Symposium on Theoretical Aspects of Computer Science 2008 (Bordeaux), pp. 409-420
www.stacs-conf.org

ON GEOMETRIC SPANNERS OF EUCLIDEAN AND UNIT DISK GRAPHS

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Abstract. We consider the problem of constructing bounded-degree planar geometric spanners of Euclidean and unit-disk graphs. It is well known that the Delaunay subgraph is a planar geometric spanner with stretch factor $C_{del} \approx 2.42$; however, its degree may not be bounded. Our first result is a very simple linear time algorithm for constructing a subgraph of the Delaunay graph with stretch factor $\rho = 1 + 2\pi(k \cos \frac{\pi}{k})^{-1}$ and degree bounded by k, for any integer parameter $k \geq 14$. This result immediately implies an algorithm for constructing a planar geometric spanner of a Euclidean graph with stretch factor $\rho \cdot C_{del}$ and degree bounded by k, for any integer parameter $k \geq 14$. Moreover, the resulting spanner contains a Euclidean Minimum Spanning Tree (EMST) as a subgraph. Our second contribution lies in developing the structural results necessary to transfer our analysis and algorithm from Euclidean graphs to unit disk graphs, the usual model for wireless ad-hoc networks. We obtain a very simple distributed, *strictly-localized* algorithm that, given a unit disk graph embedded in the plane, constructs a geometric spanner with the above stretch factor and degree bound, and also containing an EMST as a subgraph. The obtained results dramatically improve the previous results in all aspects, as shown in the paper.

Introduction

Given a set of points P in the plane, the Euclidean graph E on P is defined to be the complete graph whose vertex-set is P. Each edge AB connecting points A and B is assumed to be embedded in the plane as the straight line segment AB; we define its cost to be the Euclidean distance $|AB|$. We define the unit disk graph U to be the subgraph of E consisting of all edges AB with $|AB| \leq 1$.

Let G be a subgraph of E. The cost of a simple path $A = M_0, M_1, ..., M_r = B$ in G is $\sum_{j=0}^{r-1} |M_j M_{j+1}|$. Among all paths between A and B in G, a path with the smallest cost is defined to be a *smallest cost path* and we denote its cost as $c_G(A, B)$. A spanning subgraph H of G is said to be a *geometric spanner* of G if there is a constant ρ such that for every

1998 ACM Subject Classification: C.1.4, F.2.2, G.2.2.
Key words and phrases: Geometric spanner, euclidean graph, unit disk graph, wireless ad-hoc networks.
The work of the first author was supported in part by a DePaul University Competitive Research Grant.

two points $A, B \in G$ we have: $c_H(A, B) \leq \rho \cdot c_G(A, B)$. The constant ρ is called the *stretch factor* of H (with respect to the underlying graph G).

The problem of constructing geometric spanners of Euclidean graphs has recently received a lot of attention due to its applications in computational geometry, wireless computing, and computer graphics (see, for example, the recent book [13] for a survey on geometric spanners and their applications in networks). Dobkin et al. [9] showed that the Delaunay graph is a planar geometric spanner of the Euclidean graph with stretch factor $(1+\sqrt{5})\pi/2 \approx 5.08$. This ratio was improved by Keil et al [10] to $C_{del} = 2\pi/(3\cos(\pi/6)) \approx 2.42$, which currently stands as the best upper bound on the stretch factor of the Delaunay graph. Many researchers believe, however, that the lower bound of $\pi/2$ shown in [7] is also an upper bound on the stretch factor of the Delaunay graph. While Delaunay graphs are good planar geometric spanners of Euclidean graphs, they may have unbounded degree.

Other geometric (sparse) spanners were also proposed in the literature including the Yao graphs [16], the Θ-graphs [10], and many others (see [13]). However, most of these proposed spanners either do not guarantee planarity, or do not guarantee bounded degree.

Bose et al. [2, 3] were the first to show how to extract a subgraph of the Delaunay graph that is a planar geometric spanner of the Euclidean graph with stretch factor ≈ 10.02 and degree bounded by 27. In the context of unit disk graphs, Li et al. [11, 12] gave a distributed algorithm that constructs a planar geometric spanner of a unit disk graph with stretch factor C_{del}; however, the spanner constructed can have unbounded degree. Wang and Li [14, 15] then showed how to construct a bounded-degree planar spanner of a unit disk graph with stretch factor $max\{\pi/2, 1+\pi\sin(\alpha/2)\} \cdot C_{del}$ and degree bounded by $19+2\pi/\alpha$, where $0 < \alpha < 2\pi/3$ is a parameter. Very recently, Bose et. al [5] improved the earlier result in [2, 3] and showed how to construct a subgraph of the Delaunay graph that is a geometric spanner of the Euclidean graph with stretch factor: $max\{\pi/2, 1+\pi\sin(\alpha/2)\} \cdot C_{del}$ if $\alpha < \pi/2$ and $(1+2\sqrt{3}+3\pi/2+\pi\sin(\pi/12)) \cdot C_{del}$ when $\pi/2 \leq \alpha \leq 2\pi/3$, and whose degree is bounded by $14 + 2\pi/\alpha$. Bose et al. then applied their construction to obtain a planar geometric spanner of a unit disk graph with stretch factor $max\{\pi/2, 1 + \pi\sin(\alpha/2)\} \cdot C_{del}$ and degree bounded by $14 + 2\pi/\alpha$, for any $0 < \alpha \leq \pi/3$. This was the best bound on the stretch factor and the degree.

