On the tail decay of M/G/1-type Markov renewal processes

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Abstract

The tail decay of M/G/1-type Markov renewal processes is studied. The Markov renewal process is transformed into a Markov chain so that the problem of tail decay is reformulated in terms of the decay of the coefficients of a suitable power series. The latter problem is reduced to analyze the analyticity domain of the power series.

1 Introduction

Consider a Markov Renewal Process (MRP) $(X_n, \tau_n)_n$ of M/G/1-type on the state space $E = \{(i, j, x) : \quad i \geq 0,\ 1 \leq j \leq m,\ x \geq 0\}$ defined by the kernel

$$K(x) = \begin{bmatrix} \widetilde{B}_0(x) & \widetilde{B}_1(x) & \widetilde{B}_2(x) & \widetilde{B}_3(x) & \dots \\ \widetilde{A}_{-1}(x) & \widetilde{A}_0(x) & \widetilde{A}_1(x) & \widetilde{A}_2(x) & \dots \\ & \widetilde{A}_{-1}(x) & \widetilde{A}_0(x) & \widetilde{A}_1(x) & \ddots \\ & & \widetilde{A}_{-1}(x) & \widetilde{A}_0(x) & \ddots \\ 0 & & & \ddots & \ddots \end{bmatrix}, \quad x \geq 0,$$

where the $m \times m$ matrices $\widetilde{A}_k(x)$ and $\widetilde{B}_k(x)$ are defined as follows:

$$\begin{aligned} \widetilde{A}_k(x) &= P\{X_{n+1} \in \ell_{i+k},\ \tau_{n+1} - \tau_n \leq x | X_0, \dots, X_n, \\ &\qquad \tau_0, \dots, \tau_n, X_n \in \ell_i,\ i \geq 1\}, \quad k \geq -1, \\ \widetilde{B}_k(x) &= P\{X_{n+1} \in \ell_k,\ \tau_{n+1} - \tau_n \leq x | X_0, \dots, X_n, \\ &\qquad \tau_0, \dots, \tau_n, X_n \in \ell_0\}, \quad k \geq 0. \end{aligned}$$

Here ℓ_i, for $i = 0, 1, \dots$, denotes the i-th *level*, i.e., the set of pairs $\{(i, j), \quad j = 1, \dots, m\}$.

For $n \geq 0$ let $\widetilde{G}(n, x)$ be the probability that in n steps and in time at most x the system goes from level ℓ_1 to level ℓ_0.

An important issue arising in the applications is the analysis of the speed of convergence to zero of the sequence $\{\widetilde{G}(n, x)\}_n$, as n tends to infinity, or in

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other words, the analysis of the tail decay of the sequence $\{\widetilde{G}(n,x)\}_n$. In order to simplify the problem, one may apply the Laplace-Stiltjes transform to $\widetilde{A}_k(x)$ and to $\widetilde{G}(n,x)$, by defining

$$A_n(s) = \int_0^\infty e^{-sx}\widetilde{A}_n(dx), \quad n \geq -1,$$
$$G(w,s) = \sum_{n=1}^\infty w^n \int_0^\infty e^{-sx}\widetilde{G}(n,dx), \quad s \in \mathbb{C}^+0.$$

In this way, one finds that

$$G(w,s) = w \sum_{n=-1}^{\infty} A_n(s) G(w,s)^{n+1}, \quad |w| \leq 1, s \in \mathbb{C}^+.$$

Now, let us define

$$G(w) = \lim_{s\to 0^+} G(w,s), \quad |w| \leq 1,$$
$$A_n = \lim_{s\to 0^+} A_n(s), \quad n \geq -1.$$

It is proved in [4] that the matrices A_n for $n \geq -1$ are nonnegative, and that $\sum_{n=-1}^\infty A_n$ is a stochastic matrix. Therefore, the matrices A_n define the homogeneous part of the transition matrix of an M/G/1-type Markov chain. Moreover, the matrix $G(w)$ solves the matrix equation

$$G(w) = w \sum_{n=-1}^{\infty} A_n G(w)^{n+1}$$

and $|G(w,s)| \leq |G(w)|$ for any $s \in \mathbb{C}^+$. It is known (see [4]) that, if $0 \leq w \leq 1$, the matrix $G(w)$ is the minimal nonnegative solution of the matrix equation

$$X = w \sum_{n=-1}^{\infty} A_n X^{n+1}.$$

Moreover, the function $G(w)$ is analytic for $|w| < 1$ and its power series expansion $G(w) = \sum_{n=0}^\infty w^n G_n$, is such that $G_n \geq 0$ for any $n \geq 0$, and is convergent for $|w| = 1$. The coefficients G_n have the following probabilistic interpretation: G_n is the probability that in n steps the system goes from level ℓ_1 to level ℓ_0.

