Solving Periodic Timetable Optimisation Problems by Modulo Simplex Calculations

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Abstract. In the last 15 years periodic timetable problems have found much interest in the combinatorial optimization community. We will focus on the optimisation task to minimise a weighted sum of undesirable slack times. This problem can be formulated as a mixed integer linear problem, which for real world instances is hard to solve. This is mainly caused by the integer variables, the so-called modulo parameter. At first we will discuss some results on the polyhedral structure of the periodic timetable problem. These ideas allow to define a modulo simplex basic solution by calculating the basic variables from modulo equations. This leads to a modulo network simplex method, which iteratively improves the solution by changing the simplex basis.

Key words: periodic event scheduling problem, integer programming, modulo network simplex

1 Introduction

In the last 15 years periodic timetable problems have found much interest in the combinatorial optimization community. Most results presented in [6, 10, 3, 4, 7, 8, 2] are based on a periodic event scheduling model published by Serafini and Ukovich 1989 ([11]).

The associated periodic event activity networks allow a flexible modelling of fixed interval timetables in public transport. A lot of practical requirements, like sequencing of trains, safety headway distances and limits for rolling stock can be incorporated into this network theoretical model. In this paper we will focus on the optimisation task to minimise a weighted sum of undesirable slack times, e.g., waiting time for passengers.

Define a *railway system* as a system of lines $\mathcal{L}$ and stations $\mathcal{S}$. Each line $L \in \mathcal{L}$ is understood to be a *transportation chain*, where the trains of L are serving a certain sequence of stations (see e.g. [12]). If line L serves stations S, then define (L, arr, S) and (L, dep, S) to be the arrival (departure) event of L at S.

A *schedule* assigns *event times* $\pi_i \in \mathbb{R}$ to all events $i = (L, dep, S)$ or $i = (L, arr, S)$. An activity $a : i \to j$ is a time consuming process, which then will consume the amount $x_a := \pi_j - \pi_i$. of time. A line can be understood as an alternating sequence of

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– run activities : $(L, dep, S) \rightarrow (L, arr, S')$ and
– stop activities : $(L, arr, S) \rightarrow (L, dep, S)$.

Run and stop activities are associated with time spans $\Delta_a = [\ell_a, u_a]$, where ℓ_a is the minimum running or stopping time and u_a is an upper bound. [1] A schedule $\boldsymbol{\pi}$ is said to be *feasible*, if $x_a = \pi_j - \pi_i \in \Delta_a$ for all $a : i \rightarrow j$. Apart from running and stopping activities, in real world problems there are many other types of constraints arising from operational, safety- or marketing-related restrictions. Almost all practical requirements can be formulated in terms of span constraints $\ell_a \leq \pi_j - \pi_i \leq u_a$ defined on a suitable arc $a : i \rightarrow j$ of the event network. Some examples are:

– *Headway constraints:* Trains using the same parts of the infrastructure have to keep a certain safety distance. This distance can be expressed as a time difference between the arrival or departure times of the lines at the stations.
– *Traveller connection constraints:* In general, there are some stations where travellers have to change from one train to another. In this case, these travellers would like to have a short waiting time at the station. Again, this constraint is a time difference constraint between arrival and departure times of lines.

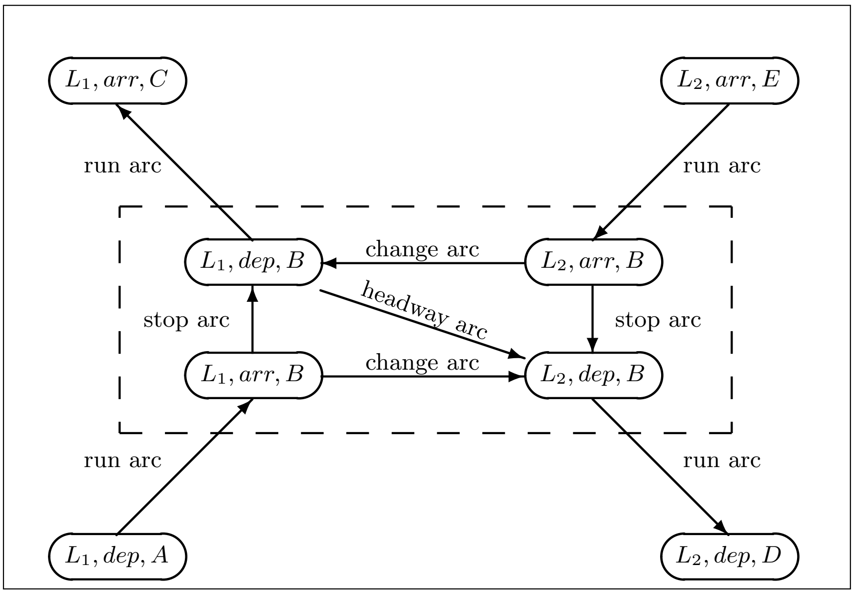


Fig. 1. Event-activity network

[1] If the running time is fixed, a running activity and the following stop activity can be simply described by one combined constraint.

