Interval Arithmetic and Standardization

Jürgen Wolff von Gudenberg
University of Würzburg

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Abstract

Interval arithmetic is arithmetic for continuous sets. Floating-point intervals are intervals of real numbers with floating-point bounds. Operations for intervals can be efficiently implemented. Hence, the time is ripe for standardization. In this paper we present an interval model that is mathematically sound and closed for the 4 basic operations. The model allows for exception free interval arithmetic, if we carefully distinguish between clean and reliable interval arithmetic on one side and rounded floating-point arithmetic on the other side. Elementary functions for intervals can be defined. In some application areas loose evaluation of functions, i.e. evaluation over an interval which is not completely contained in the function domain, is recommended, In this case, however, a discontinuity flag has to be set to inform that Brouwer's fixed point theorem is no longer applicable.

1 Real Interval Arithmetic

1.1 Real Interval Arithmetic

Real interval arithmetic is defined as arithmetic on continuous (in the sense of complete, not discrete) sets.

Definition 1 *For intervals $A = [a_1, a_2]$ and $B = [b_1, b_2] \in \mathbb{IR}$, arithmetic operations are defined as set operations*

$$A \circ B := \{a \circ b \mid a \in A, b \in B\}$$

for operations $\circ \in \{+, -, \cdot, /\}$, $0 \notin B$ in case of division.

Remark 1 *Since the operations are continuous, the result is an interval, $A \circ B \in \mathbb{IR}$*

This definition can be extended to elementary functions.

Definition 2 $f(X) = \{f(x) \mid x \in X\}$ *denotes the range of values of the function $f : D_f \subseteq \mathbb{R} \to \mathbb{R}$ over the interval $X \subseteq D_f$.*

Remark 2 *If f is continuous, the range is an interval.*

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Remark 3 *The range is independent from the specific expression that describes the function. That is not the case for the interval evaluation.*

Definition 3 *The* ***interval evaluation*** $\mathbf{f} : \mathbb{IR} \to \mathbb{IR}$ *(of a real function f over an interval X) is defined as the function that is obtained by replacing every operator and every elementary function by its interval arithmetic counterpart under the assumption that all operations are executable without exceptions.*

Remark 4 *The variables in an interval evaluation now denote intervals.*
An obvious generalisation for functions of multiple variables may be defined.

Two basic principles are mandatory for every definition of interval arithmetic.
The containment principle is known as the fundamental theorem of interval arithmetic [1].

Principle 1 *If the interval evaluation is defined, we have*

$$f(X) \subseteq \mathbf{f}(X)$$

The second priciple, the inclusion isotonicity is related.

Principle 2 *If $X \subseteq Y$, we have*

$$\mathbf{f}(X) \subseteq \mathbf{f}(Y)$$

2 Floating-point Interval Arithmetic

When we proceed to the set of floating-point intervals, our topic for standardization, we are still talking about continuous sets, only the endpoints are floating-point numbers. For the sake of clarity and to emphasize the difference, we introduce a separate notation for floating-point interval arithmetic.

Definition 4 *Let F denote the set of floating-point numbers, then $\mathbb{I}F \subset \mathbb{IR}$ denotes the set of floating-point intervals $A = [a_1, a_2]$ where $a_1, a_2 \in F$ and $a_1 \leq a_2$*

Remark 5 $A = [a_1, a_2] = \{a \in \mathbb{R} \mid a_1 \leq a \leq a_2\}$

Definition 5 *The* ***floating-point interval evaluation*** $\diamond\mathbf{f} : \mathbb{I}F \to \mathbb{I}F$ *of the function expression f is defined as the function that is obtained by replacing every operator and every elementary function by its floating-point interval arithmetic counterpart under the assumption that all operations are executable without exceptions.*

The containment principle guarantees that every real number in the original range of values of a continuous function is contained in the result of the floating-point interval evaluation of the same function over the same argument interval.

Principle 3 *If the floating-point interval evaluation for f is defined, we have*

$$f(x) \in \diamond\mathbf{f}(X),\ \forall x \in X \cap D_f$$

Remark 6 *The floating-point interval evaluation is defined, if f is continuous and $X \subseteq D_f$.*

A clean semantics that respects the two basic principles of containment and inclusion isotonicity is mandatory. It can be obtained when we implement the well-known formulae involving only the endpoints and use directed roundings. In the following we denote operations with directed rounding by putting the rounding symbol around the operator. ∇ or Δ for downwards or upwards, respectively. Alternatively, one may indicate rounding towards $-\infty$ by a $\check{\circ}$ and rounding towards $+\infty$ by a $\hat{\circ}$ over the operator symbol $\circ$.