Our goal is to estimate the decay rate of the coefficients G_n of the power series $G(w)$. From a classical result in complex analysis [3, Theorem 2.2f], if $r > 1$ is the convergence radius of $G(w)$, then for any matrix norm, one has $\|G_n\| = O(\theta^n)$ for any θ such that $1/r < \theta < 1$. Therefore, our problem is reduced to estimating the convergence radius of $G(w)$.

We prove that, under suitable mild assumptions, the matrix power series $G(w)$ is analytic for $|w| < R$ and convergent for $|w| \leq R$. Here $R = \sigma/\theta(\sigma)$, where σ is the unique solution of the equation $\theta'(z)z = \theta(z)$ in the interval $(0, r_a)$, where $r_a > 1$ is the convergence radius of $A(z) = \sum_{n=-1}^{+\infty} z^{n+1} A_n$ and $\theta(z)$ is the spectral radius of $A(z)$.

The numerical value of R can be computed by relying on any effective iterative algorithm for solving a nonlinear scalar equation where at each step of the iteration the spectral radius of $A(z)$ must be computed together with its derivative.

2 Tail decay analysis

In this section we study the tail decay of the matrix power series $G(w) = \sum_{n=0}^{\infty} w^n G_n$. We recall that $G(w)$, for $|w| \leq 1$, solves the matrix equation

$$X = w \sum_{n=-1}^{\infty} A_n X^{n+1}. \quad (1)$$

Moreover, $G_0 = 0$, $G_n \geq 0$ for any $n \geq 1$ and, if $0 < w \leq 1$, $G(w)$ is the minimal nonnegative solution of (1).

We denote by r_a the convergence radius of the matrix power series $A(z) = \sum_{n=-1}^{+\infty} z^{n+1} A_n$, and we assume that the matrix $A(1)$ is irreducible and stochastic. We will denote by $\theta(z)$ the spectral radius of $A(z)$. It is proved in [2, Lemma 1] that, since $A(1)$ is irreducible, $\theta(z)$ is a real analytic function for $0 < z < r_a$. Moreover, $\theta(z)$ is strictly increasing for $0 < z < r_a$.

The tail decay analysis of $G(w)$ is performed by estimating the speed of convergence to zero of G_n as $n \to \infty$. To do this, we determine the analyticity domain of $G(w)$ on the complex plane. Indeed, from a classical result on analytic functions (see [3, Theorem 2.2f]), if $G(w)$ is analytic for $|w| < R$, where $R > 1$, then $\|G_n\| = O(\sigma^n)$ for any $1/R < \sigma < 1$ and for any matrix norm.

Define the scalar function

$$\phi_w(z) = w\theta(z), \quad w \in \mathbb{C},$$

and recall that the drift of an M/G/1-type Markov chain is defined by $\mu = \boldsymbol{\alpha}^T \boldsymbol{a}$, where $\boldsymbol{\alpha}^T$ is the stationary probability vector of $A(1)$, $\boldsymbol{a} = \sum_{n=-1}^{+\infty} nA_n\boldsymbol{e}$, and $\boldsymbol{e}$ is the vector of all ones.

We assume throughout that one of the following two conditions holds:

1. $\mu \geq 0$
2. $\mu < 0$, $r_a > 1$ and $\det(A(z) - zI)$ has a zero outside the closed unit disk.

