Current results and open questions on PH and MAP characterization*

L. Bodrog[1], A. Heindl[2], A. Horváth[3], G. Horváth[1], M. Telek[1]
[1] Department of Telecommunications,
Budapest University of Technology and Economics, Hungary
[2] Department of Computer Science 7,
University of Erlangen-Nuremberg, Germany
[3] Departimento di Informatica, Universitá di Torino, Italy

Abstract

This paper summarizes some recent results of the authors on the characterization and the canonical representation of order n phase type distributions (PH(n)) and Markov Arrival processes (MAP(n)). These results make possible a unique and minimal representation of PH(n) and MAP(n) and opens the way for constructing efficient fitting methods for MAP(n).

Especially, this summary is based on [10, 1, 9, 2].

Keywords: Phase Type Distribution, Markov Arrival processes, Canonical Form, Moment Bounds.

1 Phase type distributions

Continuous phase type distributions are defined as the time to absorption in a continuous time Markov chain with n transient and an absorbing state. The probability distribution function of an order n phase type distribution (PH(n)) is

$$F(t) = Pr(\mathcal{X} \le t) = 1 - v e^{\boldsymbol{H} t} \mathbb{1} \ ,$$

where the row vector v is the *initial (probability) vector* of the Markov chain, the $n \times n$ square matrix $\boldsymbol{H}$ is the *transient generator* of the Markov chain and $\mathbb{1}$ is the *closing vector*. The probability density function, its Laplace transform and the moments of the distribution are

$$f(t) = v e^{\boldsymbol{H} t} (-\boldsymbol{H}) \mathbb{1} \ ,$$

$$f^*(s) = E(e^{-s\mathcal{X}}) = v (s\boldsymbol{I} - \boldsymbol{H})^{-1} (-\boldsymbol{H}) \mathbb{1} \ ,$$

and

$$\mu_n = E(\mathcal{X}^n) = n! v (-\boldsymbol{H})^{-n} \mathbb{1} \ .$$

Unfortunately, the $(v, \boldsymbol{H})$ pair is not a unique *representation* of the distribution.

Example 1 *Consider the following vector matrix pairs.*

$$v = \begin{bmatrix} 0.1 & 0.5 & 0.4 \end{bmatrix}, \quad \boldsymbol{H} = \begin{bmatrix} -5 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 0 & -4 \end{bmatrix}$$

and

$$z = \begin{bmatrix} -1.1 & 2.5 & -0.4 \end{bmatrix}, \quad \boldsymbol{G} = \begin{bmatrix} -11 & 10 & -1 \\ -6.6 & 6 & -1 \\ -15 & 20 & -6 \end{bmatrix}$$

The $(v, \boldsymbol{H})$ *and the* $(z, \boldsymbol{G})$ *vector matrix pairs represent the same distribution, since*

$$F(t) = 1 - v e^{\boldsymbol{H} t} \mathbb{1} = 1 - z e^{\boldsymbol{G} t} \mathbb{1} \ .$$

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Definition 1 *$(z, \boldsymbol{G})$ is* similar *to $(v, \boldsymbol{H})$ if there is a square matrix $\boldsymbol{B}$, such that $\boldsymbol{B}^{-1}$ exists, $z = v\boldsymbol{B}$, $\boldsymbol{G} = \boldsymbol{B}^{-1}\boldsymbol{H}\boldsymbol{B}$ and $\boldsymbol{B}\mathbb{1} = \mathbb{1}$.*

Any vector matrix pair *similar* to $(v, \boldsymbol{H})$ defines the same distribution, since

$$F(t) = 1 - ze^{\boldsymbol{G}t}\mathbb{1} = 1 - v\boldsymbol{B}e^{\boldsymbol{B}^{-1}\boldsymbol{H}\boldsymbol{B}t}\mathbb{1} = 1 - v\boldsymbol{B}\boldsymbol{B}^{-1}e^{\boldsymbol{H}t}\boldsymbol{B}\mathbb{1} = 1 - ve^{\boldsymbol{H}t}\mathbb{1} \ .$$

In the previous example the

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 5 & 0 \\ 2 & 0 & -1 \end{bmatrix} .$$

similarity matrix is used.