2.1 Representaion

A finite floating-point interval is represented by two floating-point numbers, the first a_1 denotes the lower bound, the second a_2 the upper bound. For a valid interval we have $a_1 \leq a_2$.
An unbounded interval has its lower bound set to $-\infty$ or its upper bound set to $+\infty$.
The empty set is denoted by $[+\infty, -\infty]$
All other representations, in particular two valid numbers with $a_1 > a_2$, denote invalid intervals.

2.2 Arithmetic Operations

We first consider the operations addition, subtraction, multiplication, and division by an interval which does not contain zero.

$$[a_1, a_2] + [b_1, b_2] = [a_1 \mathbin{\overset{+}{\nabla}} b_1, a_2 \mathbin{\overset{+}{\Delta}} b_2],$$

addition

$$[a_1, a_2] - [b_1, b_2] = [a_1 \mathbin{\overset{-}{\nabla}} b_2, a_2 \mathbin{\overset{-}{\Delta}} b_1],$$

subtraction

Remark 7 *Note that these formulas hold for unbounded intervals and the emptyset, as well.*

Remark 8 *The division by an interval containing zero raises an exception.*

Different ways of handling that exception are suggested below.

2.3 Set Operations

We have intersection and convex (or interval) hull, as well as membership and inclusion tests.

	$0 \leq b_1$	$b_1 < 0 \leq b_2$	$b_2 < 0$
$0 \leq a_1$	$[a_1 \dot\bigtriangledown b_1, a_2 \dot\bigtriangleup b_2]$	$[a_2 \dot\bigtriangledown b_1, a_2 \dot\bigtriangleup b_2]$	$[a_2 \dot\bigtriangledown b_1, a_1 \dot\bigtriangleup b_2]$
$a_1 < 0 \leq a_2$	$[a_1 \dot\bigtriangledown b_2, a_2 \dot\bigtriangleup b_2]$	$[\min(a_1 \dot\bigtriangledown b_2, a_2 \dot\bigtriangledown b_1),$ $\max(a_1 \dot\bigtriangleup b_1, a_2 \dot\bigtriangleup b_2)]$	$[a_2 \dot\bigtriangledown b_1, a_1 \dot\bigtriangleup b_1]$
$a_2 < 0$	$[a_1 \dot\bigtriangledown b_2, a_2 \dot\bigtriangleup b_1]$	$[a_1 \dot\bigtriangledown b_2, a_1 \dot\bigtriangleup b_1]$	$[a_2 \dot\bigtriangledown b_2, a_1 \dot\bigtriangleup b_1]$

Table 1: The 9 cases of interval multiplication

	$0 < b_1$	$b_2 < 0$
$0 \leq a_1$	$[a_1 \not\bigtriangledown b_2, a_2 \not\bigtriangleup b_1]$	$[a_2 \not\bigtriangledown b_2, a_1 \not\bigtriangleup b_1]$
$a_1 < 0 \leq a_2$	$[a_1 \not\bigtriangledown b_1, a_2 \not\bigtriangleup b_1]$	$[a_2 \not\bigtriangledown b_2, a_1 \not\bigtriangleup b_2]$
$a_2 < 0$	$[a_1 \not\bigtriangledown b_1, a_2 \not\bigtriangleup b_2]$	$[a_2 \not\bigtriangledown b_1, a_1 \not\bigtriangleup b_2]$

Table 2: The 6 cases of interval division with $0 \notin B$.

$$
\begin{aligned}
[a_1, a_2] \cap [b_1, b_2] &= [max(a_1, b_1), min(a_2, b_2)] \\
[a_1, a_2] \cup [b_1, b_2] &= [min(a_1, b_1), max(a_2, b_2)]
\end{aligned}
$$

$$
\begin{aligned}
a \in [a_1, a_2] \Leftrightarrow [a_1, a_2] \ni a &\Leftrightarrow a_1 \leq a \wedge a \leq a_2 \\
[a_1, a_2] \subseteq [b_1, b_2] \Leftrightarrow [b_1, b_2] \supseteq [a_1, a_2] &\Leftrightarrow a_1 \geq b_1 \wedge a_2 \leq b_2
\end{aligned}
$$

Remark 9 *Note that these formulas hold for unbounded intervals and the emptyset, as well.*

3 Unbounded Intervals and Division by Zero

3.1 The Set Approach

In section 2.1 we introduced intervals with one endpoint $+\infty$ or $-\infty$. Two interpretations are possible.