We prove the following

Proposition 1. *The following properties hold:*

1. *the equation $z\theta'(z) = \theta(z)$ has a unique solution σ in $(0, r_a)$;*
2. *the equation $\phi_w(z) = z$ has:*
   - *a solution of multiplicity two in $(0, r_a)$ if $w = \sigma/\theta(\sigma)$,*
   - *two distinct solutions in $(0, r_a)$ for any $0 < w < \sigma/\theta(\sigma)$.*

Proof. Define the function $f_w(t) = \log \phi_w(e^t) - t$, for $-\infty < t < \log r_a$. In [2] it is shown that the function $\log \theta(e^t)$, for $-\infty < t < \log r_a$ is a convex increasing function of t. Moreover, according to the results of [2], the equation $f_1(t) = 0$ has in $(-\infty, \log r_a)$: two distinct solutions $t_1 < 0$ and $t_2 = 0$ if $\mu > 0$, a solution of multiplicity two in $t = 0$ if $\mu = 0$, two distinct solutions $t_1 = 0$ and $t_2 > 0$

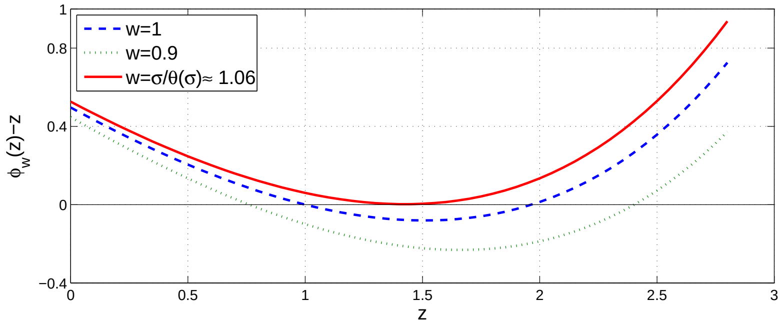

Figure 1: Plot of the function $\phi_w(z) - z$ for different values of w, case $\mu < 0$

if $\mu < 0$. In all the cases, for convexity, the function $f_1(t) = \log(\theta(e^t)) - t$ has a unique global minimum $t_1 \le t^* \le t_2$ in $-\infty < t < \log r_a$. Moreover, t^* is the unique solution to the equation $f_1'(t) = 0$ in $(-\infty, \log r_a)$. By computing the derivative, we find that t^* is the unique solution to $e^t\theta'(e^t) = \theta(e^t)$ in $(-\infty, \log r_a)$. By setting $\sigma = e^{t^*}$, we conclude that σ is the unique solution to $z\theta'(z) = \theta(z)$ in the interval $(0, r_a)$. Concerning the second part, observe that $\phi_w(z) = z$ has a solution in $0 < z < r_a$ if and only if the equation $f_w(t) = 0$ has a solution in $-\infty < t < \log r_a$. Since $f_w(t) = f_1(t) + \log w$, the equation $f_w(t) = 0$ has a solution of multiplicity two if $f_w(t^*) = 0$, two distinct solutions if $f_w(t^*) < 0$. Since $\sigma = e^{t^*}$, one finds that $f_w(t^*) = 1/\theta'(\sigma) = \sigma/\theta(\sigma)$. □

Figure 1 shows the behaviour of the function $\phi_w(z) - z$ for different values of w.

By Proposition 1, if $0 < w \le R$, the equation $\phi_w(z) = z$ has at least a solution in $(0, r_a)$. Let us donote by $\xi(w)$ the smallest $\xi(w) \in (0, r_a)$ such that $\phi_w(\xi(w)) = \xi(w)$. Since $A(1)$ is irreducible, also $A(\xi(w))$ is irreducible and by the Perron-Frobenius theorem [1], there exists a positive vector $\boldsymbol{v}(w)$ such that $wA(\xi(w))\boldsymbol{v}(w) = \xi(w)\boldsymbol{v}(w)$.