Definition 2 *Any general $(z, \boldsymbol{G})$ pair is referred to as a* matrix representation *of $F(t)$.*

Definition 3 *A $(v, \boldsymbol{H})$ matrix representation is referred to as a* Markovian representation *if v is a probability vector ($v_i \geq 0$, $\sum_i v_i = 1$) and $\boldsymbol{H}$ is a valid transient generator matrix ($H_{ii} < 0$, $H_{ij} \geq 0$ for $i \neq j$, $\exists \boldsymbol{H}^{-1}$).*

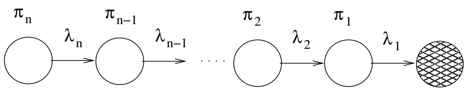
There are negative consequences of the non-uniqueness of Markovian representations of PH(n). E.g. it is difficult to check the equivalence of two different Markovian representations; optimization methods using general Markovian representations for distribution fitting with PH(n) are less efficient.

To overcome these difficulties we need to define a unique Markovian representation for all PH(n) which is commonly referred to as *canonical representation.* A proper canonical representation is:

- unique,
- Markovian,
- simple (preferably it contains the minimal number of parameters),
- augmented with a function that
  - transforms from any matrix representation to the canonical form if possible,
  - indicates if it is not possible.

Currently canonical forms are available for acyclic PH distributions of any order and for order 2 and 3 phase type distributions.

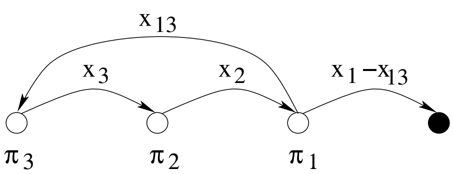
Any acyclic PH(n) (APH(n)) distributions can be transformed to the following form [4],



where $\lambda_i \geq \lambda_{i+1} > 0$, $\forall i \in \{1, \ldots, n-1\}$. The problem of transforming a non-Markovian representation of a distribution to a Markovian canonical form was not considered before. [4] presents a procedure which transforms any acyclic Markovian representation to this canonical from, but this transformation is not applicable for cyclic Markovian and non-Markovian representations of APH(n) distributions. Recently [7, 8] presented procedures for this purpose.

Since any cyclic PH(2) distribution can be transformed to an equivalent APH(2) [3], the APH(2) canonical form can be applied for the whole PH(2) class.

Based on the results of [6], a canonical representation of the PH(3) class is presented in [9]. It has the following structure



where $x_1 \geq x_2 \geq x_3 > 0$, $0 \leq x_{13} \leq x_1$, $0 \leq \pi_1, \pi_2, \pi_3$, $\pi_1 + \pi_2 + \pi_3 = 1$. This general form (containing 6 independent parameters) splits into 3 possible cases with 5 independent parameters. These cases are

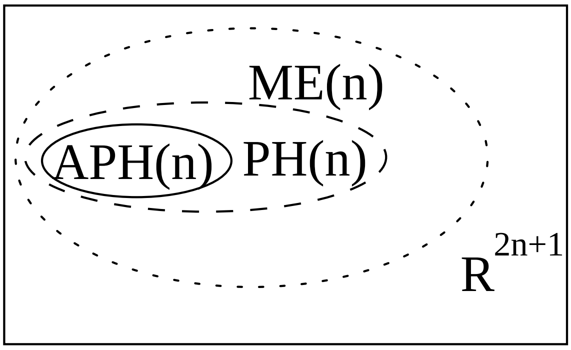
- $x_{13} = 0$

- $x_1 = x_2$
- $\pi_2 = 0$

It is a nice feature of this canonical form that it is identical with the one proposed in [4] when the distribution is an APH(3).

The minimal, unique, and Markovian canonical form makes possible the use of efficient fitting methods, but it is restricted to the above mentioned PH classes. For fitting with other PH classes it is still beneficial to find a minimal and unique representation of PH distributions. Potential candidates are the first $2n-1$ moments of the distribution which we call *moments representation* and the $2n-1$ coefficients of the normalized rational Laplace transform of the density function which we call *Laplace representation*. For fitting purposes we use the moments representation, which carries more practical information about the distribution.

Since the moments representation is not closely related to a Markovian representation of the distribution it is necessary to check if a set of $2n-1$ moments defines a valid PH distribution or not. Indeed we need to investigate the borders of the following classes of distributions



where ME(n) stands for the order n matrix exponential distribution.

The following two-step numerical procedure can identify if a set of $2n-1$ moments falls in the APH(n) or in the PH(n) classes.

- Convert the moments representation to a matrix representation by the procedure proposed by A. van de Liefvoort in [11],
- check if the obtained matrix representation can be transformed to an acyclic Markovian representation, which ensures the APH(n) membership, or to an order n Markovian representation, which ensures the PH(n) membership.

The analysis of the ME(n)\PH(n) membership is more complex. It requires to check if the distribution function defined by the matrix representation is monotone increasing.