1. ∞ is not a valid point in the interval, it just states that the interval is unbounded [6].

2. We consider closed intervals, hence ∞ is a part of it [9, 8].

In both cases, we do not allow intervals with lower bound $+\infty$ or upper bound $-\infty$.
We now define division by an interval containing zero. Rewriting the definition, we obtain:

$$A/B := \{a/b \mid a \in A, b \in B\} = \{x \mid bx = a \wedge a \in A \wedge b \in B\}$$

Applying this formula eight distinct cases can be set out. In the following table in column 3 we display the 2 bounds, that are returned by the operation. Since no valid intervals are returned, if 0 is in the interior of B, we add a 4-th column with a set interpretation.

case	$A = [a_1, a_2]$	$B = [b_1, b_2]$	$result$	A/B
1	$0 \in A$	$0 \in B$	$-\infty, +\infty$	$(-\infty, +\infty)$
2	$0 \notin A$	$B = [0, 0]$	$+\infty, -\infty$	$\emptyset$
3	$a_2 < 0$	$b_1 < b_2 = 0$	$a_2 \nabla b_1, +\infty$	$[a_2 \nabla b_1, +\infty)$
4	$a_2 < 0$	$b_1 < 0 < b_2$	$a_2 \nabla b_1, a_2 \Delta b_2$	$(-\infty, a_2 \Delta b_2] \cup [a_2 \nabla b_1, +\infty)$
5	$a_2 < 0$	$0 = b_1 < b_2$	$-\infty, a_2 \Delta b_2$	$(-\infty, a_2 \Delta b_2]$
6	$a_1 > 0$	$b_1 < b_2 = 0$	$-\infty, a_1 \Delta b_1$	$(-\infty, a_1 \Delta b_1]$
7	$a_1 > 0$	$b_1 < 0 < b_2$	$a_1 \nabla b_2, a_1 \Delta b_1$	$(-\infty, a_1 \Delta b_1] \cup [a_1 \nabla b_2, +\infty)$
8	$a_1 > 0$	$0 = b_1 < b_2$	$a_1 \nabla b_2, +\infty$	$[a_1 \nabla b_2, +\infty)$

Table 3: The eight cases of interval division with $A, B \in IS$, and $0 \in B$.

Since, in case 1, $0 \in A$ and $0 \cdot x = 0, \forall x \in \mathbb{R}$ we have $\mathbb{R} = (-\infty, \infty)$ as the solution set, whereas in case 2 there is no $x \in \mathbb{R}$ with $0 \cdot x = a$ for $a \in A$. The other cases are derived by limit processes, or by the arithmetic conventions for infinities.

Remark 10

- *If 0 is in the interior of B, the solution set consists of 2 unbounded intervals.*
- *Alternatively the whole line $\mathbb{R}$ can be returned, but that would loose valuable information.*

3.2 Discussion

Let us further discuss the 2 alternatives. The former is consistent with the definition of interval arithmetic as set arithmetic. The subintervals are used in the interval Newton method as 2 sets possibly containing zeros. The middle part $(a_1 \nabla b_1, a_1 \Delta b_2)$ is cut out, since it cannot contain a zero. Hence, the process proceeds, whereas the whole $\mathbb{R}$, swollows this information and the process stops.
The latter solution has two obvious advantages. The system is closed, i.e. in any case a valid interval is returned, and the containment principle also holds for floating-point

results, that are no real numbers, but the symbols $\pm\infty$. In this case we have to replace the empty set in row 2 by the whole set, again a huge oversetimation.
Let us assume to return the two bounds in reverse order and raise an exception when one of the cases 4 or 7 occurs. Then there are several options.

1. The user (programmer) catches the exception and builds the two intervals.

   That means that interval division cannot be applied inside an expression. Even if it is the last operation, we do not know in advance, whether the result consists of one or two intervals.

2. The user (programmer) catches the exception and builds the interval $[-\infty, +\infty]$.

   That is hopelessly slow.

3. Interval division in general delivers 2 intervals, one of which most often is empty.

   This would lead to a definition of interval pair arithmetic and complicate the standard unnecessarily.