Proposition 2. *Consider the matrix sequence $X^{(k)}(w)$ defined by*

$$X^{(k+1)}(w) = w \sum_{n \ge -1} A_n (X^{(k)}(w))^{n+1}, \quad k \ge 0, \tag{2}$$

with $X^{(0)}(w) = 0$. Then:

1. *$X^{(k)}(w)$ is analytic for $|w| < R$ and convergent for $|w| = R$, where $R = \sigma/\theta(\sigma)$;*
2. *$X^{(k)}(w) = \sum_{n=0}^{k} w^n G_n + \sum_{n=k+1}^{\infty} w^n E_n^{(k)}$;*
3. *if $0 < w \le R$, then $0 \le X^{(k)}(w) \le X^{(k+1)}(w)$ and $X^{(k)}(w)\boldsymbol{v}(w) \le \xi(w)\boldsymbol{v}(w)$ for any $k \ge 0$;*

4. *the sequence* $\{X^{(k)}(w)\}_k$ *converges uniformly to* $G(w)$ *in any closed disk* $\{|w| \le r\}$, *with* $0 < r \le R$.

Proof. The analyticity of $X^{(k)}(w)$ is proved by induction on k. More specifically, we prove that, for any $k \ge 0$, $X^{(k)}(w)$ is a matrix power series in w with nonnegative matrix coefficients, and that $X^{(k)}(R)\boldsymbol{v}(R) \le \xi(R)\boldsymbol{v}(R)$. For the nonnegativity of the matrix coefficients and for the positivity of $\boldsymbol{v}(R)$, this latter property implies that $X^{(k)}(w)$ is analytic for $|w| < R$ and convergent for $|w| = R$. For $k = 0$ then $X^{(k)}(w) = 0$ and therefore the property trivially holds. Assume that $X^{(k)}(w)$ is a power series with nonnegative matrix coefficients such that $X^{(k)}(R)\boldsymbol{v}(R) \le \xi(R)\boldsymbol{v}(R)$. Then $X^{(k+1)}(w)$ is a formal matrix power series, since $X^{(k+1)}(w) = wA(X^{(k)}(w))$, where $A(z) = \sum_{n\ge -1} A_n z^{n+1}$; moreover the matrix coefficients of $X^{(k+1)}(w)$ are an infinite sum of nonnegative matrices since $A_n \ge 0$ for any $n \ge -1$. Observe that, by inductive hypothesis, one has

$$X^{(k+1)}(R)\boldsymbol{v}(R) = R \sum_{n\ge -1} A_n (X^{(k)}(R))^{n+1}\boldsymbol{v}(R) \le \\ R \sum_{n\ge -1} \xi(R)^{n+1} A_n \boldsymbol{v}(R) = \xi(R)\boldsymbol{v}(R),$$

hence $X^{(k+1)}(R)\boldsymbol{v}(R) \le \xi(R)\boldsymbol{v}(R)$.

Let us prove Part 2. For $|w| \le 1$ let $E^{(k)}(w) = X^{(k)}(w) - G(w)$. We prove by induction on k that $E^{(k)}(w) = \sum_{n=k+1}^{\infty} w^n E_n^{(k)}$. If $k = 0$ one has $E^{(0)}(w) = X^{(0)}(w) - G(w) = \sum_{n=1}^{\infty} w^n G_n$, since $G_0 = X^{(0)}(w) = 0$, therefore the property holds. Assume that $E^{(k)}(w) = \sum_{n=k+1}^{\infty} w^n E_n^{(k)}$. One has

$$E^{(k+1)}(w) = w \sum_{n\ge -1} A_n ((X^{(k)}(w))^{n+1} - G(w)^{n+1}).$$

By using the identity $X^n - Y^n = \sum_{i=0}^{n-1} X^i (X - Y) Y^{n-i-1}$, $n \ge 1$, from the latter equation we obtain

$$E^{(k+1)}(w) = w \sum_{n\ge -1} A_n \sum_{i=0}^{n} (X^{(k)}(w))^i E^{(k)}(w) G(w)^{n-i}.$$

Therefore $E^{(k+1)}(w) = \sum_{n=k+2}^{\infty} w^n E_n^{(k+1)}$ for suitable matrix coefficients $E_n^{(k+1)}$.

Part 3 can be easily proved by induction on k.