We have two new results in this paper. We develop structural results about Delaunay graphs that allow us to present a very simple linear-time algorithm that, given a Delaunay graph, constructs a subgraph of the Delaunay graph with stretch factor $1+2\pi(k\cos(\pi/k))^{-1}$ (with respect to the Delaunay graph) and degree at most k, for any integer parameter $k \geq 14$. This result immediately implies an $O(n \lg n)$ algorithm for constructing a planar geometric spanner of a Euclidean graph with stretch factor of $(1 + 2\pi(k\cos(\pi/k))^{-1}) \cdot C_{del}$ and degree at most k, for any integer parameter $k \geq 14$ (n is the number of vertices in the graph). We then translate our work to unit disk graphs and present our second result: a very simple and *strictly-localized* distributed algorithm that, given a unit-disk graph embedded in the plane, constructs a planar geometric spanner of the unit disk graph with stretch factor $(1 + 2\pi(k\cos(\pi/k))^{-1}) \cdot C_{del}$ and degree bounded by k, for any integer parameter $k \geq 14$. This efficient distributed algorithm exchanges no more than $O(n)$ messages in total, and runs in $O(\Delta \lg \Delta)$ local time at a node of degree Δ. We show that both spanners include a Euclidean Minimum Spanning Tree as a subgraph.

Both algorithms significantly improve previous results (described above) in terms of the stretch factor and the degree bound. To show this, we compare our results with previous results in more detail. For a degree bound $k = 14$, our result on Euclidean graphs imply a bound of at most 3.54 on the stretch factor. As the degree bound k approaches ∞, our bound on the stretch factor approaches $C_{del} \approx 2.42$. The very recent results of Bose et al. [5] achieve a lowest degree bound of 17, and that corresponds to a bound on the stretch factor of at least 23. If Bose et al. [5] allow the degree bound to be arbitrarily large (i.e., to approach ∞), their bound on the stretch factor approaches $(\pi/2) \cdot C_{del} > 3.75$. Our stretch factor and degree bounds for unit disk graphs are the same as our results for Euclidean graphs. The smallest degree bound derived by Bose et al. [5] is 20, and that corresponds to a stretch factor of at least 6.19. If Bose et al. [5] allow the degree bound to be arbitrarily large, then their bound on the stretch factor approaches $(\pi/2) \cdot C_{del} > 3.75$. On the other hand, the smallest degree bound derived in Wang et al. [14, 15] is 25, and that corresponds to a bound of 6.19 on the stretch factor. If Wang et al. [14, 15] allow the degree bound to be arbitrarily large, then their bound on the stretch factor approaches $(\pi/2) \cdot C_{del} > 3.75$. Therefore, even the worst bound of at most 3.54 on the stretch factor corresponding to our lowest bound on the degree $k = 14$, beats the best bound on the stretch factor of at least 3.75 corresponding to arbitrarily large degree in both Bose et al. [5] and Wang et al. [14, 15]!

1. Definitions and Background

We start with the following well known observation:

Observation 1.1. A subgraph H of graph G has stretch factor ρ if and only if for every edge $XY \in G$: the length of a shortest path in H from X to Y is at most $\rho \cdot |XY|$.

For three non-collinear points X, Y, Z in the plane we denote by $\bigcirc XYZ$ the circumscribed circle of triangle $\triangle XYZ$. A *Delaunay triangulation* of a set of points P in the plane is a triangulation of P in which the circumscribed circle of every triangle contains no point of P in its interior. It is well known that if the points in P are *in general position* (i.e., no four points in P are cocircular) then the Delaunay triangulation of P is unique [8]. In this paper—as in most papers in the literature—we shall assume that the points in P are in general position; otherwise, the input can be slightly perturbed so that this condition is satisfied. The *Delaunay graph* of P is defined as the plane graph whose point-set is P and whose edges are the edges of the Delaunay triangulation of P. An alternative definition that we end up using is:

Definition 1.2. An edge XY is in the Delaunay graph of P if and only if there exists a circle through points X and Y whose interior contains no point in P.

It is well known that the Delaunay graph of a set of points P is a spanning subgraph of the Euclidean graph defined on P (i.e., the complete graph on point-set P) whose stretch factor is bounded by $C_{del} = 4\sqrt{3}\pi/9 \approx 2.42$ [10].

Given integer parameter $k > 6$, the *Yao subgraph* [16] of a plane graph G is constructed by performing the following *Yao step* at every point M of G: place k equally-separated rays out of M (arbitrarily defined), thus creating k closed cones of size $2\pi/k$ each, and choose the shortest edge in G out of M (if any) in each cone. The Yao subraph consists of edges in G chosen by *either* endpoint. Note that the degree of a point in the Yao subgraph of G may be unbounded.

