Some Recent Results on Pie Cutting

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For cake cutting, cuts are parallel to an axis or segment and yield rectangular pieces. As such, cutting a cake is viewed as dividing a line segment. For pie cutting, cuts are radial from the center of a disc to the circumference and yield sectors or wedge-shaped pieces. As such, cutting a pie is viewed as dividing a circle. This difference in topology is apparent in the minimal number of cuts necessary to divide the desserts. For n players, it takes a minimal number of $n-1$ cuts to divide a linear cake and n cuts to divide a circular pie so that each player receives a single piece. Though, there is clearly a relationship between cutting a cake and cutting a pie. Once a circular pie has a single cut, then it can be straightened out into a segment, looking like a cake. Isn't a cake just a pie that has been cut? Gale (1993) suggested that this topology was a significant difference. This note is to summarize and compare some of the recent results on pie cutting that appear in Barbanel and Brams (2007) and Brams, Jones, and Klamler (2007). In particular, the geometric structure presented in Barbanel and Brams (2007) is used to prove and to explain results in Brams, Jones, and Klamler (2007).

Pie cutting or dividing a circle is more appropriate for some applications. For example, dividing a shoreline between more than two vendors would result in disconnected shorelines for a vendor if a cake-cutting procedure were to be used (see Figure 1). The circular structure is also useful when staffing employees (*e.g.*, emergency room physicians, police, *etc.*) or allocating machine time (*e.g.*, production plants, computer time, *etc.*) over 24-hour periods.

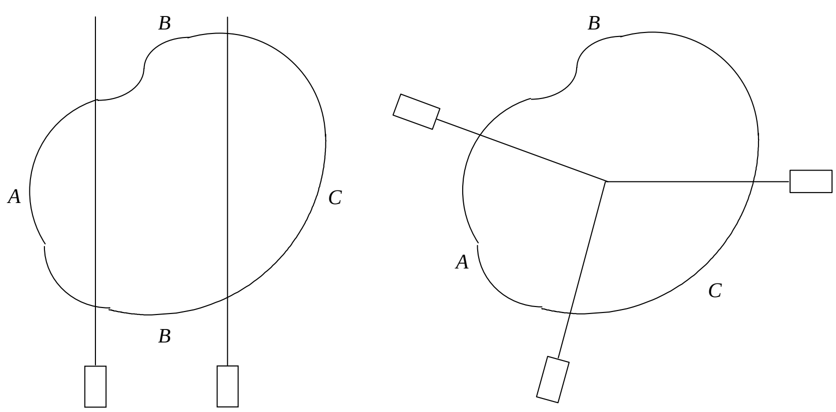


Figure 1: Dividing a shoreline with a cake-cutting and a pie-cutting procedure.

Barbanel and Brams (2007), Brams, Jones, and Klamler (2007), and Thomson (2007) are concerned with properties of minimal cut divisions of pies. An allocation of pie into n wedge-shaped sectors among n players is

• *envy-free* if no player prefers another sector to her own,

• *undominated* if every other possible allocation of sectors either gives the same values or else decreases the value assigned to at least one player, and

• *equitable* if all players value the sectors they receive equally.

Gale (1993) asked whether there always exists an envy-free allocation of pie for n players that is undominated using the minimal number of n cuts. For 2 players, Barbanel and Brams (2007) prove the existence of an envy-free, undominated and equitable allocation and provide a procedure to yield an envy-free and undominated allocation. Thomson (2007) also proves existence, providing a different geometric framework than Barbanel and Brams to study pie cutting. Brams, Jones, and Klamler (2007) extend Gale's question to unequal entitlements by extending envy-freeness to unequal entitlements. They provide a computational procedure to yield such a proportional allocation, requiring players to know their value functions or measures. Both the procedures in Barbanel and Brams (2007) and Brams, Jones, and Klamler (2007) are reviewed below. For 3 or more players, Barbanel and Brams (2007) provide an envy-free procedure for 3 players whereas Brams, Jones, and Klamler (2007) provide counterexamples for the extended version of Gale's question for 3 players.