Non-periodic timetable problems are very easy to solve by shortest path calculations. For fixed interval timetables, where all departure and arrival events will be repeated periodically, such a simple model is no more appropriate. The reasons are manifold: A priori it is not clear between which trains passengers are changing or in which sequence trains are leaving or entering stations. All this can only be decided after the time ordering of all events is known. A *periodic schedule* assigns periodic *event times* $\pi_i \in \mathbb{R}$ to all events, which will take place at all time points $\pi_i + zT\,(z \in \mathbb{Z})$. The integer multiples z of the period are called *modulo parameter* and code the periodic sequence of all events.

For reasons of simplicity we assume one common period T for the complete system. Different periods for the lines can be handled by using the least common multiple (compare for [5]).

A solution of the *periodic* timetable problem is defined by a vector $\boldsymbol{\pi} \in \mathbb{R}^n$, which defines for each event i one point of time π_i, such that i will be periodically repeated at all times $\pi_i + z_iT\,(z_i \in \mathbb{Z})$.

Define ℓ_a and u_a to be the minimum and maximum allowed process times of a constraint $a : i \to j$. Then a periodic timetable $\boldsymbol{\pi}$ is feasible, if

$$\forall a : i \to j \in \mathcal{A} : \exists z_a \in \mathbb{Z} : \ \ell_a \le \pi_j - \pi_i - z_aT \le u_a. \tag{1}$$

Lower and upper slack time measures that amount of time for which the tension $\pi_j - \pi_i$ on this arc may be increased or decreased and is defined by

$$\begin{aligned} y_a^{low} &:= [x_a - \ell_a]_T = x_a - \ell_a - z_aT \text{ for a suitable } z_a \in \mathbb{Z} \\ y_a^{upp} &:= [u_a - x_a]_T = u_a - x_a + z_aT \text{ for a suitable } z_a \in \mathbb{Z}. \end{aligned}$$

The modulo operator is defined by $[t]_T := \min \{\, t + zT \mid t + zT \ge 0\}$ and fulfills $0 \le [t]_T < T$.

Since lower and upper slack times may be exchanged by inverting the direction of the arc a, the problem to minimize the slack time in a periodic timetable can be defined in terms of lower slack time y_a^{low}. In summary, the **periodic timetable slack problem** can be formulated as the mixed integer program

$$\min \left\{ \sum_{a:i \to j} \boldsymbol{\omega}_a(\pi_j - \pi_i - \ell_a - z_aT) \;\middle|\; \forall a \in \mathcal{A} : \ \ell_a \le \pi_j - \pi_i - z_aT \le u_a;\ z_a \in \mathbb{Z} \right\} \tag{2}$$

The resulting planning problems are known to be NP-hard.

2 The Periodic Timetable Polyhedron

At first we will briefly summarize the basic concepts and notations of network flow models.

The incidence matrix of a network is an $n \times m$ matrix $\Theta = (\theta_{ia})$ which contains one row for each arc a and one column i for each node:

$$\theta_{ai} = \begin{cases} 1, & \text{if } a : j \to i \\ -1, & \text{if } a : i \to j \\ 0, & \text{else} \end{cases}$$

A potential $\boldsymbol{\pi} \in \mathbb{R}^n$ associates with each node $i = 1, .., n$ a real value $\pi_i \in \mathbb{R}$.

$$\mathcal{Q} := conv.hull \left(\left\{ \begin{pmatrix} \boldsymbol{\pi} \\ \boldsymbol{z} \end{pmatrix} \middle| \; \boldsymbol{\ell} \leq \Theta^t \boldsymbol{\pi} - T\boldsymbol{z} \leq \boldsymbol{u};\; \boldsymbol{z} \in \mathbb{Z}^m; \boldsymbol{\pi} \in \mathbb{R}^n \right\} \right)$$

is said to be the periodic timetable polyhedron.

The potential difference $x_a := \pi_j - \pi_i$ is said to be the tension on arc $a : i \to j$ and can be expressed as $\Theta^t \boldsymbol{\pi} = \boldsymbol{x}$. Adding a co-tree arc a to the arcs of a spanning tree $\mathcal{T}$, defines a uniquely determined cycle c. Its incidence vector $\boldsymbol{\gamma}_c = (\gamma_{ca})$ is defined by

$$\gamma_{ca} := \begin{cases} 1 & \text{, if the cycle contains arc } a \text{ in positive direction} \\ -1 & \text{, if the cycle contains arc } a \text{ in negative direction} \\ 0 & \text{, else.} \end{cases}$$

The network matrix $\Gamma = (\gamma_{ca})$ of a tree $\mathcal{T}$ contains for each co-tree arc the incidence vector of the associated cycle as one row. $\boldsymbol{x} \in \mathbb{R}^m$ is a tension (i.e. there exists a potential $\boldsymbol{\pi} \in \mathbb{R}^n$ with $\Theta^t \boldsymbol{\pi} = \boldsymbol{x}$), if and only if there holds $\Gamma \boldsymbol{x} = \boldsymbol{0}$. A *periodic tension* $\boldsymbol{x}$ fulfils $\Gamma \boldsymbol{x} \equiv_T \boldsymbol{0}$.