Two edges MX, MY incident on a point M in a graph G are said to be *consecutive* if one of the angular sectors determined by MX and MY contains no neighbors of M.

2. Bounded Degree Spanners of Delaunay Graphs

Let P be a set of points in the plane and let E be the complete, Euclidean graph defined on point-set P. Let G be the Delaunay graph of P. This section is devoted to proving the following theorem:

Theorem 2.1. *For every integer $k \geq 14$, there exists a subgraph G' of G such that G' has maximum degree k and stretch factor $1 + 2\pi(k \cos \frac{\pi}{k})^{-1}$.*

A linear time algorithm that computes G' from G is the key component of our proof. This very simple algorithm essentially performs a *modified Yao step* (see Section 2.3) and selects up to k edges out of every point of G. G' is simply the spanning subgraph of G consisting of edges chosen by *both* endpoints.

In order to describe the modified Yao step, we must first develop a better understanding of the structure of the Delaunay graph G. Let CA and CB be edges incident on point C in G such that $\angle BCA \leq 2\pi/k$ and CA is the shortest edge within the angular sector $\angle BCA$. We will show how the above theorem easily follows if, for every such pair of edges CA and CB:

1. we show that there exists a path p from A to B in G of length $|p|$, such that: $|CA| + |p| \leq (1 + 2\pi(k \cos \frac{\pi}{k})^{-1})|CB|$, and
2. we modify the standard Yao step to include the edges of this path in G', in addition to including the edges picked by the standard Yao step but without increasing the number of edges chosen at each point beyond k.

This will ensure that: for any edge $CB \in G$ that is not included in G' by the modified Yao step, there is a path from C to B in G', whose edges are all included in G' by the modified Yao step, and whose cost is at most $(1+2\pi(k \cos \frac{\pi}{k})^{-1})|CB|$. In the lemma below, we prove the existence of this path and show some properties satisfied by edges of this path; we will then modify the standard Yao step to include edges satisfying these properties.

Lemma 2.2. *Let $k \geq 14$ be an integer, and let CA and CB be edges in G such that $\angle BCA \leq 2\pi/k$ and CA is the shortest edge in the angular sector $\angle BCA$. There exists a path $p : A = M_0, M_1, ..., M_r = B$ in G such that:*

(i) $|CA| + \sum_{i=0}^{r-1} |M_i M_{i+1}| \leq (1 + 2\pi(k \cos \frac{\pi}{k})^{-1})|CB|$.

(ii) *There is no edge in G between any pair M_i and M_j lying in the closed region delimited by CA, CB and the edges of p, for any i and j satisfying $0 \leq i < j - 1 \leq r$.*

(iii) $\angle M_{i-1} M_i M_{i+1} > (\frac{k-2}{k})\pi$, *for $i = 1, \cdots, r - 1$.*

(iv) $\angle CAM_1 \geq \frac{\pi}{2} - \frac{\pi}{k}$.

We break down the proof of the above lemma into two cases: when $\triangle ABC$ contains no point of G in its interior, and when there are points of G inside $\triangle ABC$. We define some additional notation and terminology first. We define the circle $(O) = \bigcirc ABC$ with center O, and set $\Theta = \angle BCA$. Note that $\angle AOB = 2\Theta \leq 4\pi/k$. We will use $\overset{\frown}{AB}$ to denote the arc of (O) determined by points A and B and facing $\angle AOB$. We will make use of the following easily verified Delaunay graph property:

Proposition 2.3. *If CA and CB are edges of G then the region inside (O) subtended by chord CA and away from B and the region inside (O) subtended by chord CB and away from A contain no points.*

2.1. The Outward Path

We consider first the case when no points of G are inside $\triangle ABC$. Since both CA and CB are edges in G and by Proposition 2.3, the region of (O) subtended by chord AB closer to C has no points of G in its interior. Keil and Gutwin [10] showed that, in this case, there exists a path between A and B in G inside the region of (O) subtended by chord AB away from C, whose length is bounded by the length of $\widehat{AB}$ (see Lemma 1 in [10]). We find it convenient to use a recursive definition of their path (for more details, we refer the reader to [10]):

1. **Base case:** If $AB \in G$, the path consists of edge AB.
2. **Recursive step:** Otherwise, a point must reside in the region of (O) subtended by chord AB and away from C. Let T be such a point with the property that the region of $\bigcirc ATB$ subtended by chord AB closer to T is empty. We call T an *intermediate point* with respect to the pair of points (A, B). Let (O_1) be the circle passing through A and T whose center O_1 lies on segment AO and let (O_2) be the circle passing through B and T whose center O_2 lies on segment BO. Then both (O_1) and (O_2) lie inside (O), and $\angle AO_1T$ and $\angle TO_2B$ are both less than $\angle AOB \leq 4\pi/k$. Moreover, the region of (O_1) subtended by chord AT that contains O_1 is empty, and the region of (O_2) subtended by chord BT and containing O_2 is empty. Therefore, we can recursively construct a path from A to T and a path from T to B, and then concatenate them to obtain a path from A to B.