For pie cutting, players' measures are assumed to be finitely additive probability measures over unit disk. This assures that the sum of values of the parts is the value of the whole and subpieces are of no greater value than larger pieces that contain them. Further, measures are assumed to be non-atomic when projected onto the normalized circumference $[0, 1]$. Hence, a radial (border) cut has no value. Finally, measures are assumed to be *absolutely continuous* with respect to each other (so that whenever a piece of pie has positive measure for one, it has positive measure for all). Let player i's measure be given by the value function v_i.

Sectors of pie are denoted by the radial cuts that intersect the unit interval circumference. Hence, a sector $[\alpha, \beta]$ with $\alpha < \beta$, is the counterclockwise sector between α and β. If $\alpha > \beta$, define $[\alpha, \beta] = [\alpha, 1] \cup [0, \beta]$. It follows that $[\alpha, \beta] \cup [\beta, \alpha] = [0, 1]$ so that $[\alpha, \beta]$ and $[\beta, \alpha]$ are complementary pieces. Denote player n's value of $[\alpha, \beta]$ by $v_n([\alpha, \beta])$. Fro 2 players, refer to the players as A and B.

1 Pie Cutting for Two Players

Barbanel and Brams (2007) consider the geometry of 2-player valuations:

$$V = \{(v_A([\alpha, \beta]), v_B([\beta, \alpha])) \mid \text{for all } (\alpha, \beta) \in [0, 1] \times [0, 1]\}.$$

They show that V is closed, symmetric with respect to $(0.5, 0.5)$, and satisfies types of path connectedness and convexity (Figure 2). The implication is that the line $v_A = v_B$ intersects a point on the efficient frontier of V. Further, because of the geometric properties of V, this point indicates the existence of an envy-free, undominated and equitable allocation of pie into 2 pieces.

Barbanel and Brams (2007) also provide a procedure to yield an envy-free and undominated allocation of pie between 2 players. This moving knife procedure can be viewed by

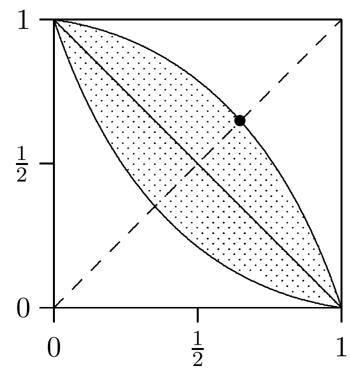


Figure 2: The point • in the shaded V demonstrates the existence of an envy-free, undominated, and equitable allocation of pie between 2 players.

their geometric structure. Under their procedure, one player rotates two radial knives counterclockwise around the circumference keeping half the value of the pie between the knives. After a complete rotation, the other player determines the position to maximize her piece. Geometrically, as player A rotates the knife, the value achieved by the players for the resulting allocation moves along the vertical line in Figure 3. Player B selects her maximum piece, represented by • in Figure 3. This answers Gale's question for 2 players. Thomson (2007) also answers Gale's question for 2 players and provides a different geometric framework.

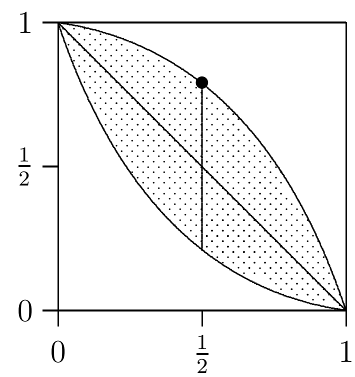


Figure 3: The point • in the shaded V is the 2-player valuation that results from the Barbanel and Brams (2007) moving-knife procedure.

Eliminating the requirement for the measures to be absolutely continuous, Barbanel and Brams (2007) show that there may not exist an envy-free and undominated allocation. Thomson (2007) considers different sets of assumptions on the players' measures, as well as additional properties of fairness.