A spanning tree structure $\mathcal{T} = \mathcal{T}^\ell + \mathcal{T}^u$ is a spanning tree, whose tree arcs are partitioned into those arcs $\mathcal{T}^\ell$ and $\mathcal{T}^u$, where the tension is restricted to be at its lower or upper bound, respectively [2]. Each spanning tree structure determines a unique potential $\boldsymbol{\pi}^{(\mathcal{T})}$, which fullfills $\pi_j^{(\mathcal{T})} - \pi_i^{(\mathcal{T})} = \ell_a$ for $(a : i \to j) \in \mathcal{T}^\ell$ and $\pi_j^{(\mathcal{T})} - \pi_i^{(\mathcal{T})} = u_a$ for $(a : i \to j) \in \mathcal{T}^u$. The spanning tree structure is said to be *feasible*, if the generated potential is feasible with respect to the span constraints for all arcs (1).

By using $\boldsymbol{b} :\equiv_T -\Gamma \boldsymbol{\ell}$ and $\boldsymbol{\delta} := \boldsymbol{u} - \boldsymbol{\ell}$, the periodic slack space is defined by

$$\mathcal{Y} := \{ \boldsymbol{y} \in \mathbb{Z}^m \mid \Gamma \boldsymbol{y} \equiv_T \boldsymbol{b}; \boldsymbol{0} \leq \boldsymbol{y} \leq \boldsymbol{\delta} \}$$

and the optimisation task is to determine $\min \{ \boldsymbol{\omega}^t \boldsymbol{y} \mid \boldsymbol{y} \in \mathcal{Y} \}$.

If the modulo parameters z_a are fixed, optimisation problem (2) becomes

$$\min \left\{ \sum_{a:i \to j} \boldsymbol{\omega}_a (\pi_j - \pi_i - \ell_a - z_a T) \;\middle|\; \forall a \in \mathcal{A} :\; \ell_a \leq \pi_j - \pi_i - z_a T \leq u_a \right\}$$

[2] This definition differs from that definition given in [1]. This is caused by the circumstances, that the dual timetable problem is a modified minimum cost flow problem **without** capacity on the arc flow values.

$$= \min \left\{ \sum_{a:i\to j} \boldsymbol{\omega}_a(\pi_j - \pi_i - \ell'_a) \;\middle|\; \forall a \in \mathcal{A}: \; \ell'_a = \ell_a + z_a T \le \pi_j - \pi_i \le u'_a = u_a + z_a T \right\}$$
$$= \min \left\{ \boldsymbol{\omega}^t \left(\Theta^t \boldsymbol{\pi} - \boldsymbol{\ell}'\right) \mid \boldsymbol{\ell}' \le \Theta^t \boldsymbol{\pi} \le \boldsymbol{u}' \right\} \tag{3}$$

the dual of a minimum cost flow problem (see [1]). The extreme points of the feasible region of this problem are associated with spanning tree structures. The network simplex method described in [1] interprets the core concept of the simplex method appropriately as network operations. In particular, each optimal basis can be characterized by the underlying spanning tree structure.

If $\boldsymbol{z}^{\mathcal{T}}$ denotes the associated modulo parameter, then $\begin{pmatrix} \boldsymbol{\pi}^{(\mathcal{T})} \\ \boldsymbol{z}^{\mathcal{T}} \end{pmatrix}$ is called a periodic basic solution with respect to the spanning tree structure $\mathcal{T}$. The following theorem is due to [6].

Theorem 21 (Extreme Points and Spanning Tree Structures) $\begin{pmatrix} \boldsymbol{\pi} \\ \boldsymbol{z} \end{pmatrix} \in \mathcal{Q}$ *is an extremal point of* $\mathcal{Q}$, *if and only, if it is a periodic basis solution with respect to a spanning tree structure.*

The orthogonal complement of the tension space is known to be the space of all flows ([9]), i.e. it holds $\{\boldsymbol{x} \mid \Gamma\boldsymbol{x} = \boldsymbol{0}\}^{\perp} = \{\boldsymbol{\varphi} \mid \Theta\boldsymbol{\varphi} = \boldsymbol{0}\}$. The space of all periodic tensions is defined by

$$\mathcal{X} := \{\boldsymbol{x} \in \mathbb{Z}^m \mid \Gamma\boldsymbol{x} \equiv_T \boldsymbol{0}\}$$

In the periodic case, we obtain

$$\{\boldsymbol{x} \in \mathbb{Z}^m \mid \Gamma\boldsymbol{x} \equiv_T \boldsymbol{0}\}^{\perp_T} = \{\boldsymbol{\varphi} \in \mathbb{Z}^m \mid \Theta\boldsymbol{\varphi} \equiv_T \boldsymbol{0}\} \tag{4}$$

The following structural characterization of valid inequalities is due to [4] and are discussed in more detail in [3].

Lemma 2.1 Let $\mathcal{Q} \neq \emptyset$. Then $\boldsymbol{\vartheta}^t\boldsymbol{\pi} - \boldsymbol{f}^t\boldsymbol{z} \ge r$ can only be a valid inequality for the polyhedron

$$\mathcal{Q} := conv.hull\left(\left\{ \begin{pmatrix} \boldsymbol{\pi} \\ \boldsymbol{z} \end{pmatrix} \;\middle|\; \boldsymbol{\ell} \le \Theta^t\boldsymbol{\pi} - T\boldsymbol{z} \le \boldsymbol{u}; \; \boldsymbol{z} \in \mathbb{Z}^m; \boldsymbol{\pi} \in \mathbb{R}^n \right\} \right)$$

with $\boldsymbol{\vartheta}^t\boldsymbol{\pi}^{(0)} - \boldsymbol{f}^t\boldsymbol{z}^{(0)} = r$ for at least one $\begin{pmatrix} \boldsymbol{\pi}^{(0)} \\ \boldsymbol{z}^{(0)} \end{pmatrix} \in \mathcal{Q}$, if and only if $\boldsymbol{f}$ is a flow with balance $\boldsymbol{\vartheta}$, i.e. it holds $T\boldsymbol{\vartheta} = \Theta\boldsymbol{f}$ and

$$Tr = \min\left\{ \boldsymbol{f}^t\boldsymbol{x} \;\middle|\; x \in \mathcal{X} \right\}$$