Definition 2.4. We call the path constructed above the *outward path* between A and B.

Keil and Gutwin [10], from this point on, use a purely geometric argument (with no use of Delaunay graph properties) to show that the length of the obtained path $A = M_0, M_1, \cdots, M_r = B$ (where each point M_p, for $p = 1, \cdots, r-1$, is an intermediate point with respect to a pair (M_i, M_j), where $0 \leq i < p < j \leq r$) is smaller than the length of $\widehat{AB}$. Figure 1 illustrates an outward path between A and B.

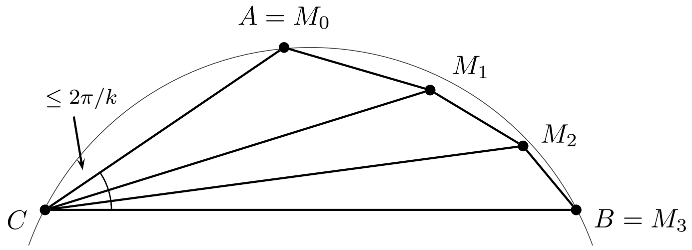

Figure 1: Illustration of an outward path.

Proposition 2.5. *In every recursive step of the outward path construction described above, if M_p is an intermediate point with respect to a pair of points (M_i, M_j), then:*

(a) *there is a circle passing through C and M_p that contains no point of G, and*

(b) *circles $\bigcirc CM_iM_p$ and $\bigcirc CM_jM_p$ contain no points of G except, possibly, in the region subtended by chords M_iM_p and M_pM_j, respectively, away from C.*

Proof. We assume, by induction, that there are circles (O_{M_i}) and (O_{M_j}) passing through C and M_i, and C and M_j, respectively, containing no points of G, and that the circle $(O) = \bigcirc CM_iM_j$ contains no point of G in the interior of the region R' subtended by chord M_iM_j closer to C. (This is certainly true in the base case because $CA, CB \in G$, by Proposition 2.3, and by our initial assumptions).

Since M_iM_j is not an edge in G, the point M_p chosen in the construction is the point with the property that the region R of $\bigcirc M_iM_pM_j$ subtended by chord M_iM_j away from C, contains no point of G. Then the circle passing through C and M_p and tangent to $\bigcirc M_iM_pM_j$ at M_p is completely inside $(O_{M_i}) \cup (O_{M_j}) \cup R \cup R'$, and therefore devoid of points of G. This proves part (a).

The region of $\bigcirc CM_iM_p$ subtended by chord M_iM_p and containing C is inside $(O_{M_i}) \cup R \cup R'$, and therefore contains no point of G in its interior. The same is true for the region of $\bigcirc CM_jM_p$ subtended by chord M_jM_p and containing C, and part (b) holds as well. ∎

We are now ready to prove Lemma 2.2 in the case when no point of G lies inside $\triangle ABC$. In this case we define the path in Lemma 2.2 to be the outward path between A and B.

Proof of Lemma 2.2 for the case of outward path.

(*i*) With $\Theta = \angle BCA$, we have $|\overset{\frown}{AB}| = 2\Theta \cdot |OA|$ and $\sin\Theta = |AB|/(2|OA|)$. We note that $|CA| + |\overset{\frown}{AB}|$ is largest when $|CA| = |CB|$, i.e. when CA and CB are symmetrical with respect to the diameter of $\bigcirc CAB$ passing through C; this follows from the fact that the perimeter of a convex body is not smaller than the perimeter of a convex body containing it (see page 42 in [1]). If $|CA| = |CB|$, $\sin\frac{\Theta}{2} = \frac{|AB|}{2|CB|}$. Using elementary trigonometry, it follows from the above facts and from $|CA| \leq |CB|$ that:

$$\begin{aligned} |CA| + |\overset{\frown}{AB}| &\leq |CB| + 2\Theta\cdot|OA| = |CB| + (\frac{\Theta}{\sin\Theta})\cdot|AB| = |CB| + (\frac{\Theta}{\cos\frac{\Theta}{2}})\cdot|CB| \\ &\leq (1 + 2\pi(k\cos\frac{\pi}{k})^{-1})|CB|. \end{aligned}$$

The last inequality follows from $\Theta \leq 2\pi/k$ and $k > 2$.

(*ii*) If M_iM_j was an edge in G then, for every p between i and j, the circle $\bigcirc M_iM_pM_j$ would not contain C. This, however, contradicts part (a) of Proposition 2.5.

(*iii*) If the outward path contains a single intermediate point M_1, then since M_1 lies inside $(O) = \bigcirc CAB$, $\angle AM_1B \geq \pi - \angle AOB/2 \geq \pi - 2\pi/k = (k-2)\pi/k$ (note that $\angle AOB = 2\cdot\angle ACB$), as desired. Now the statement follows by induction on the number of steps taken to construct the outward path between A and B, using the fact (proved in [10]) that each angle $\angle M_{i-1}O_iM_{i+1}$ at the center of the circle (O_i) defining the intermediate point M_i, is bounded by $\angle AOB$.