Brams, Jones, and Klamler (2007) extend the problem by considering allocations of pie in which n players have *unequal entitlements*, represented by the vector $(p_1, \ldots, p_n)$ such that $p_i > 0$ and $\sum_{i=1}^{n} p_i = 1$. Each player i is entitled to p_i of the pie. Robertson and Webb (1998) consider unequal entitlements of cake, as well as the unequal division of discrete, non-divisible items. Denote an allocation by $S = (S_1, \ldots, S_n)$ where S_i is player i's piece. An allocation S is *proportional* if $\frac{v_i(S_i)}{p_i} = \frac{v_j(S_j)}{p_j}$ for all i and j. Envy-freeness can be extended for unequal entitlements: S is *envy-free* if $\frac{v_i(S_i)}{p_i} \geq \frac{v_i(S_j)}{p_j}$ for all i and j. In words, this assures that no player thinks another player received a disproportionally large piece, based on the latter player's entitlement. A player i finds an allocation is *acceptable* if $v_i(S_i) \geq p_i$. For $n = 2$ players, an acceptable allocation is envy-free. An allocation S is *dominated* by T if $v_i(T_i) \geq v_i(S_i)$ for all i and $v_j(T_j) > v_j(S_j)$ for some j. For 2 players, an allocation S that is proportional and maximizes the value that one player receives, *i.e.*, $v_1(S_1)$, is undominated.

For two players, Brams, Jones, and Klamler (2007) prove that there exists an acceptable allocation of pie in which each player receives a piece valued at exactly his or her entitlement, using a minimal number of 2 cuts (Theorem 1).

Theorem 1. Brams, Jones, and Klamler (2007) *For any $p \in [0, 1]$, the pie can be cut with two radial cuts such that A values one piece at p and B values the other at $1 - p$, according to their respective measures.*

The proof is analytic with a geometric interpretation, requiring some terminology from real analysis. A continuous function $f : I \to I$ has a *horizontal chord* of length p if there exists an x and $x + p$ in I such that $f(x) = f(x + p)$ (see Figure 4). In general, a continuous function on $[0, 1]$ with $f(0) = f(1)$ does not have a chord of length p for all $p \in [0, 1]$. The existence of an allocation satisfying Theorem 1 is related to horizontal chords of every length $p \in [0, 1]$, requiring the construction of a continuous function over $[0, 2]$, instead of $[0, 1]$. See Boas (1960) for additional information on horizontal chords.

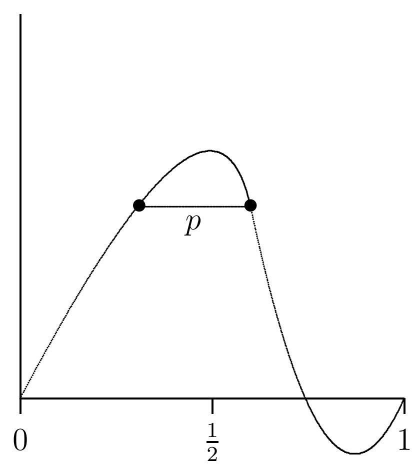


Figure 4: A continuous function with a horizontal chord of length p.

Let v_A and v_B be the value functions for players A and B, respectively. Define $x(t)$ by $v_A([0, x(t)] = t$ and let $g(t) = v_B([0, x(t)])$. Further, define $f(t) = g(t) - t$. Then, f is a continuous function on $[0, 1]$ with $f(0) = f(1)$. But, f can be extended to the interval $[0, 2]$ by defining $f(t) = f(t - 1)$ for $t \in [1, 2]$ (see Figure 5). The endpoints of a chord of length p will be the cutpoints on the radius so that the allocation is of value p to player A and of value $1 - p$ to player B.