4. There are 2 division operators, one for each alternative.

   (a) return the two bounds in reverse order

   (b) return $(-\infty, +\infty)$

   Now, the user who is aware of her application can choose the appropriate operation.

5. Only the second, simple alternative is provided.

   The user has to compute the bounds of his subintervals with explicit floating-point operations. The cases 4 or 7 have to be checked by hand.

As a conclusion of our discussion, we favor the closed, simple approach.

4 The Closed, Simple Approach

We discard the containment of floating-point symbols in case 2, but we tolerate an overestimation in cases 4 or 7. We can simplify the table.

Remark 11 *The C++ proposal [2] uses the same division table. The approach replaces the interval evaluation by the so-called range closure.*

4.1 Range and Topological Closure

As function evaluation we compute the range over the original domain and return the closure of that set:

case	$A=[a_1,a_2]$	$B=[b_1,b_2]$	A/B
1	$0\in A$	$0\in B$	$(-\infty,+\infty)$
2	$0\notin A$	$B=[0,0]$	$\emptyset$
3	$a_2<0$	$b_1<b_2=0$	$[a_2 \bigtriangledown b_1,+\infty)$
4	$a_2<0$	$b_1<0<b_2$	$(-\infty,+\infty)$
5	$a_2<0$	$0=b_1<b_2$	$(-\infty,a_2 \bigtriangleup b_2]$
6	$a_1>0$	$b_1<b_2=0$	$(-\infty,a_1 \bigtriangleup b_1]$
7	$a_1>0$	$b_1<0<b_2$	$(-\infty+\infty)$
8	$a_1>0$	$0=b_1<b_2$	$[a_1 \bigtriangledown b_2,+\infty)$

Table 4: The eight cases of closed interval division with $A,B\in\mathbb{IF}$, and $0\in B$.

Definition 6 *:*
Let $f: D_f\subseteq\mathbb{R}\to\mathbb{R}$, then the containment set by range closure $f_r^*: \wp\mathbb{R}^*\mapsto\wp\mathbb{R}^*$ defined by

$$f_r^*(X) := \{f(x)|x\in X\cap D_f\}\cup\{\lim_{x\to x^*} f(x)|x\in X\cap D_f\}\subseteq\mathbb{R}^* \tag{1}$$

contains the extended range of f, where $\mathbb{R}^*=\mathbb{R}\cup\{-\infty\}\cup\{\infty\}$.

Remark 12 *In order to fulfill the containment principle, it is sufficient to consider sequences that are contained in X. X as well as the computed range closure may be unbounded intervals.*

The topological closure follows the same definition with the exception that in equation 2 the constraint has been changed, so that more sequences are considered. That means that more accumulation points are taken into account.

Definition 7 *:*
Let $f: D_f\subseteq\mathbb{R}\to\mathbb{R}$, then the containment set by topological closure $f_r^*: \wp\mathbb{R}^*\mapsto\wp\mathbb{R}^*$ defined by

$$f_r^*(X) := \{f(x)|x\in X\cap D_f\}\cup\{\lim_{x\to x^*} f(x)|x\in X\cap D_f\}\cup\{\lim_{x\to x^*} f(x)|x\in D_f, x^*\in X\} \tag{2}$$

contains the extended range of f, where $\mathbb{R}^*=\mathbb{R}\cup\{-\infty\}\cup\{\infty\}$.

This topological closure delivers $[-\infty,+\infty]$ in case 2, since there are positive as well as negative sequences converging to zero. In general it holds that all IEEE no-numbers like $\pm\infty$ are contained in the containment set. Hence, the interpretation of the containment priciple 1 becomes easier.

Corollary 4

$$f(X)\subseteq\mathbf{f}(X)\subseteq\mathbf{f}_r^*(X)\subseteq\mathbf{f}^*(X) \tag{3}$$

4.2 Exception-free Arithmetic

As we stated above a floating-point interval is a set of real numbers where the endpoints are floating-point numbers. A floating-point interval thus is completely different from a floating-point number that usually denotes a more or less crude approximation of a real number. We interpret the bounds of an interval as sharp in the sense that lower or upper bounds are true bounds and do not carry some rounding noise in the relevant direction. Therefore it is not recommended to provide mixed operations between floating-point numbers and intervals. A sophisticated user, however, may define those operations, either by overloading the operators or, preferably, by explicitly invoking a constructor. If we follow these rules, we can show that `NaNs` or signed zeros do not need a special treatment, since they will never occur and the infinity symbols are only used to describe sets, i.e. intervals.