(*iv*) This follows from the fact that $\angle CAM_1 \geq \angle CAB \geq \pi/2 - \pi/k$. The last inequality is true because $|CA| \leq |CB|$ and $\angle BCA \leq 2\pi/k$ in $\triangle CAB$.

∎

2.2. The Inward Path

We consider now the case when the interior of $\triangle ABC$ contains points of G. Let S be the set of points consisting of points A and B plus all the points interior to $\triangle ABC$ (note that $C \notin S$). Let $CH(S)$ be the points on the convex hull of S. Then $CH(S)$ consists of points $N_0 = A$ and $N_s = B$, and points $N_1, ..., N_{s-1}$ of G interior to $\triangle ABC$. We have the following proposition:

Proposition 2.6. *For every* $i = 1, \cdots, s-1$:
(a) $CN_i \in G$,
(b) $|CN_i| \leq |CN_{i+1}|$, *and*
(c) $\angle N_{i-1}N_iN_{i+1} \geq \pi$, *where* $\angle N_{i-1}N_iN_{i+1}$ *is the angle facing point* C.

Proof. These follow follow from the following facts: CA and CB are edges in G, CA is the shortest edge in its cone, and hence $|CA| \leq |CN_i|$, for $i = 0, \cdots, s$, and points $N_0, \cdots, N_s$ are on $CH(S)$ in the listed order. ∎

Since $|CN_i| \leq |CN_{i+1}|$ and no point of G lies inside $\triangle N_iCN_{i+1}$ (N_i and N_{i+1} are on $CH(S)$), CN_i is the shortest edge in the angular sector $\angle N_iCN_{i+1}$. Since $\angle N_iCN_{i+1} \leq \angle BCA \leq 2\pi/k$, by Lemma 2.2 there exists an outward path P_i between N_i and N_{i+1}, for every $i = 0, 1, \cdots, s-1$, satisfying all the properties of Lemma 2.2. Let $A = M_0, M_1, \cdots, M_r = B$ be the concatenation of the paths P_i, for $i = 0, \cdots, r-1$.

Definition 2.7. We call the path $A = M_0, M_1, \cdots, M_r = B$ constructed above the *inward path* between A and B.

Figure 2 illustrates an inward path between A and B.

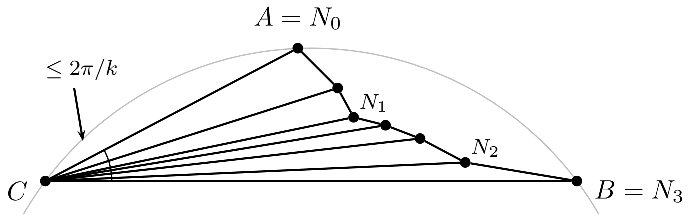

Figure 2: Illustration of an inward path.

We now prove Lemma 2.2 in the case when there are points of G interior to $\triangle ABC$. In this case we define the path in Lemma 2.2 to be the inward path between A and B.

Proof of Lemma 2.2 for the case of inward path.
(i) Define A'' to be a point on the half-line $[CA$ such that $|CA''| = |CB|$, and let $(O'') = \bigcirc CA''B$. Denote by α'' the length of the arc of $\bigcirc CA''B$ subtended by chord $A''B$ and facing $\angle A''CB$. For every $i = 0, 1, \cdots, s-1$, we define arc α_i to be the arc of $\bigcirc CN_iN_{i+1}$ subtended by chord N_iN_{i+1} and facing $\angle N_iCN_{i+1}$. For every $i = 0, 1, ..., s-1$, we define N'_i to be the point on the half-line $[CN_i$ such that $|CN'_i| = |CN_{i+1}|$, (O_i) to be the circle $\bigcirc CN'_iN_{i+1}$, and α'_i to be the arc of (O_i) subtended by chord N'_iN_{i+1} and facing $\angle N'_iCN_{i+1}$. Finally, for every $i = 0, \cdots, s-1$, we define N''_i to be the point of intersection of the half-line $[CN_i$ and circle (O''), and α''_i to be the arc of (O'') subtended by chord $N''_iN''_{i+1}$ and facing $\angle N''_iCN''_{i+1}$. As shown in section 2.1, the length of the outward path P_i between N_i and N_{i+1} is bounded by the length of α_i. Since the convex body C_1 delimited by CN_i, CN_{i+1} and α_i is contained inside the convex body C_2 delimited by CN'_i, CN_{i+1} and α'_i,

by [1], the perimeter of C_1 is not larger than that of C_2. Denoting by $|P_i|$ the length of path P_i, we get:

$$|P_i| \leq |N_i N_i'| + \alpha_i', \quad i = 1, \cdots, s-1. \tag{2.1}$$

Since (O_i) and (O'') are concentric circles (of center C), and the radius of (O_i) is not larger than that of (O''), we have $\alpha_i' \leq \alpha_i''$, for $i = 0, \cdots, s-1$. It follows from Inequality (2.1) that:

$$|P_i| \leq |N_i N_i'| + \alpha_i'', \quad i = 1, \cdots, s-1. \tag{2.2}$$

Using Inequalities (2.1) and (2.2) we get:

$$|CA| + \sum_{i=0}^{s-1} |P_i| \leq |CA| + \sum_{i=0}^{s-1} |N_i N_i'| + \sum_{i=0}^{s-1} \alpha_i''. \tag{2.3}$$

Noting that $\sum_{i=0}^{s-1} |N_i N_i'| = |CB| - |CA|$, that $\sum_{i=0}^{r-1} \alpha_i'' = \alpha''$, and using the same argument as in part (i) of Lemma 2.2) completes the proof.