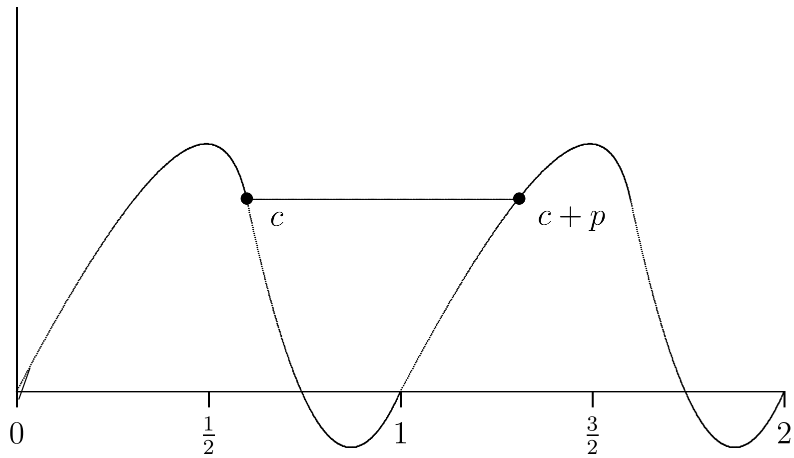


Figure 5: Extending a function onto $[0, 2]$ so that all horizontal chords of length $p \in [0, 1]$ exist.

Lemma 1. Brams, Jones, and Klamler (2007) *A continuous function* $f : [0, 2] \to \Re$ *such that* $f(t+1) = f(t)$ *for all* $t \in [0, 1]$ *has horizontal chords of all lengths* $p \leq 1$*. For* $p \in [0, 1]$*, there exists a* $c \in [0, 1)$ *such that* $f(c + p) - f(c) = 0$.

The existence of a chord of length p ensures that the pie can be cut at $x(c)$ and $x(c+p)$ so that $v_A([x(c), x(c+p)]) = v_B([x(c), x(c+p)] = p$. The result of Theorem 1 can be viewed from the geometric perspective of Barbanel and Brams (2007), as explained below.

Of course, Theorem 1 makes no claim about whether or not the resulting allocation is undominated. Brams, Jones, and Klamler (2007) re-state Gale's question to unequal entitlements: Under unequal entitlements, does there exist an envy-free and undominated allocation of pie using a minimal number of cuts?

Lemma 1 demonstrates that there exists an allocation of pie so that each player receives a piece valued at his or her entitlement with no surplus left over. Using a variation of the moving-knife procedure from Barbenal and Brams (2007), a player can rotate two knives keeping p between the two knives. If player B selects the piece that he values the most, then he will value it at least as much as $1-p$. In general, he will value the maximum valued piece more than $1-p$. Hence, it is possible for player A to receive a piece that she values at p and for player B to receive a piece that he values at $1 - p$, but with a surplus piece, as pictured in Figure 6. (Players A and B can be required to have their pieces share a border.)

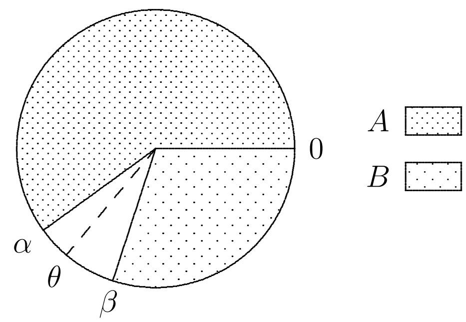


Figure 6: Player A receives $[0, \alpha]$ valued at p while player B receives $[\beta, 1]$ valued at $1-p$. The surplus (white space) can be divided in proportion to the players' entitlements by cutting at some θ between α and β.

Theorem 2. Brams, Jones, and Klamler (2007) *For two players with entitlements p and $1-p$, there exists an envy-free, efficient, and proportional allocation using 2 radial cuts such that each player receives at least its entitled share, given the players' measures are absolutely continuous.*

The proof of Theorem 2 in Brams, Jones, and Klamler (2007) uses the insight from Figure 6; and, this insight can be viewed as a procedure. However, it can also be proved from the geometric framework of Barbanel and Brams (2007). Recall that $V = \{(v_A([\alpha, \beta]), v_B([\beta, \alpha])) \mid \text{for all } (\alpha, \beta) \in [0,1] \times [0,1]\}$. Then, the line through the origin of slope $\dfrac{p_B}{p_A}$ intersects the ordered pair of values for the allocations from Theorems 1 and 2. See Figure 7, for the example where player A is entitled to 0.6 and player B is entitled to 0.4. This provides a positive solution to Gale's question for unequal entitlements for two players.

