant function $\frac{1}{2}$ as primitive constants, the initial subalgebra will be the ring of polynomials whose coefficients are dyadic rationals,[14] which is dense by the (classical) Stone-Weierstraß theorem.

Another way to deal with non-dense initial subalgebra is to replace $\langle\emptyset\rangle_{\mathcal{A}}$ in Theorem 1 with a chosen dense subalgebra $\mathcal{D} \leq \mathcal{A}$, but then the statement is that there is at most one effectively complete subalgebra of $\nabla\mathcal{A}$ containing $\nabla\mathcal{D}$ for which $d(x, y) < q$ is semidecidable.

The next theorem complements Theorem 1 by giving conditions for existence of subalgebras.

Theorem 2. *Let $\mathcal{A}$ be a classical complete premetric Σ-algebra. Suppose the relation $d(x, y) < q$ is semidecidable on $\langle\emptyset\rangle_{\nabla\mathcal{A}}$ and the operations of $\langle\emptyset\rangle_{\nabla\mathcal{A}}$ are effectively locally uniformly continuous. Then $\nabla\mathcal{A}$ has an effective complete subalgebra on which the relation $d(x, y) < q$ is semidecidable.*

Again, we omit the proof. We know from the previous theorem that the desired subalgebra must be the completion of $\langle\emptyset\rangle_{\nabla\mathcal{A}}$, from which a concrete representation can be computed: because $\langle\emptyset\rangle_{\nabla\mathcal{A}}$ is essentially represented by a Gödel numbering of closed terms, its completion is represented by sequences of (Gödel codes of) closed terms that are fast Cauchy.

6 Applications

In this section we apply the results to two common scenarios.

6.1 Discrete premetric spaces

The simplest kind of complete premetric algebras are the discrete ones. Let $\mathcal{A}$ be a classical Σ-algebra and define the *discrete premetric* on $|\mathcal{A}|$ by

$$d(x, y) \leq q \iff (q < 1 \implies x = y),$$

which of course corresponds to the metric that takes on only values 0 and 1. In the discrete premetric every set is complete and every map is uniformly continuous. Therefore, half of the conditions in Theorems 1 and 2 are trivially satisfied. Furthermore, a discrete premetric is semidecidable on $\mathcal{B} \leq \nabla\mathcal{A}$ if, and only if, equality is semidecidable on $\mathcal{B}$, because $x = y \iff d(x, y) < 1$ and $d(x, y) < q \iff (q > 1 \lor x = y)$. Thus we obtain the following result.

Proposition 8. *Suppose $\mathcal{A}$ is a* finitely generated *classical Σ-algebra. Up to isomorphism, there is at most one effective structure on $\mathcal{A}$ for which the operations and the generators are effective, and equality is semidecidable. Furthermore, if there is such an effective structure, it is isomorphic to the effective subalgebra $\langle\{a_1, \ldots, a_n\}\rangle_{\nabla\mathcal{A}}$ of $\nabla\mathcal{A}$ generated by the generators $a_1, \ldots, a_n$ for $\mathcal{A}$.*

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[14] A dyadic rational is one of the form $n/2^k$.

More precisely, the first part of the proposition states that there is at most one realizability relation $\Vdash_{\mathcal{A}}$ on the set $|\mathcal{A}|$ which turns the classical algebra $\mathcal{A}$ into an effective one[15] such that equality is semidecidable. The second part gives an explicit description of the effective structure, and also implies that the resulting assembly is modest.

In the context of type 1 effectivity Proposition 8 was first proved by Mal'cev, see [7] and [8, Theorem 4.1.2]. He actually considered two versions, one with general recursive functions and another with partial recursive functions. Our result corresponds to the partial recursive case because all partial recursive functions are representable in a PCA.

6.2 The real numbers

The real numbers form a classical ordered field, and a classical complete premetric space with the usual premetric $d(x, y) \leq q \iff |x - y| \leq q$. A slight complication is division because it is a partial operation. The journal version of this extended abstract will include a proper treatment of partial operations. For now, we circumvent division by viewing the real numbers as a ring $\mathcal{R} = (\mathbb{R}, 0, 1, \frac{1}{2}, +, -, \times)$ with a primitive constant $\frac{1}{2}$. The initial subalgebra is the ring of dyadic rationals, which is dense in $\mathbb{R}$. The relation $|x - y| < q$ is semidecidable, even decidable when x and y are dyadic rationals and q a rational. The operations are easily seen to be effectively locally uniformly continuous. Thus the conditions of both main theorems are satisfied. Up to isomorphism there is exactly one effectively complete effective subring $\mathbf{R} \leq \nabla\mathcal{R}$ on which the relation $d(x, y) < q$ is semidecidable. We may replace semidecidability of $d(x, y) < q$ with semidecidability of the strict order relation $x < y$ because $d(x, y) < q \iff -q < x - y < q$ and

$$x < y \iff \\ \exists\, q, r \in \mathbb{Q}\,.\, \exists\, k \in \mathbb{N}\,.\, \big(d(x, q) < 2^{-k} \wedge d(y, r) < 2^{-k} \wedge q + 2^{-k+2} < r\big)\,.$$

The dyadic rationals have *approximate division*: for all $k \in \mathbb{N}$ and dyadic rationals a and $b \neq 0$ there exists a dyadic rational c such that $d(a, bc) < 2^{-k}$. The completion of a premetric ring with approximate division is always a field, constructively speaking. By putting all these observations together we get the following result.

Proposition 9. *Up to isomorphism, there is exactly one effectively complete effective subfield of the real numbers for which the strict linear order is is semidecidable.*

When the proposition is specialized to type 2 effectivity it gives Hertling's result [2] about type 2 representations of reals, while the interpretation in type 1 effectivity corresponds to a result of Moschovakis [1].

[15] This means that the operations and generators are realized by elements of $\mathbb{A}^{\#}$.

7 Conclusion

The relation d on a premetric space (X, d) induces a uniform structure on X whose (basic) entourages are $E_q = \{(x, y) \in X \times X \mid d(x, y) \leq q\}$, for rational $q > 0$. This suggests that one should look for a generalization to uniform spaces. We would first need a suitable constructive treatment of uniform spaces and their completions.

Another direction which might be worth investigating follows the work of Blanck et al. [9] who formulated general results about stability of effective algebras in type 1 effectivity. Their theorems do not translate into our settings easily, because they assume a structure which is not metric, but rather like that of sequential or limit spaces. Again, to incorporate such results we would require a constructive theory of limit spaces and their completions.

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