(ii) Since $CN_p \in G$ for $p = 1, \cdots, s-1$ by part (a) of Proposition 2.6, by planarity of G, if such an edge between two points M_i and M_j exists, then M_i and M_j must belong to an outward path between two points N_p and N_{p+1} of $CH(S)$. But this contradicts part (ii) of Lemma 2.2 for the case of the outward path applied to N_p and N_{p+1}.

(iii) For each $i = 0, \cdots, r$, either $M_i = N_j \in CH(S)$, or M_i is an intermediate point on the outward path between two points N_p and N_q in $CH(S)$. In the former case $\angle M_{i-1} M_i M_{i+1} \geq \angle N_{j-1} M_i N_{j+1} \geq \pi \geq (k-2)\pi/k$ for $k \geq 14$ (N_{j-1} and N_j are points before and after $M_i = N_j$ on $CH(S)$), by part (c) of Proposition 2.6. In the latter case $\angle M_{i-1} M_i M_{i+1} \geq (k-2)\pi/k$ by the proof of part (iii) of Lemma 2.2 applied to the outward path between N_p and N_q.

(iv) This follows from $|CA| = |CM_0| \leq |CM_1|$ and $\angle ACM_1 \leq \angle ACB \leq 2\pi/k$, in triangle $\triangle CAM_1$.

■

2.3. The Modified Yao Step

We now augment the *Yao step* so edges forming the paths described in Lemma 2.2 are included in G', in addition to the edges chosen in the standard Yao step. Lemma 2.2 says that consecutive edges on such paths form moderately large angles. The modified Yao step will ensure that consecutive edges forming large angles are included in G'. The algorithm is described in Figure 3.

Since the algorithm selects at most k edges incident on any point M and since only edges chosen by both endpoints are included in G', each point has degree at most k in G'.

Before we complete the proof of Theorem 2.1, we show that the running time of the algorithm is linear. Note first that all edges incident on point M of degree Δ can be mapped to the k cones around M in linear time in Δ. Then, the shortest edge in every cone can be found in time $O(\Delta)$ (step 2. in the algorithm). Since k is a constant, selecting the $\ell/2$ edges clockwise (or counterclockwise) from a sequence of $ell < k$ empty cones around M (step 3.1.) can be done in $O(\Delta)$ time. Noting that the total number of edges in G is linear in the number of vertices completes the analysis.

To complete the proof of Theorem 2.1, all we need to do is show:

Algorithm **Modified Yao step**

INPUT: *A Delaunay graph G; integer $k \geq 14$*

OUTPUT: *A subgraph G' of G of maximum degree k*

1. define k disjoint cones of size $2\pi/k$ around every point M in G;
2. in every non-empty cone, select the shortest edge incident on M in this cone;
3. **for** every maximal sequence of $\ell \geq 1$ consecutive empty cones:
   3.1. **if** $\ell > 1$ **then** select the first $\lfloor \ell/2 \rfloor$ unselected incident edges on M clockwise from the sequence of empty cones and the first $\lceil \ell/2 \rceil$ unselected edges incident on M counterclockwise from the sequence of empty cones;
   3.2. **else** (i.e., $\ell = 1$) let MX and MY be the incident edges on M clockwise and counterclockwise, respectively, from the empty cone; **if** either MX or MY is selected **then** select the other edge (in case it has not been selected); **otherwise** select the shorter edge between MX and MY breaking ties arbitrarily;
4. G' is the spanning subgraph of G consisting of edges selected by both endpoints.

Figure 3: The modified Yao Step.

Lemma 2.8. *If edge CB is not selected by the algorithm, let CA be the shortest edge in the cone out of C to which CB belongs. Then the edges of the path described in Lemma 2.2 are included in G' by the algorithm.*

Proof. For brevity, instead of saying that the algorithm **Modified Yao Step** selects an edge MX out of a point M, we will say that M selects edge MX. To get started, it is obvious that C will select edge CA.

By part *(iv)* of Lemma 2.2, the angle $\angle CAM_1 \geq \pi/2 - \pi/k \geq 6\pi/k$ for $k \geq 14$. Therefore, at least two empty cones must fall within the sector $\angle CAM_1$ determined by the two consecutive edges CA and AM_1, and edges AC and AM_1 will both be selected by A. Since edge CA is also selected by point C, edge $AC \in G'$.