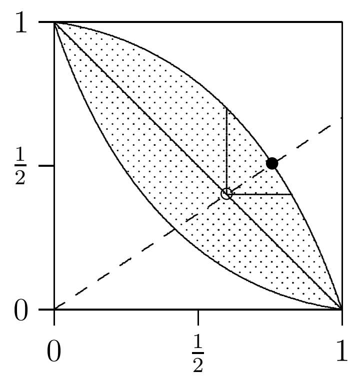


Figure 7: For $(p_1, p_2) = (0.6, 0.4)$, Theorem 1 yields $\circ$; Theorem 2 yields $\bullet$.

Brams, Jones, and Klamler (2007) provide a procedure to yield an acceptable, proportional allocation for 2 players with rational entitlements. However, as in Figure 6, the allocation may not be efficient, resulting in a surplus or left over piece. The procedure requires multiple moving knives, according to the denominator of the reduced fractional entitlements. Figure 8 accompanies the description of the procedure.

Two-Player Procedure for Entitlements $\left(\frac{k}{n}, \frac{n-k}{n}\right)$ (Brams, Jones, and Klamler, 2007)
1. Select a point on the circumference of the pie at random. Denote the radius from the center of the pie to this point as 0 radians (angle 0).
2. Player A, unobserved by player B, marks $n-1$ additional angles that, together with angle 0, divide the pie into n sectors (dashed lines in figure).
3. Player B places one knife along the radius at angle 0 and places $n-1$ knives from the center of the pie to the circumference at $n-1$ angles that, together with angle 0, divide the pie into n sectors (solid lines in figure).
4. Player B rotates the n knives counterclockwise in such a way that the knives continue to define n sectors.

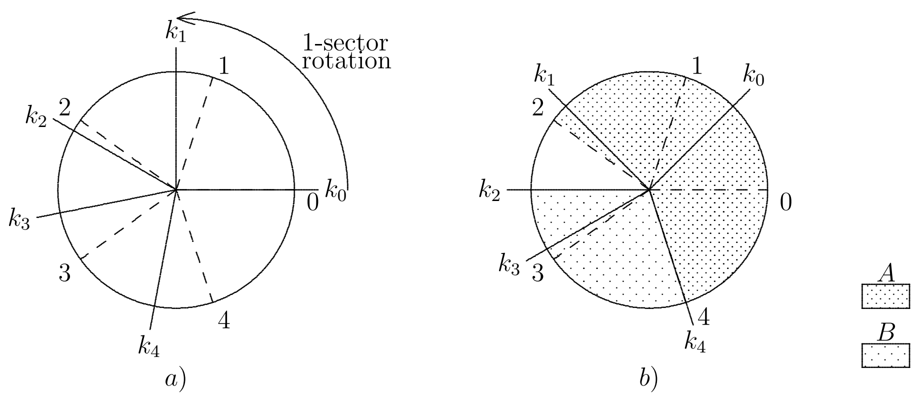


Figure 8: Players A and B use the proportional pie-cutting procedure for rational-number entitlements to divide the pie in a ratio of 3 : 2. b) Player B rotates its knives until player A stops the rotation when knife k_4 coincides with A's angle 4. Player A receives the 3 consecutive sectors counterclockwise from knife k_4, and player B receives the pie between knives k_2 and k_4.

5. Player A stops the rotation when one of player B's knives is coincident with one of player A's n angles (k_4 and 4 in figure), and there are k consecutive sectors in the counterclockwise direction from this knife, according to A's angles, that do not intersect $n-k$ consecutive sectors in the clockwise direction from this knife, according to B's knives.
6. Player A reveals its angles. The pie is cut in three places: the two radii defining the boundary of A's k consecutive sectors; and at the knife that, together with the knife coincident with A's angle, forms the boundary of the $n-k$ consecutive sectors, according to B's knives (see figure).
7. If A does not call stop before player B's knives traverse one sector, then neither player receives any of the pie.