∎

Theorem 22 *There exists a matrix* F, *where each of its rows is a periodic tension (i. e.* $\Theta F \equiv_T \boldsymbol{0}$*) and a right hand side* $\boldsymbol{r}$, *such that*

$$conv.hull\left(\{\boldsymbol{x} \in \mathbb{Z}^m \mid \exists \boldsymbol{z} \in \mathbb{Z}^m : \Gamma\boldsymbol{x} - T\boldsymbol{z} = \boldsymbol{0}; \boldsymbol{\ell} \le \boldsymbol{x} - T\boldsymbol{z} \le \boldsymbol{u}\}\right)$$

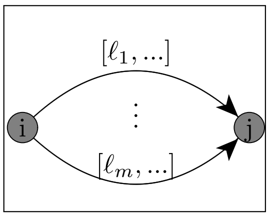
$$= \{ \boldsymbol{x} \mid F\boldsymbol{x} \geq \boldsymbol{r} \} ,$$

or equivalently

$$\begin{aligned} & conv.hull\,(\{ \boldsymbol{y} \in \mathbb{Z}^m \mid \exists \boldsymbol{z} \in \mathbb{Z}^m : \Gamma \boldsymbol{y} - T\boldsymbol{z} = \boldsymbol{b}; \boldsymbol{0} \leq \boldsymbol{y} - T\boldsymbol{z} \leq \boldsymbol{\delta} \}) \\ = & \{ \boldsymbol{y} \mid F\boldsymbol{y} \geq \tilde{\boldsymbol{r}} := \boldsymbol{r} - F\boldsymbol{\ell} \} \end{aligned}$$

■

An example for the construction of such inequalities is as follows. Consider a system of parallel arcs connecting two nodes i and j. The unbounded periodic timetable slack problem (without upper bounds on the arcs) deals with timetables from the set



$$P^*(\boldsymbol{\ell}) := \left\{ \begin{pmatrix} \pi_i \\ \pi_j \\ z_1 \\ \vdots \\ z_m \end{pmatrix} \in \mathbb{Z}^{m+2} \;\middle|\; \forall a = 1, \ldots m : \ell_a \leq \pi_j - \pi_i - z_a T \right\}$$

Without loss of generality, a non degenerate[3] lower bound vector $\boldsymbol{\ell}$ can be assumed to be normalized in the sense, that

$$0 \leq \ell_1 = [\ell_1]_T < \ell_2 = [\ell_2]_T < \ldots < \ell_m = [\ell_m]_T < T \tag{5}$$

Lemma 2.2 For $\ell \in \mathbb{Z}^m$ with

$$0 \leq \ell_1 < \ell_2 < \ldots < \ell_m < T \tag{6}$$

define the vector $\boldsymbol{f}$ by

$$f_a := \begin{cases} \ell_1 - \ell_m + T & \text{if } a = 1 \\ \ell_a - \ell_{a-1} & \text{if } a > 1 \end{cases} \tag{7}$$

Then there holds

1. $\forall a = 1, \ldots, m : \; 0 \leq f_a < T$
2. $\sum_{a=1}^{m} f_a = T$
3. $\forall a' : \sum_{a=a'+1}^{m} f_a = \ell_m - \ell_{a'}$

[3] $\boldsymbol{\ell}$ is called non degenerate, if $[\ell_a]_T \neq [\ell_{a'}]_T$ forall $a \neq a'$

Especially, $\boldsymbol{f}$ is a periodic flow with node mass balance $\vartheta_A = -T$ and $\vartheta_B = T$. The inequality

$$\pi_B - \pi_A - \boldsymbol{f}^t \boldsymbol{z} = \pi_B - \pi_A - \sum_{a=1}^{m} f_a z_a \geq f_0 := \ell_m \tag{8}$$

is a valid for $P^*(\boldsymbol{\ell})$. ■

Each of the considered arcs may be replaced by a chain of arcs, resulting in a system of paths between i and j. Consider a spanning tree. Then each tree arc $a : i \rightarrow j$ generates a cut and for each arc within this cut we find a path from i to j. Hence, there is a natural i, j path system, which can be used to generate cutting planes or equivalently rows of the matrix F.

3 The Modulo Simplex Method

For reasons of simplicity, in the following we only describe the case that the tension is restricted to be at its lower bound. This is no loss of generality, since upper bounds can be modelled as lower bounds on inverse directed arcs. Within the simplex method this means that the corresponding non-basic variable is set to the upper bound. Feasibility check and calculation of the modified cost of basis exchanges can be done straightforward.