Definition 8 *We consider the system of (extended) floating-point intervals* $\mathbb{I}F := \{[a_1, a_2] \mid a_1 \leq a_2\} \cup \{[a_1, +\infty) \mid a_1 < +\infty\} \cup \{(-\infty, a_2] \mid a_2 > -\infty\} \cup \{(-\infty, +\infty)\} \cup \{\emptyset\}$ *Note that* a_1, a_2 *are floating-point numbers but the set definitions are to be read for all real numbers.*

Theorem 5 *The system* $\mathbb{I}F$ *is closed under the 4 basic operations given by the following tables.*

The proof of the theorem may be picked from the tables, see also [5, 6], For the empty set all operations deliver the empty set.

Addition	$(-\infty, b_2]$	$[b_1, b_2]$	$[b_1, +\infty)$	$(-\infty, +\infty)$
$(-\infty, a_2]$	$(-\infty, a_2 \overset{\triangle}{+} b_2]$	$(-\infty, a_2 \overset{\triangle}{+} b_2]$	$(-\infty, +\infty)$	$(-\infty, +\infty)$
$[a_1, a_2]$	$(-\infty, a_2 \overset{\triangle}{+} b_2]$	$[\mathbf{a_1} \overset{\bigtriangledown}{+} \mathbf{b_1}, \mathbf{a_2} \overset{\triangle}{+} \mathbf{b_2}]$	$[a_1 \overset{\bigtriangledown}{+} b_1, +\infty)$	$(-\infty, +\infty)$
$[a_1, +\infty)$	$(-\infty, +\infty)$	$[a_1 \overset{\bigtriangledown}{+} b_1, +\infty)$	$[a_1 \overset{\bigtriangledown}{+} b_1, +\infty)$	$(-\infty, +\infty)$
$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$

Table 5: Addition of extended intervals.

Subtraction	$(-\infty, b_2]$	$[b_1, b_2]$	$[b_1, +\infty)$	$(-\infty, +\infty)$
$(-\infty, a_2]$	$(-\infty, +\infty)$	$(-\infty, a_2 \overset{\triangle}{-} b_1]$	$(-\infty, a_2 \overset{\triangle}{-} b_1]$	$(-\infty, +\infty)$
$[a_1, a_2]$	$[a_1 \overset{\bigtriangledown}{-} b_2, +\infty)$	$[\mathbf{a_1} \overset{\bigtriangledown}{-} \mathbf{b_2}, \mathbf{a_2} \overset{\triangle}{-} \mathbf{b_1}]$	$(-\infty, a_2 \overset{\triangle}{-} b_1]$	$(-\infty, +\infty)$
$[a_1, +\infty)$	$[a_1 \overset{\bigtriangledown}{-} b_2, +\infty)$	$[a_1 \overset{\bigtriangledown}{-} b_2, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$
$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$

Table 6: Subtraction of extended intervals.

Note that in line 4 we consider the multiplication with an exact zero $0 \in \mathbb{R}$.

