Matrix Analytic Methods in Branching Processes

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Abstract

We examine the question of solving the extinction probability of a particular class of continuous-time multi-type branching processes, named Markovian binary trees (MBT). The extinction probability is the minimal nonnegative solution of a fixed point equation that turns out to be quadratic, which makes its resolution particularly clear.

We analyze first two linear algorithms to compute the extinction probability of an MBT, of which one is new, and, we propose a quadratic algorithm arising from Newton's iteration method for fixed-point equations.

Finally, we add a catastrophe process to the initial MBT, and we analyze the resulting system. The extinction probability turns out to be much more difficult to compute; we use a $G/M/1$-type Markovian process approach to approximate this probability.

Keywords: Branching Processes; Matrix Analytic Methods; Extinction Probability, Catastrophe Process.

1 Introduction

We consider the resolution of quadratic equations of the form

$$\boldsymbol{x} = \boldsymbol{a} + B\,(\boldsymbol{x} \otimes \boldsymbol{x}), \tag{1}$$

where $\boldsymbol{x}$ is an unknown n-vector, $\boldsymbol{a}$ is a vector of size n and B is a matrix of size $n \times n^2$, all of them having real components, such that $\mathbf{0} \le \boldsymbol{a} \le \mathbf{1}$, $\boldsymbol{a} \ne \mathbf{0}$, $B \ge 0$, and $\boldsymbol{a} + B\,\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ denotes a vector of 1's.

Equations of this type occur in the analysis of the extinction probability of a particular family of multi-type branching processes. Branching processes are stochastic models of growth for populations consisting of several types of individuals who may produce offsprings during their lifetime, each individual behaving *independently* of each other in the standard case. For the most part, applications are found in biology and epidemiology [3, 7] but also in telecommunication systems [6, 12].

For the special class under study, the life of each individual is controlled by a transient Markovian arrival process [10], called the *phase* process, on the state space $\{1, \ldots, n\}$. At different epochs, an individual may give birth to one child at a time, and the child's life is controlled by an independent replica of the same phase process. Eventually, some event in the phase process causes the death of the individual. The Markovian phase process assumption allows one to model a large variety of situations, so that the only real restriction lies in the constraint that births occur singly. The quantity of basic interest is the extinction probability of the branching process, that is, the probability that all the individuals will eventually have died out at some time.

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These branching processes have been investigated under the name Markovian Binary Trees (MBT) in Bean, Kontoleon and Taylor [8, 11], and in Hautphenne, Latouche and Remiche [4, 5]. We refer to these papers for a detailed description of MBTs and only give the essential aspects here.

In order to compute the extinction probability, it is not necessary to keep track of the time which elapses between births, but only on the fact that there is a birth or not. For that reason, we need not give the formal description of the phase Markovian process itself, but we only have to define the following quantities: a_i is the probability that an individual who is in phase i eventually dies without any additional offspring; $B_{i,jk}$ is the probability that an individual who is in phase i eventually produces a child, that the child starts its life in phase j, and that the parent switches to phase k after the birth; q_i, to be computed, is the conditional probability that an MBT becomes extinct, given that it begins with a unique individual in phase i, for $1 \leq i, j, k \leq n$. These quantities are organized in the column vectors $\boldsymbol{a}$ and $\boldsymbol{q}$ and the matrix B.

An MBT eventually becomes extinct if and only if the initial individual dies without any offspring or if it produces a child and both the child and the parent processes eventually become extinct. This shows that $\boldsymbol{q}$ is a solution of (1); actually, $\boldsymbol{q}$ is its minimal nonnegative solution.

The MBT will be called subcritical, supercritical or critical if the spectral radius $\rho[M]$ of the nonnegative matrix

$$M = B\,(\mathbf{1} \otimes I + I \otimes \mathbf{1}) = B\,(\mathbf{1} \oplus \mathbf{1}) \tag{2}$$

is strictly less than one, strictly greater than one, or equal to one, respectively (Athreya and Ney [1, Chapter 5]). In the subcritical and critical cases, $\boldsymbol{q} = \mathbf{1}$, while in the supercritical case $\boldsymbol{q} \leq \mathbf{1}$, $\boldsymbol{q} \neq \mathbf{1}$. Thus, as far as computing $\boldsymbol{q}$ is concerned, the supercritical case is the only interesting case and we usually assume that such is the case.

Here, we first consider the extinction probability problem for a standard MBT, and after for an MBT undergoing some catastrophe process. In the last case, the individuals do not behave independently of each others anymore, which makes the analysis of the process much more difficult.

2 Extinction probability of a standard MBT

Bean, Kontoleon and Taylor [8, 11] analyse two linearly convergent algorithms to solve (1). The first one is named the *depth* algorithm and is obtained by using fixed point (or functional) iterations on (1). The second algorithm in [8, 11], named the *order* algorithm, is based on a rewrite of (1); that equation may also be written as

$$\boldsymbol{x} = [I - B\,(\boldsymbol{x} \otimes I)]^{-1}\,\boldsymbol{a} \tag{3}$$

or equivalently as

$$\boldsymbol{x} = [I - B\,(I \otimes \boldsymbol{x})]^{-1}\,\boldsymbol{a}. \tag{4}$$

The order algorithm uses fixed point iterations on (3) *or* on (4). A third linearly convergent algorithm, is presented in Hautphenne, Latouche and Remiche [4]; it is called the *thicknesses* algorithm and is obtained by using fixed point iterations alternatively on (3) *and* on (4). It offers some advantages in that it better exploits possible dissymmetries in the structure of B. The convergence rates of these three linear algorithms are studied and compared in [4].

In [5], we apply Newton's method to (1) and we show that the resulting sequence is quadratically and globally convergent.

The successive approximations of the four algorithms have a probabilistic interpretation: each iteration may be interpreted as the extinction probability of the branching process, under a set of constraints which become weaker at each step. This is described in Hautphenne, Latouche and Remiche [4, 5].

3 Extinction probability with catastrophes

Let us now add a catastrophe process to the initial MBT. Assume that the catastrophes occur following a Poisson Process, that they arrive independently of the evolution of the MBT, and that any particle in a given phase at the time of the catastrophe is killed with a probability depending on this phase.

This yields to dependencies in the evolution of the individuals. The extinction probability $\boldsymbol{q}^*$ is thus much more difficult to obtain. We used a $G/M/1$-type Markovian process approach to numerically compute $\boldsymbol{q}^*$. This approach has to be improved since it is numerically unstable when the number of phases in the MBT is large. We are currently working on that question.

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