We consider the periodic timetable slack problem

$$\min \left\{ \boldsymbol{\omega}'^t \boldsymbol{y} \;\middle|\; \boldsymbol{y} \in \mathcal{Y} := \{ \boldsymbol{y} \in \mathbb{Z}^m \mid \exists \boldsymbol{z} \in \mathbb{Z}^m : \Gamma \boldsymbol{y} - T\boldsymbol{z} = \boldsymbol{b}; \boldsymbol{0} \leq \boldsymbol{y} - T\boldsymbol{z} \leq \boldsymbol{\delta} \} \right\}.$$

The integrality of the modulo parameter $\boldsymbol{z}$ makes the problem hard. For this reason we will eliminate those variables and keep them implicitly in the model by using modulo calculations. The modulo simplex method explores the extreme points of the polyhedron $conv.hull\,(\mathcal{Y})$.

The tree and co-tree arcs of the underlying spanning tree split the network matrix $\Gamma = [N_{\mathcal{T}}, E_{\mathcal{T}}^{co}]$ into its basic (= co-tree) and non-basic (= tree) components. Therefore a periodic basic solution is given by $\begin{pmatrix} \boldsymbol{y}_{\mathcal{T}} \\ \boldsymbol{y}_{\mathcal{T}}^{co} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{b} \end{pmatrix}$, which is feasible if $\boldsymbol{b} \leq \boldsymbol{\delta}$. Any periodic tension $\boldsymbol{x}$ (with $\Gamma \boldsymbol{x} \equiv_T \boldsymbol{0}$) leads to a new solution $\boldsymbol{y}' := [\boldsymbol{y} + \boldsymbol{x}]_T = \boldsymbol{y} + \boldsymbol{x} - \boldsymbol{z}' T$ of $\Gamma \boldsymbol{y}' \equiv_T \boldsymbol{b}$ and stays feasible, if $\boldsymbol{y}' \leq \boldsymbol{\delta} := \boldsymbol{u} - \boldsymbol{\ell}$. In the following we will describe the problem by the use of a simplex tableau like structure. Consider the network matrix $\Gamma = [N, E]$ with respect to a spanning tree. The tree arcs are denoted by $a_1, ..., a_{r-1}$ and the co-tree arcs are given by $a_r, ..., a_m$. Then the slack space is given by the modulo equations

The resulting objective is given by

$$\boldsymbol{\omega} = \sum_{i=r}^{m} \boldsymbol{\omega}_i b_i$$

$$\begin{array}{ccc|cccc}
\gamma_{r1}y_1 + \dots + & \gamma_{r\,r-1}y_{r-1} & & + y_r & & & \equiv_T b_r \\
\vdots & \vdots & & & \ddots & & \vdots \\
\gamma_{i1}y_1 + \dots + & \gamma_{i\,r-1}y_{r-1} & & + & y_i & & \equiv_T b_i \\
\vdots & \vdots & & & & \ddots & \vdots \\
\gamma_{m1}y_1 + \dots + & \gamma_{m\,r-1}y_{r-1} & & + & & y_m & \equiv_T b_m
\end{array}$$

	a_1	....	a_j	...	a_r	...	a_i	...	a_m	rhs
a_r	γ_{r1}	...	γ_{rj}	...	1	...	0	...	0	b_r
$\vdots$	$\vdots$	...	$\vdots$	....		$\ddots$				$\vdots$
a_i	γ_{i1}	...	$\boxed{\gamma_{ij}}$	...	0	...	1	...	0	b_i
$\vdots$	$\vdots$	...	$\vdots$	....				$\ddots$		$\vdots$
a_m	γ_{m1}	...	γ_{mj}	....	0	...	0	...	1	b_m
obj.										$\sum_{i=r}^{m} \omega_i b_i$

A basis exchange can be described by exchanging a leaving co-tree arc a_i with an entering tree arc a_j, which belongs to the uniquely determined co-tree cycle of the actual tree. The resulting cut $\boldsymbol{\eta}^{(a_i,a_j)}$ is given by adjoining the leaving tree component to the a_i-associated column of N. Each $\alpha \in \mathbb{Z}$ with $\alpha\boldsymbol{\eta}^{(a_i,a_j)} \leq \boldsymbol{\delta} = \boldsymbol{u} - \boldsymbol{\ell}$ defines by $\boldsymbol{y}' := \boldsymbol{y} + \alpha\boldsymbol{\eta}^{(a_i,a_j)}$ a new solution.

Exchanging co-tree arc i with tree arc j leads to the new solution

	a_1	....	a_i	...	a_r	...	a_j	...	a_n	rhs
a_r	$\gamma_{rl} - \frac{\gamma_{i1}\gamma_{rj}}{\gamma_{ij}}$	...	0	...	1	...	$-\frac{\gamma_{rj}}{\gamma_{ij}}$	...	0	$\left[b_r - \frac{\gamma_{rj}}{\gamma_{ij}} b_i\right]_T$
$\vdots$	$\vdots$	...	$\vdots$	....		$\ddots$				$\vdots$
a_j	$\frac{\gamma_{i1}}{\gamma_{ij}}$	...	1	...	0	...	$\frac{1}{\gamma_{ij}}$	...	0	$\left[\frac{b_i}{\gamma_{ij}}\right]_T$
$\vdots$	$\vdots$	...	$\vdots$	....				$\ddots$		$\vdots$
a_m	$\gamma_{m1} - \frac{\gamma_{i1}\gamma_{mj}}{\gamma_{ij}}$	...	0	....	0	...	$-\frac{\gamma_{mj}}{\gamma_{ij}}$	...	1	$\left[b_m - \frac{\gamma_{mj}}{\gamma_{ij}} b_i\right]_T$
obj.										$\tilde{\omega}_{ij} = \omega + \Delta\omega_{ij}$

.