Concerning Part 4, let $0 < r \le R$ and let $\|\cdot\|_1$ be the 1-norm. Since the matrix coefficients of $E^{(k)}(w)$ are nonnegative, one has

$$\sup_{|w|\le r} \|E^{(k)}(w)\|_1 = \sup_{|w|\le r} \left\| \sum_{n=k+1}^{\infty} w^n E_n^{(k)} \right\|_1 = \left\| \sum_{n=k+1}^{\infty} r^n E_n^{(k)} \right\|_1$$

and the latter quantity converges to zero as $k \to \infty$. □

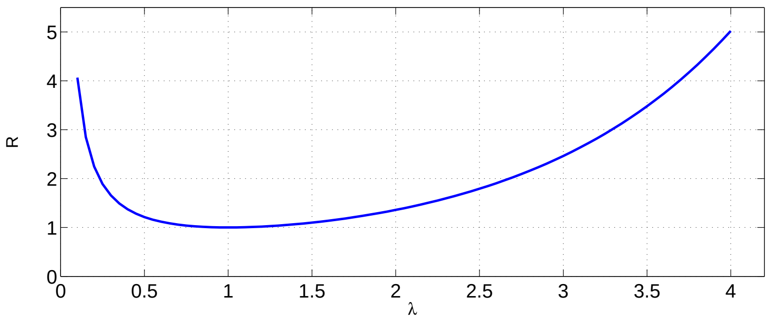

Figure 2: Convergence radius of $G(w)$ for Example 1, as a function of λ

Theorem 3. *The matrix function $G(w)$ is analytic for $|w| < R$ and convergent for $|w| \leq R$, where $R = \sigma/\theta(\sigma)$ and σ is defined in Proposition 1. Morever, if $0 < w \leq R$, then the spectral radius of $G(w)$ is $\xi(w)$.*

Proof. $G(w)$ is a power series with nonnegative matrix coefficients, therefore it is sufficient to prove that $G(R)$ is bounded. This latter fact follows from Proposition 2, since the sequence $\{X^{(k)}(R)\}_k$ is bounded and convergent to $G(R)$. In order to prove that $\rho(G(w)) = \xi(w)$, we consider the matrix $B(w) = D^{-1}G(w)D$, where $D = \mathrm{Diag}(\boldsymbol{v}(w))$ and $\boldsymbol{v}(w)$ is a positive vector such that $G(w)\boldsymbol{v}(w) = \xi(w)\boldsymbol{v}(w)$. It easily follows that $B(w)\boldsymbol{e} = \xi(w)\boldsymbol{e}$, this implies that $||B(w)||_\infty = \xi(w)$. Since $\rho(B(w)) \leq ||B(w)||_\infty = \xi(w)$, and since $\xi(w)$ is an eigenvalue of $B(w)$ one has $\rho(G(w)) = \rho(B(w)) = \xi(w)$. $\square$

From the proof of Proposition 1, the unique solution σ in $(0, r_a)$ of the equation $z\theta'(z) = \theta(z)$ is $\sigma = e^{t^*}$, where t^* is the global minimum of the function $f_1(t) = \log(\theta(e^t)) - t$ in $(-\infty, \log r_a)$. Morever $t_1 < t^* < t_2$ if $\mu \neq 0$, and $t^* = t_1 = t_2$ if $\mu = 0$, where t_1 and t_2 are the solutions to the equation $f_1(t) = 0$ in $(-\infty, \log r_a)$.

Example 1. Consider the simple scalar case of a Poisson distribution of parameter $\lambda > 0$, where $A_n = e^{-\lambda}\frac{\lambda^{n+1}}{(n+1)!}$, for $n \geq -1$. We may verify that $G(w) = \sum_{i=1}^{\infty} w^i e^{i(1-\lambda)}\lambda^{i-1}$, therefore the convergence radius of $G(w)$ is $R_G = \frac{e^{\lambda-1}}{\lambda}$. Since $A(z)$ is a scalar function, $\theta(z) = A(z) = e^{\lambda(z-1)}$, and the equation $z\theta'(z) = \theta(z)$ becomes $\lambda z e^{\lambda(z-1)} = e^{\lambda(z-1)}$, therefore $\sigma = \lambda^{-1}$. According to Theorem 3 $R = \sigma/\theta(\sigma) = \lambda^{-1}/A(\lambda^{-1})$, which coincides with R_G. Figure 2 illustrates the convergence radius of $G(w)$ as a function of λ: we observe that for $\lambda = 1$ (the null recurrent case) the radius is minimum, and equal to 1, while it diverges as $\lambda \to 0$ and $\lambda \to +\infty$.

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