Multiplication	$[b_1, b_2]$ $b_2 \leq 0$	$[b_1, b_2]$ $b_1 < 0 < b_2$	$[b_1, b_2]$ $b_1 \geq 0$	$[0, 0]$	$(-\infty, b_2]$ $b_2 \leq 0$	$(-\infty, b_2]$ $b_2 \geq 0$	$[b_1, +\infty)$ $b_1 \leq 0$	$[b_1, +\infty)$ $b_1 \geq 0$	$(-\infty, +\infty)$
$[a_1, a_2], a_2 \leq 0$	$[\mathbf{a_2 \bigtriangledown b_2, a_1 \bigtriangleup b_1}]$	$[\mathbf{a_1 \bigtriangledown b_2, a_1 \bigtriangleup b_1}]$	$[\mathbf{a_1 \bigtriangledown b_2, a_2 \bigtriangleup b_1}]$	$[0, 0]$	$[a_2 \bigtriangledown b_2, +\infty)$	$[a_1 \bigtriangledown b_2, +\infty)$	$(-\infty, a_1 \bigtriangleup b_1]$	$(-\infty, a_2 \bigtriangleup b_1]$	$(-\infty, +\infty)$
$a_1 < 0 < a_2$	$[\mathbf{a_2 \bigtriangledown b_1, a_1 \bigtriangleup b_1}]$	$[\mathbf{min(a_1 \bigtriangledown b_2, a_2 \bigtriangledown b_1)},$ $\mathbf{max(a_1 \bigtriangleup b_1, a_2 \bigtriangleup b_2)}]$	$[\mathbf{a_1 \bigtriangledown b_2, a_2 \bigtriangleup b_2}]$	$[0, 0]$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$
$[a_1, a_2], a_1 \geq 0$	$[\mathbf{a_2 \bigtriangledown b_1, a_1 \bigtriangleup b_2}]$	$[\mathbf{a_2 \bigtriangledown b_1, a_2 \bigtriangleup b_2}]$	$[\mathbf{a_1 \bigtriangledown b_1, a_2 \bigtriangleup b_2}]$	$[0, 0]$	$(-\infty, a_1 \bigtriangleup b_2]$	$(-\infty, a_2 \bigtriangleup b_2]$	$[a_2 \bigtriangledown b_1, +\infty)$	$[a_1 \bigtriangledown b_1, +\infty)$	$(-\infty, +\infty)$
$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$
$(-\infty, a_2], a_2 \leq 0$	$[a_2 \bigtriangledown b_2, +\infty)$	$(-\infty, +\infty)$	$(-\infty, a_2 \bigtriangleup b_1]$	$[0, 0]$	$[a_2 \bigtriangledown b_2, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, a_2 \bigtriangleup b_1]$	$(-\infty, +\infty)$
$(-\infty, a_2], a_2 \geq 0$	$[a_2 \bigtriangledown b_1, +\infty)$	$(-\infty, +\infty)$	$(-\infty, a_2 \bigtriangleup b_2]$	$[0, 0]$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$
$[a_1, +\infty), a_1 \leq 0$	$(-\infty, a_1 \bigtriangleup b_1]$	$(-\infty, +\infty)$	$[a_1 \bigtriangledown b_2, +\infty)$	$[0, 0]$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$
$[a_1, +\infty), a_1 \geq 0$	$(-\infty, a_1 \bigtriangleup b_2]$	$(-\infty, +\infty)$	$[a_1 \bigtriangledown b_1, +\infty)$	$[0, 0]$	$(-\infty, a_1 \bigtriangleup b_2]$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$[a_1 \bigtriangledown b_1, +\infty)$	$(-\infty, +\infty)$
$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$[0, 0]$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$

Table 7: Multiplication of extended intervals.

Division $\mathbf{0 \notin B}$	$[b_1, b_2]$ $b_2 < 0$	$[b_1, b_2]$ $b_1 > 0$	$(-\infty, b_2]$ $b_2 < 0$	$[b_1, +\infty)$ $b_1 > 0$
$[a_1, a_2], a_2 \leq 0$	$[\mathbf{a_2} \bigtriangledown \mathbf{b_1}, \mathbf{a_1} \bigtriangleup \mathbf{b_2}]$	$[\mathbf{a_1} \bigtriangledown \mathbf{b_1}, \mathbf{a_2} \bigtriangleup \mathbf{b_2}]$	$[0, a_1 \bigtriangleup b_2]$	$[a_1 \bigtriangledown b_1, 0]$
$[a_1, a_2], a_1 < 0 < a_2$	$[\mathbf{a_2} \bigtriangledown \mathbf{b_2}, \mathbf{a_1} \bigtriangleup \mathbf{b_2}]$	$[\mathbf{a_1} \bigtriangledown \mathbf{b_1}, \mathbf{a_2} \bigtriangleup \mathbf{b_1}]$	$[a_2 \bigtriangledown b_2, a_1 \bigtriangleup b_2]$	$[a_1 \bigtriangledown b_1, a_2 \bigtriangleup b_1]$
$[a_1, a_2], a_1 \geq 0$	$[\mathbf{a_2} \bigtriangledown \mathbf{b_2}, \mathbf{a_1} \bigtriangleup \mathbf{b_1}]$	$[\mathbf{a_1} \bigtriangledown \mathbf{b_2}, \mathbf{a_2} \bigtriangleup \mathbf{b_1}]$	$[a_2 \bigtriangledown b_2, 0]$	$[0, a_2 \bigtriangleup b_1]$
$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$
$(-\infty, a_2], a_2 \leq 0$	$[a_2 \bigtriangledown b_1, +\infty)$	$(-\infty, a_2 \bigtriangleup b_2]$	$[0, +\infty)$	$(-\infty, 0]$
$(-\infty, a_2], a_2 \geq 0$	$[a_2 \bigtriangledown b_2, +\infty)$	$(-\infty, a_2 \bigtriangleup b_1]$	$[a_2 \bigtriangledown b_2, +\infty)$	$(-\infty, a_2 \bigtriangleup b_1]$
$[a_1, +\infty), a_1 \leq 0$	$(-\infty, a_1 \bigtriangleup b_2]$	$[a_1 \bigtriangledown b_1, +\infty)$	$(-\infty, a_1 \bigtriangleup b_2]$	$[a_1 \bigtriangledown b_1, +\infty)$
$[a_1, +\infty), a_1 \geq 0$	$(-\infty, a_1 \bigtriangleup b_1]$	$[a_1 \bigtriangledown b_2, +\infty)$	$(-\infty, 0]$	$[0, +\infty)$
$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$