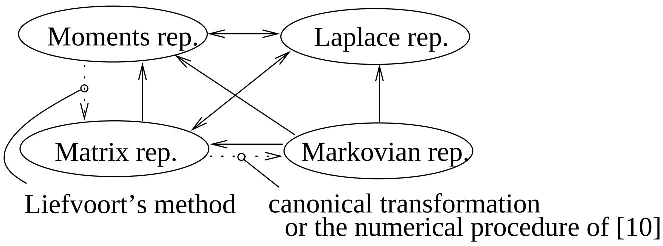
Based on the available canonical forms and transformation methods we have the following cases:

- $n = 2$
  - $ME(2) \equiv PH(2) \equiv APH(2)$ and the border is "known", because canonical form with associated transformation method exists.
- $n = 3$
  - The APH - PH border is "known" (APH canonical form exists).
  - The PH - ME border is "known" (PH(3) canonical form exists).
  - The outer border of the ME class is "less known", i.e. there is no efficient explicit procedure to check if a set of $2n-1$ moments defines a ME distribution or not. A set of available approaches is presented in [5].
- $n > 3$
  - The APH - PH border is "known" (APH canonical form).

– The PH - ME border is "almost known", i.e. there is a numerical optimization method [10] which transforms a matrix representation to a Markovian representation if possible, in the majority of the cases.

– The outer border of ME is "less known".

We can summarize the available transformations between representations of PH distributions as follows.



2 Markov Arrival Processes

Markov Arrival Processes of order n (MAP(n)) are defined as arrival processes governed by a CTMC of n states (or phases). The most common description of these processes are through the $\boldsymbol{D_0}$ and the $\boldsymbol{D_1}$ matrices of size $n \times n$.

- $\boldsymbol{D_0}$ contains the phase transition rates without arrivals,
- $\boldsymbol{D_1}$ contains the phase transition rates with arrivals.

Having these matrices the parameters of the MAP(n) process are computed as follows.

- The double transform of the number of arrivals in $(0, t)$, N_t, is:

$$f(s, z) = \mathcal{L}_{t \to s}(E(z^{N_t})) = \pi(s\boldsymbol{I} - \boldsymbol{D_0} - z\boldsymbol{D_1})^{-1}\mathbb{1} ,$$

- the joint density of the first $k+1$ inter arrival times, $X_0, X_1, \ldots, X_k$, is:

$$f(x_0, x_1, \ldots, x_k) = \pi e^{\boldsymbol{D_0}x_0}\boldsymbol{D_1}e^{\boldsymbol{D_0}x_1}\boldsymbol{D_1}\ldots e^{\boldsymbol{D_0}x_k}\boldsymbol{D_1}\mathbb{1} ,$$

- and the joint moments of these inter arrival times are:

$$E(X_0^{i_0}X_1^{i_1}\ldots X_k^{i_k}) = \pi i_0!(-\boldsymbol{D_0})^{i_0}\boldsymbol{P}\, i_1!(-\boldsymbol{D_0})^{i_1}\boldsymbol{P}\ldots i_k!(-\boldsymbol{D_0})^{i_k}\mathbb{1}$$

  where $\boldsymbol{P} = (-\boldsymbol{D_0})^{-1}\boldsymbol{D_1}$, and π is the solution of $\pi\boldsymbol{P} = \pi$ and $\pi\mathbb{1} = 1$.

Similar to the Markovian representations of a PH distribution the $(\boldsymbol{D_0}, \boldsymbol{D_1})$ representation of a MAP is not unique. A similarity transform with matrix $\boldsymbol{B}$ (having the same properties as in the previous section) results in a different matrix representation of the same MAP, i.e.

$$(\boldsymbol{D_0}, \boldsymbol{D_1}) \equiv (\boldsymbol{B}^{-1}\boldsymbol{D_0}\boldsymbol{B}, \boldsymbol{B}^{-1}\boldsymbol{D_1}\boldsymbol{B}).$$

In order to efficiently fit arrival processes with MAP(n) we also need canonical forms, which contain a minimal number of parameters and Markovian.