By part *(iii)* of Lemma 2.2, for every $i = 1, 2, \cdots, r-1$, the angle $\angle M_{i-1}M_iM_{i+1} \geq (k-2)\pi/k \geq 10\pi/k$ for $k \geq 12$, and hence at least four cones fall within the angular sector $\angle M_{i-1}M_iM_{i+1}$. Since by part *(ii)* of Lemma 2.2 M_iC is the only possible edge inside the angular sector $\angle M_{i-1}M_iM_{i+1}$, it is easy to see that regardless of the position of these four cones with respect to edge M_iC, M_i ends up selecting all edges M_iM_{i-1}, M_iM_{i+1} and M_iC in steps 2 and/or 3 of the algorithm. Since we showed above that A selects edge AM_1, this shows that all edges M_iM_{i+1}, for $i = 0, \cdots, r-2$, are selected by both their endpoints, and hence must be in G'. Moreover, edge $M_{r-1}M_r = M_{r-1}B$ is selected by point M_{r-1}.

We now argue that edge BM_{r-1} will be selected by B. First, observe that $|BM_{r-1}| \leq |\overset{\frown}{AB}| < |CB|$. Let CD be the other consecutive edge to CB in G (other than CM_{r-1}). Because C does not select B, it follows that $\angle M_{r-1}CD \leq 6\pi/k$. Otherwise, since CM_{r-1} and CB are in the same cone, two empty cones would fall within the sector $\angle BCD$ and C would select B. Since CB is an edge in G, by the characterization of Delaunay edges [8], $\angle CM_{r-1}B + \angle CDB \leq \pi$. By considering the quadrilateral $CDBM_{r-1}$, we have $\angle M_{r-1}CD + \angle DBM_{r-1} \geq \pi$. This, together with the fact that $\angle M_{r-1}CD \leq 6\pi/k$, imply that $\angle DBM_{r-1} \geq (k-6)\pi/k \geq 8\pi/k$, for $k \geq 14$. Therefore, $\angle DBM_{r-1}$ contains at least three cones of size $2\pi/k$ out of B. If one of these cones falls within the angular sector $\angle CBM_{r-1}$ then, since $|M_{r-1}B| < |CB|$, BM_{r-1} must have been selected out of B.

Suppose now that $\angle CBM_{r-1}$ contains no cone inside and hence $\angle CBM_{r-1} < 4\pi/k$. If one of these three cones within sector $\angle DBM_{r-1}$ contains edge CB, then the remaining two cones must fall within $\angle DBC$ and BM_{r-1} will get selected out of B when considering the sequence of at least two empty cones contained within $\angle CBD$. Suppose now that all three empty cones fall within $\angle CBD$. Then we have $\angle CBD \geq 6\pi/k$.

If $\angle M_{r-1}CD \geq 4\pi/k$, then since $M_{r-1}C$ and CB belong to the same cone, the sector $\angle BCD$ must contain an empty cone. Because D is exterior to $\bigcirc CBM_{r-1}$, $\angle CBM_{r-1} < 4\pi/k$, and $\angle M_{r-1}CB \leq 2\pi/k$, it follows that $\angle CDB < \angle M_{r-1}CB + \angle CBM_{r-1} < 6\pi/k < \angle DBC$. Therefore, by considering the triangle $\triangle CDB$, we note that $|CB| < |CD|$. But then edge CB would have been selected by C in step 3 since the sector $\angle BCD$ contains an empty cone, a contradiction.

It follows that $\angle M_{r-1}CD \leq 4\pi/k$, and therefore $\angle M_{r-1}BD \geq (k-4)\pi/k \geq 10\pi/k$ for $k \geq 14$. This means that at least four cones are contained inside sector $\angle DBM_{r-1}$. It is easy to check now that regardless of the placement of the edge BC with respect to these cones, edge BM_{r-1} is always selected out of B by the algorithm. This completes the proof. ■

Corollary 2.9. *A Euclidean Minimum Spanning Tree (EMST) on P is a subgraph of G'.*

Proof. It is well known that a Delaunay graph (G) contains a EMST. If an edge CB is not in G', then, by Lemma 2.8, a path from C to B is included in G'. All edges on this path are no longer than CB, so there is a EMST not including CB. ■

Since a Delaunay graph of a Euclidean graph of n points can be computed in time $O(n \lg n)$ [8] and has stretch factor $C_{del} \approx 2.42$, we have the following theorem.

Theorem 2.10. *There exists an algorithm that, given a set P of n points in the plane, computes a plane geometric spanner of the Euclidean graph on P that contains a EMST, has maximum degree k, and has stretch factor $(1 + 2\pi(k \cos \frac{\pi}{k})^{-1}) \cdot C_{del}$, where $k \geq 14$ is an integer. Moreover, the algorithm runs in time $O(n \lg n)$.*

3. Geometric Spanners of Unit Disk Graphs

In this section we generalize our planar geometric spanner algorithm to unit disk graphs. Unit disk graphs model wireless ad-hoc and sensor networks and, for packet routing and other applications, a bounded-degree planar geometric spanner of the wireless network is often desired. Due to the limited computational power of the network devices and the requirement that the network be robust with respect to device joining and leaving the network, the construction/algorithm should ideally be *strictly-localized*: the computation performed at a point depends solely on the information available at the point and its d-hop neighbors, for some constant d (in our case $d = 2$). In particular, no global propagation of information should take place in the network.