The modified solution has cost

$$\tilde{\omega}_{ij} := \sum_{k=1}^{i-1} \omega_k \left[b_k - \frac{\gamma_{kj}}{\gamma_{ij}} b_i\right]_T + \omega_j \left[\frac{b_i}{\gamma_{ij}}\right]_T + \sum_{k=i+1}^{r} \omega_k \left[b_k - \frac{\gamma_{kj}}{\gamma_{ij}} b_i\right]_T$$

The cost difference can therefore be calculated by

$$\Delta\omega_{ij} = \tilde{\omega}_{ij} - \omega$$

$$= \sum_{k \neq i} \omega_k \left(b_k - \left[b_k - \frac{\gamma_{kj}}{\gamma_{ij}} b_i \right]_T \right) + \omega_i b_i - \omega_j \left[\frac{b_i}{\gamma_{ij}} \right]_T \quad (9)$$

The following example illustrates these considerations.

3.1 Example

Consider a problem with period $T = 20$ and underlying event network shown in Figure 2.

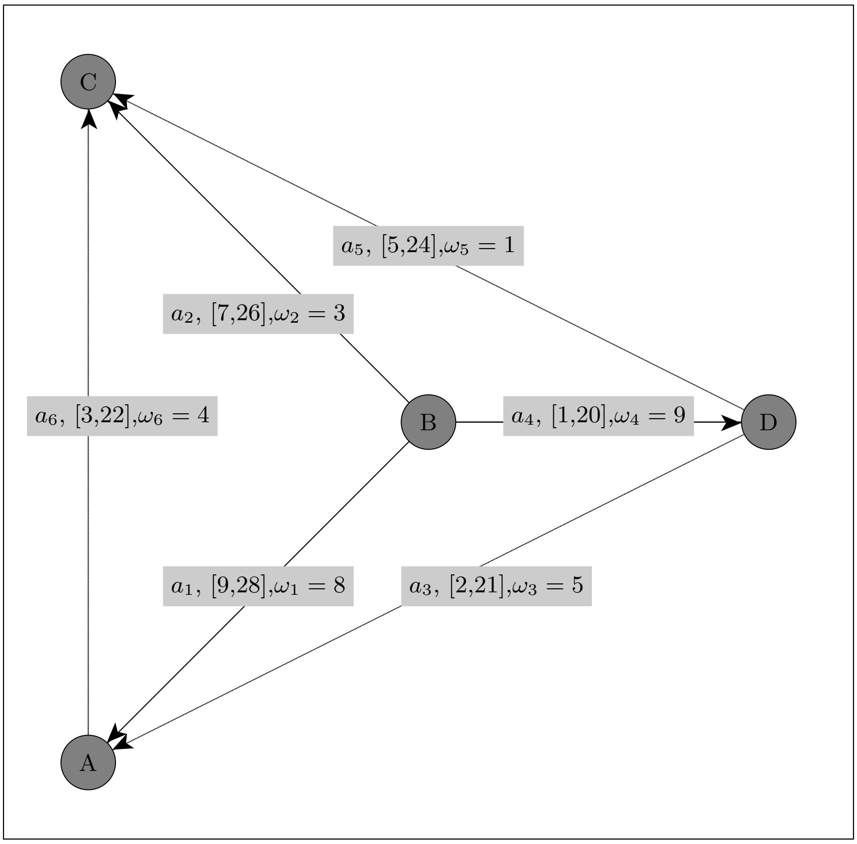


Fig. 2. Event Network

The initial spanning tree $\mathcal{T} = \mathcal{T}^\ell + \mathcal{T}^u$ with $\mathcal{T}^\ell = \{a_2, a_3, a_5\}$ and the resulting potential is given by Figure 3. This initial spanning tree structure induces the following *modulo simplex tableau* with total cost $\omega = 129$.
The following table contains for each possible basis exchange the resulting cost difference. This can be calculated by formula (9). The best gain will be received