Table 8: Division of extended intervals with $0 \notin B$.

Division $\mathbf{0 \in B}$	$B =$ $[0,0]$	$[b_1, b_2]$ $b_1 < b_2 = 0$	$[b_1, b_2]$ $b_1 < 0 < b_2$	$[b_1, b_2]$ $0 = b_1 < b_2$	$(-\infty, b_2]$ $b_2 = 0$	$(-\infty, b_2]$ $b_2 > 0$	$[b_1, +\infty)$ $b_1 < 0$	$[b_1, +\infty)$ $b_1 = 0$	$(-\infty, +\infty)$
$[a_1, a_2], a_2 < 0$	$\emptyset$	$[\mathbf{a_2} \bigtriangledown\!\!\!\!/\ \mathbf{b_1}, +\infty)$	$(-\infty, +\infty)$	$(-\infty, \mathbf{a_2} \bigtriangleup \mathbf{b_2}]$	$[0, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, 0]$	$(-\infty, +\infty)$
$[a_1, a_2], a_1 \leq 0 \leq a_2$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$
$[a_1, a_2], a_1 > 0$	$\emptyset$	$(-\infty, \mathbf{a_1} \bigtriangleup \mathbf{b_1}]$	$(-\infty, +\infty)$	$[\mathbf{a_1} \bigtriangledown\!\!\!\!/\ \mathbf{b_2}, +\infty)$	$(-\infty, 0]$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$[0, +\infty)$	$(-\infty, +\infty)$
$(-\infty, a_2], a_2 < 0$	$\emptyset$	$[a_2 \bigtriangledown\!\!\!\!/\ b_1, +\infty)$	$(-\infty, +\infty)$	$(-\infty, a_2 \bigtriangleup b_2]$	$[0, +\infty)$	$(-\infty, +\infty)$	$(-\infty, 0]$	$(-\infty, 0]$	$(-\infty, +\infty)$
$(-\infty, a_2], a_2 > 0$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$
$[a_1, +\infty), a_1 < 0$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$
$[a_1, +\infty), a_1 > 0$	$\emptyset$	$(-\infty, a_1 \bigtriangleup b_1]$	$(-\infty, +\infty)$	$[a_1 \bigtriangledown\!\!\!\!/\ b_2, +\infty)$	$(-\infty, 0]$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$[0, +\infty)$	$(-\infty, +\infty)$
$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$

Table 9: Division of extended intervals with $0 \in B$.

name	domain	range	special values
sqr	$\mathbb{R}^*$	$[0, \infty]$	
power	$\mathbb{R}^* \times \mathbb{Z}$	$\mathbb{R}^*$	power([0,0],0) = [1,1]
pow	$[0, \infty] \times \mathbb{R}^*$	$[0, \infty]$	pow([0,0],[0,0]) =$[0, \infty]$
sqrt	$[0, \infty]$	$[0, \infty]$	
exp, exp10, exp2	$\mathbb{R}^*$	$[0, \infty]$	
expm1	$\mathbb{R}^*$	$[-1, \infty]$	
log, log10, log2	$[0, \infty]$	$\mathbb{R}^*$	log $([0, 0]) = [-\infty]$

Table 10: Extended domains and ranges for the elementary functions

4.3 Intermediate Conclusion

The closed definition of interval operations is mathematically sound and fulfills the priciples of containment and inclusion isotonicity. Under the assumption that no external (hardware) event changes the data, we can guarantee that all intervals produced are valid intervals.
One may argue that we loose information, when we overstimate the union of 2 unbounded intervals by the whole line, but this information can always be explicitly computed by 2 floating-point divisions. The interval newton method needs an a priori test whether the denominator contains zero, and then the finite bounds of the 2 unbounded intervals can be determined. When, on the other hand, we deliver the 2 quotients as an improper interval, we have to check for this situation after the division and produce the 2 subintervals.
We see that programming the interval Newton method needs specific operation in any case. Hence, we propose to standardize the simpler closed system.