Unfortunately, there are much less results for the canonical representation of MAPs. The only MAP class for which canonical representation exists is the MAP(2) class [1]. If the lag-1 correlation parameter is non-negative the canonical form is

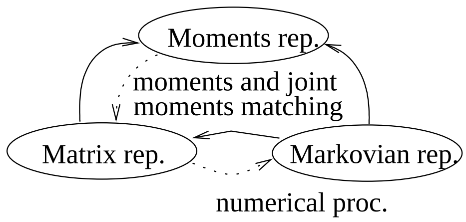
$$\boldsymbol{D_0} = \begin{bmatrix} -\lambda_1 & (1-a)\lambda_1 \\ 0 & -\lambda_2 \end{bmatrix}, \quad \boldsymbol{D_1} = \begin{bmatrix} a\lambda_1 & 0 \\ (1-b)\lambda_2 & b\lambda_2 \end{bmatrix} ,$$

and if it is positive the canonical form is

$$\boldsymbol{D_0} = \begin{bmatrix} -\lambda_1 & (1-a)\lambda_1 \\ 0 & -\lambda_2 \end{bmatrix}, \quad \boldsymbol{D_1} = \begin{bmatrix} 0 & a\lambda_1 \\ b\lambda_2 & (1-b)\lambda_2 \end{bmatrix} ,$$

where $0 < \lambda_1 \leq \lambda_2$, $0 \leq a \leq 1$ and $0 \leq b \leq 1$. This canonical form contains 4 parameters.

Higher order MAPs are also uniquely defined with properly chosen n^2 parameters [10]. A proper set of parameters are the first $2n-1$ moments of the interarrival time and the first $(n-1)^2$ joint moments of consecutive interarrival times, i.e. $E(X_0^i), i = 1, \dots, 2n-1$ and $E(X_0^i X_1^j), i, j = 1, \dots, n-1$. All together these are $(2n-1) + (n-1)^2 = n^2$ parameters. We refer to this set of parameters as the *moments representation* of MAP(n). Using the moments representation of MAP(n), for $n \geq 3$, we also have to check the limits of the MAP(n) class. It can be done using the the two-step procedure presented in [10]. The first step generates a matrix representation of the MAP(n) based on the moments representation and the second one applies the same numerical procedure to transform it to a Markovian representation.



The solid lines of the figure indicate that the moments and the joint moments can be easily computed from any matrix representation including the Markovian representations.

Example 2 *Starting from the Markovian representation*

$$[E(X_0^1), \dots, E(X_0^5)] = [0.850622, 1.79118, 5.90205, 26.122, 144.701]$$

$$\{E(X_0^i X_1^j)\} = \begin{bmatrix} 1. & 0.850622 & 1.79118 \\ 0.850622 & 0.717379 & 1.50502 \\ 1.79118 & 1.5069 & 3.15812 \end{bmatrix}$$

we apply the two-step procedure of [10] to obtain a Markovian representation of this MAP(3).

The moments representation to matrix representation procedure results in:

$$\begin{bmatrix} -5.8026 & 40.6949 & -39.1859 \\ -0.374745 & -4.10233 & 3.31133 \\ -0.408995 & 3.31859 & -4.09507 \end{bmatrix}, \begin{bmatrix} 1.34479 & -25.6148 & 28.5636 \\ 0.382524 & -0.522655 & 1.30587 \\ 0.401216 & 1.30639 & -0.522132 \end{bmatrix},$$

and the matrix representation to Markovian representation procedure:

$$\begin{bmatrix} -5.11384 & 2.04249 & 0.821793 \\ 0.00591528 & -1.64109 & 1.15513 \\ 0.487631 & 3.73933 & -7.24507 \end{bmatrix}, \begin{bmatrix} 0.097178 & 0.646453 & 1.50593 \\ 0.180526 & 0.120557 & 0.178963 \\ 2.92962 & 0.00621774 & 0.0822646 \end{bmatrix}.$$

Since the procedure resulted in a valid Markovian representation we conclude that the original set of moments and joint moments represent a MAP(3).

3 Summary of results, problems and open questions

The presented two-step method allows the moments matching of PH(n) and MAP(n) for $n \geq 3$, if the set of moments define a PH(n) or a MAP(n), but based on our experience it is often not the case with field data. In these cases either efficient fitting methods, or partial moments matching procedures are required, which are not really available yet. The research for these kind of methods is a dominant challenge on this field.

The available canonical forms of PH(3) and MAP(2) distributions already indicate that there is no hope for a single canonical structure which can describe the whole class, but we need more than one structures occasionally with very strange constraints (e.g., 2 diagonal elements of the generator matrix are equal). Indeed, our preliminary investigations suggest that the number of different structures increases rapidly with order.

The "explicit" transformation methods of canonical forms require the solution of the spectral equation, and we guess that it remains the case for the currently unsolved classes (PH(n) with $n \geq 4$ and MAP(n) with $n \geq 3$). Unfortunately, the solution of the spectral equation can become numerically hard around the limits of discriminants. Furthermore, explicit solutions are available only for $n \leq 4$. For $n \geq 5$ only numerical solutions are available.

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