The results in the previous section do not carry over to unit disk graphs because not all Delaunay graph edges on a point-set P are unit disk edges. However, if U is the unit disk graph on points in P and $UDel(U)$ is the subgraph of the Delaunay graph on P obtained by deleting edges of length greater than one unit, then $UDel(U)$ is a connected, planar, spanning subgraph of U with stretch factor bounded by C_{del} (see [11, 4]). Therefore, if we apply the results from the previous section to $UDel(U)$ and observe that all edges on the path defined in Lemma 2.2 must be unit disk edges (given that edges CA and CB are), it is easy to see that Theorem 2.1 and Theorem 2.10 carry over to unit disk graphs. The only problem, however, is that the construction of $UDel(U)$ cannot be done in a strictly-localized manner.

To solve this problem, Wang et al. [11, 12] introduced a subgraph of U denoted $LDel^{(2)}(U)$. It was shown in [11, 12] that $LDel^{(2)}(U)$ is a planar supergraph of $UDel(U)$, and hence also has stretch factor bounded by C_{del}. Moreover, the results in [6, 15] show how $LDel^{(2)}(U)$ can be computed with a strictly-localized distributed algorithm exchanging no more than $O(n)$ messages in total (n is the number of points in U), and having a local processing time of $O(\Delta \lg \Delta) = O(n \lg n)$ at a point of degree Δ. In a style similar to Definition 1.2, $LDel^{(2)}(U)$ can be defined as follows:

Definition 3.1. An edge XY of U is in $LDel^{(2)}(U)$ if and only if there exists a circle through points X and Y whose interior contains no point of U that is a 2-hop neighbor of X or Y.

We will use $G = LDel^{(2)}(U)$ as the underlying subgraph of U to replace the Delaunay graph G used in the previous section. We note that G is planar, is a supergraph of $UDel(U)$, and hence has stretch factor C_{del}. To translate our results to unit disk graphs, we need to show that the inward and outward paths are still well defined in G. In particular, we need to show that Lemma 2.2 holds for $G = LDel^{(2)}(U)$. We outline the general approach and omit the details for lack of space.

The following is equivalent to Proposition 2.3:

Lemma 3.2. *If CA and CB are edges of G then the region of $(O) = \bigcirc ABC$ subtended by chord CA and away from B and the region of (O) subtended by chord CB and away from A contain no points that are two hop neighbors of A, B and C.*

Proof. By symmetry it is enough to prove the lemma for the region of (O) subtended by chord CA and away from B. By Definition 3.1, there is a circle (O_{CA}) passing through C and A whose interior is empty of any point within two hops of C or A. The region of (O) subtended by chord CA and away from B is inside this circle, so we only need to argue that it doesn't contain two hop neighbors of B either. If it did, say point X, then any neighbor of X and B would have to be a neighbor of C or A as well, a contradiction. ∎

With this lemma in hand, the recursive construction of the outward path given in Subsection 2.1 can be applied to the graph $G = LDel^{(2)}(U)$. The following proposition for $G = LDel^{(2)}(U)$ corresponds to Proposition 2.5 for Delaunay graphs and is proven in an equivalent manner:

Proposition 3.3. *In every recursive step of the outward path construction, if M_p is an intermediate point with respect to a pair of points (M_i, M_j), then:*

(a) *there is a circle passing through C and M_p that contains no point of G that is a two-hop neighbor of C or M_p, and*
(b) *circles $\bigcirc CM_iM_p$ and $\bigcirc CM_jM_p$ contain no points of G that are two-hop neighbors of C, M_i and M_p and C, M_j, and M_p, respectively, except, possibly, in the region subtended by chords M_iM_p and M_pM_j, respectively, away from C.*

With this proposition, we can show that Lemma 2.2 holds true for $G = LDel^{(2)}(U)$ for outward paths. It holds for inward paths as well, using the same argument as in Section 2.2.

Finally, it is obvious how the **Modified Yao Step** algorithm in Section 2.3 can be easily described as a strictly-localized algorithm. We can show, therefore, the following theorem:

Theorem 3.4. *There exists a distributed strictly-localized algorithm that, given a set P of n points in the plane, computes a plane geometric spanner of the unit disk graph on P that contains a EMST, has maximum degree k, and has stretch factor $(1 + 2\pi(k \cos \frac{\pi}{k})^{-1}) \cdot C_{del}$, for any integer $k \geq 14$. Moreover, the algorithm exchanges no more than $O(n)$ messages in total, and has a local processing time of $\Delta \lg \Delta$ at a point of degree Δ.*

Due to the strictly-localized nature of the algorithm, the algorithm is very robust to topological changes (such as wireless devices moving or joining or leaving the network), an essential property for the application of the algorithm in a wireless ad-hoc environment.

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