5 Elementary Functions

Interval versions of elementary functions must deliver an enclosure of the real range. Least bit accurate versions have been proposed in [2]. The rely on the same functions as those floating-point functions in the IEEE-754 arithmetic standard.
We will discuss the aspect of extending these functions.

5.1 The continuity approach

Functions are not extended. An exception that normally leads to program termination is raised, whenever a function is to be evaluated outside its domain.

5.2 The containment set approach

In the containment set approach [9, 8], how it is implemented in `filib++` [3], e.g., infinities are treated as values. There are point intervals and operations for $\pm\infty$.
In this framework elementary functions can easily be extended, see Table 10.

$A=[a_1,a_2]$	Range	discontFlag set
$a_2<0$	$\emptyset$	yes
$a_1<0\le a_2$	$[0,\sqrt{a_2}]$	yes
$0\le a_1$	$[\sqrt{a_1},\sqrt{a_2}]$	no
$(-\infty,a_2];a_2<0$	$\emptyset$	yes
$(-\infty,a_2];0\le a_2$	$[0,\sqrt{a_2}]$	yes
$[a_1,+\infty)\,;a_1<0$	$[0,+\infty)$	yes
$[a_1,+\infty)\,;0\le a_1$	$[\sqrt{a_1},+\infty)$	yes

Table 11: Extended interval square root

Definition 9 *A function* $\mathbf{f}$ *is loosely evaluated over an interval* X*, if* $\mathbf{f}(X) := \mathbf{f}(X \cap \overline{D_f})$ *where* D_f *is the domain of* $\mathbf{f}$.

Remark 13 *A discontinuousIntervalFunction exception has to be raised, if a function* $\mathbf{f}$ *is loosely evaluated over an interval* X *with* $X \nsubseteq D_f$*. A corresponding flag [7] has to be set.*

- The default handling in this case should be to terminate.
- There is an instruction to read that flag. Hence, user defined actions can be executed.
- An alternative may be to ignore the exception.

The flag indicates that applications which rely on the continuity of the functions like verification algorithms using Brouwer's fixed-point theorem are not allowed.
As an example we show the definition of the square root in Table 11.

6 Conclusion and Further Topics

In this position paper we have proposed a definition of extended interval arithmetic that is closed and mathematically sound. It should be taken as the core of the coming interval arithmetic standard.
The standard should also specify set operations and comparisons as well as elementary functions. For the latter a discontinuity flag shall be defined that supports the loose evaluation.
Further topics of the standard shall be complete arithmetic including an exact dotproduct.

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References

[1] Alefeld, G. and Herzberger J. *Introduction to Interval Computation*, Academic Press 1983

[2] Brönnimann, H., Melquiond, G., Pion, S. *A Proposal to add Interval Arithmetic to the C++ Standard Library (revision 2)* `http://www.open-std.org/jtc1/sc22/wg21/docs/papers/2006/n2137.pdf`

[3] Lerch,M. Hofschuster, W. et al *FILIB++, a Fast Interval Library Supporting Containment Computations*, ACM TOMS Vol 32 No 2, pp.299 - 324, 2006

[4] Kulisch, U: *Advanced Arithmetic for the Digital Computer, Interval Arithmetic Revisited*, Jan 2001

[5] Kulisch, U: *Computer Arithmetic and Validity*, de Gruyter, to be published, 2008

[6] Kulisch, U: *Letter to the IEEE-754 committee concerning Interval Arithmetic* , Jan 2008

[7] Pryce, J. *Project Authorisation Request for an Interval Standard Study Group*, Feb 2008

[8] Pryce, J., Corliss, G. *Interval Arithmetic with Containment Sets* Computing 4, 2005

[9] Walster, G.W. et al.: *The "Simple" Closed Interval System*, Sun Microsystems, Feb 2000