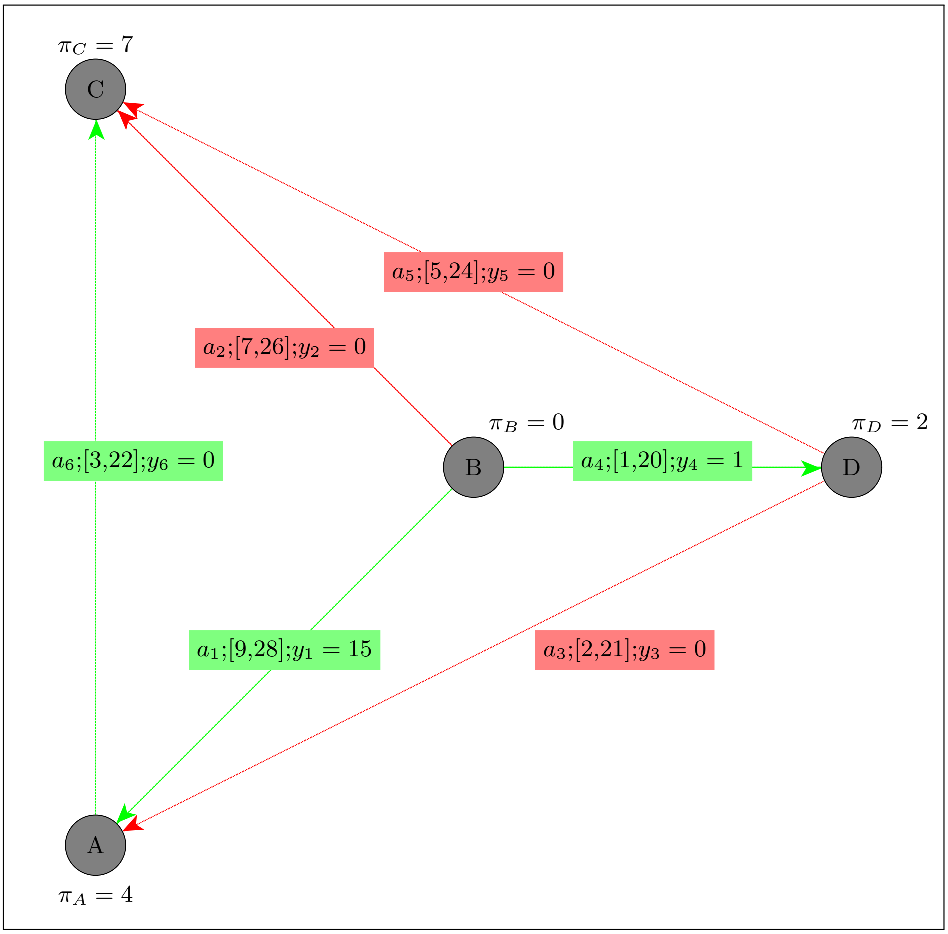


Fig. 3. Spanning tree structure (= red arcs), associated potential and slack times $y_a = [\pi_j - \pi_i - \ell_a]_T$.

by exchanging tree arc a_2 with co-tree arc a_1. Table 3 shows the new modulo simplex tableau after pivot step. The new objective is $\omega = 129 - 60 = 69$.

Table 3. Modulo Simplex Tableau of Step 1.

	a_1	a_3	a_5	a_4	a_2	a_6	b	ω
a_4	-1	1	0	1	0	0	6	9
a_2	-1	1	-1	0	1	0	5	3
a_6	0	1	-1	0	0	1	0	4
ω							69	

Table 1. Initial Modulo Simplex Tableau.

	a_2	a_3	a_5	a_4	a_1	a_6	b	ω
a_4	-1	0	1	1	0	0	1	9
a_1	-1	-1	1	0	1	0	15	8
a_6	0	1	-1	0	0	1	0	4
ω							129	

Table 2. Cost difference Δ for all possible basis exchanges.

	a_2	a_3	a_5
a_4	40	-	-12
a_1	-60	-35	0
a_6	-	0	20

The algorithm performs such modulo simplex pivot steps as long as a basis exchange will generate an improvement of the solution. Clearly, this only leads to a local minimum. Each periodic tension $\boldsymbol{\eta}$ with $\Gamma\boldsymbol{\eta} \equiv_T \mathbf{0}$ and $\boldsymbol{\eta} \leq \boldsymbol{\delta}$ defines by $\boldsymbol{y}' := \boldsymbol{y} + \boldsymbol{\eta}$ a new solution of the problem. It improves the old solution, if the new objective value gets better. In case of an improvement the modulo simplex pivoting will be applied again. This requires a basic solution, which can be simply received by solving the **non-periodic** minimum cost flow with fixed modulo parameter by the classical network simplex method.

In order to improve the local optimum after modulo simplex pivoting we apply a special class of cuts: For each node i the set of all leaving or entering arcs is a cut $\boldsymbol{\eta}^{(i)}$. Modifying the potential value of node i by $\pi_i' := \pi_i + \delta$, equals with the solution $\boldsymbol{y} + \delta\boldsymbol{\eta}^{(i)}$ after applying the $\delta-$multiple of the cut. For the class of those *single node cuts* it is obviously easy to check the improvement by enumerating all possible values for δ.

The modulo network simplex method can be summarized by

3.2 Modulo Network Simplex Algorithm

Initialisation: Determine an initial feasible tree structure $\mathcal{T} = \mathcal{T}^\ell + \mathcal{T}^u$ with feasible solution y

Single node improvement: WHILE (there exists an improving single node cut $\boldsymbol{\eta}$) **DO**

1. Apply this cut by transforming the solution $\boldsymbol{y}' := \boldsymbol{y} + \boldsymbol{\eta}$.
2. Fix the modulo parameter of this solution $\boldsymbol{y}'$ and solve the non-periodic minimum cost flow problem (see (3)) by the classical network simplex method. Then, the optimal solution becomes a tree solution.
3. Modulo-Simplex-Pivoting:
   (a) For each basis exchange pair (i, j) with $\gamma_{ij} \neq 0$ calculate the cost difference $\Delta\omega_{ij}$.
   (b) If $\Delta\omega_{ij} < 0$ and $\boldsymbol{\eta}^{(a_i,a_j)} \leq \boldsymbol{\delta} = \boldsymbol{u} - \boldsymbol{\ell}$, then improve the solution by exchanging co-tree arc a_i with tree arc a_j and continue with step (a). Otherwise terminate Modulo-Simplex-Pivoting.

The non-periodic simplex algorithm terminates, if the well known complementary slackness conditions are fullfilled. For the periodic case such a strong optimality condition cannot be given. However, sometimes it is possible to transform the periodic basic solution of a modulo simplex step into a primal feasible basic solution of a relaxation

$$\tilde{\boldsymbol{y}} \in \left\{ \boldsymbol{y} \mid \tilde{F}\boldsymbol{y} \geq \boldsymbol{r} \right\} \supseteq conv.hull \left(\{ \boldsymbol{y} \in \mathbb{Z}^m \mid \exists \boldsymbol{z} \in \mathbb{Z}^m : \Gamma\boldsymbol{y} - T\boldsymbol{z} = \boldsymbol{b}; \boldsymbol{0} \leq \boldsymbol{y} - T\boldsymbol{z} \leq \boldsymbol{\delta} \} \right)$$

If $\tilde{\boldsymbol{y}}$ is already optimal, i. e. $\boldsymbol{\omega}^t\tilde{\boldsymbol{y}} = \min \left\{ \boldsymbol{\omega}^t\boldsymbol{y} \mid \tilde{F}\boldsymbol{y} \geq \boldsymbol{r} \right\}$, then we found the optimal solution of the overall problem. Otherwise, the basis representation of $\tilde{\boldsymbol{y}}$ has **negative** reduced costs. A basis transformation of $\tilde{F}$ will exchange a tree and a co-tree arc, which then, also done for the modulo simplex, will possibly improve the solution.

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4 Computational Results For a Real World Scenario

4.1 The Traffic Sample

We applied the described algorithm to a real world traffic sample, which was derived from the south-west area of the German Railway Network (see Figure 4).

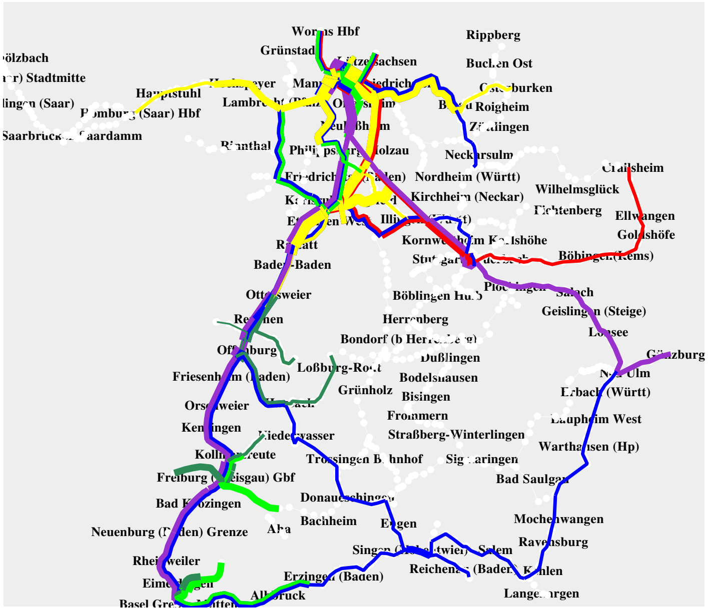

Fig. 4. The Traffic Sample contains 92 lines from the south-west area of the German railway network.

The timetabling problem contains 92 different railway lines with periods of 20, 30, 60 and 120 minutes, which results in an overall period of

$$T = lcm(20, 30, 60, 120) = 120\,\text{minutes}.$$

The resulting periodic event scheduling problem contains 669 event nodes and in total 3831 (with 3287 headway) constraints.

To solve the feasibility problem without any passenger connection constraints, we used a constraint programming approach, which finds a feasible solution within approximately one minute computation time. Next, for an origin destination matrix we applied a traffic assignment, by routing passengers on best paths. In this way we obtained for each possible connection between different lines a weight for the number of passengers using this change activity. The origin destination matrix contains only values given in percent of the total (unknown) traffic volume. For this reason, the change activity weight is primary that percentage of total volume which uses this connection. Due to the huge amount of approximately 1200 change activities with positive passenger weight, we only pick out the most important ones.

Table 4. Computational Results for the Modulo-Simplex-Algorithm

iteration	objective	description
	620952.00	initial solution from constraint propagation
	462111.00	min cost flow with fixed modulo parameter $\boldsymbol{z}$
1	436881.00	modulo-network simplex
2	415182.00	modulo-network simplex
...	...	...
35	327113.00	modulo-network simplex
36	319874.00	single node cut improvement + min cost flow
37	312342.00	modulo-network simplex
...	...	...
56	294567.00	modulo-network simplex
57	286122.00	single node cut improvement + min cost flow
58	273789.00	modulo-network simplex
...	...	...
67	254988.00	modulo-network simplex
68	254711.00	single node cut improvement + min cost flow
69	254711.00	modulo-network simplex
68	**254711.00**	**final solution**

To do this and to get integer valued weights, the percentage was multiplied by a factor 200, which results into 570 connection constrains with weights in the range between 1 and 280. The results of the modulo network method are given by table 4. In total, the method needs approximately 20 minutes computation time.

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