Brams, Jones, and Klamler (2007) prove that risk-averse players will implement the above procedure truthfully. They also view Theorem 2 as a computational procedure in which players submit their measures. If players are able to articulate their measures, then risk-averse players will submit them truthfully to a referee to arrive at the envy-free, efficient, and proportional outcome from Theorem 2. The procedure can be viewed as a referee rotating two knives, keeping the proportion of the values to players A and B to be the proportion of the entitlements. If the submitted measures were truthful, then the players will agree that the referee should stop when either player's piece is maximized. (See Figure 9).

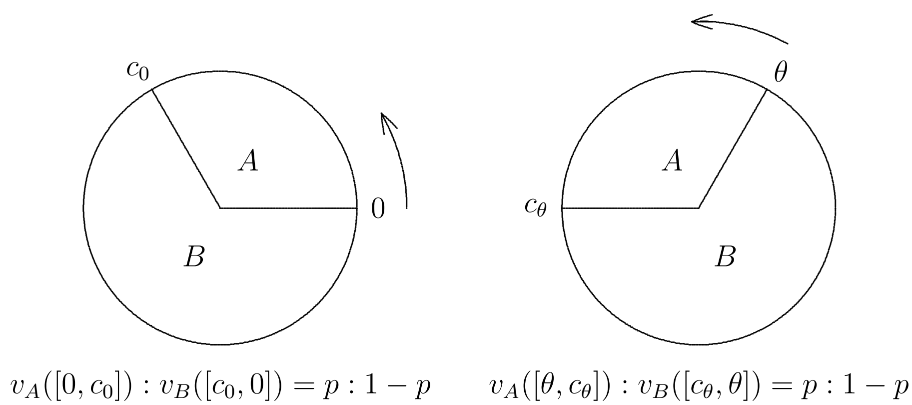


Figure 9: A referee rotates θ counterclockwise from 0 to 2π, keeping $v_A([\theta, c_\theta]) : v_B([c_\theta, \theta]) = p : 1 - p$.

2 Three or More Players

Barbanel and Brams (2007) provide an envy-free procedure for 3 players (with equal entitlements). They assume that three players, denoted by A, B, and C, have absolutely continuous measures.

Three-Player Envy-Free Procedure (Barbanel and Brams, 2007)
1. A rotates 3 radial knives continously around the pie maintaining what she believes is $\frac{1}{3} : \frac{1}{3} : \frac{1}{3}$ portions.
2. B calls "stop" when he thinks two of the pieces are tied for largest, which must occur for at least one set of positions in the rotation.
3. The players then choose pieces in the order C, B, and A.

Step 2 can occur because at some point B will think that two of the pieces are tied for largest. This follows by continuity of the players' measures and the intermediate-value theorem. Under this procedure, truthfulness ensures envy-freeness. In particular, C takes its most-valued piece, B can choose one of the tied-for-most valued pieces, and A values all the pieces the same. Realize that this does not address Gale's question for 3 players, as the above envy-free allocation may be dominated.

Brams, Jones, and Klamler (2007) consider unequal entitlements for $n = 3$ players and provide counterexamples to show that measures and entitlements exist for which there is no 3-player proportional allocation.

Example 1. *Suppose that the entitlements for 3 players are in the proportion* $3 : 1 : 1$. *Assume that the players' measures are uniform over sectors where each sector is equally valued in Figure 10. For A to receive* $\frac{3}{5}$ *of the pie, A must receive 5+ consecutive sectors. However, any 5 consecutive sectors must contain sector 1 or sector 6. Because B values sector 1 at* $\frac{4}{5}$ *and C values sector 6 at* $\frac{4}{5}$*, then B or C cannot receive a piece that it values at* $\frac{1}{5}$. *Hence, it is impossible to have a proportional, acceptable allocation between the three players.*

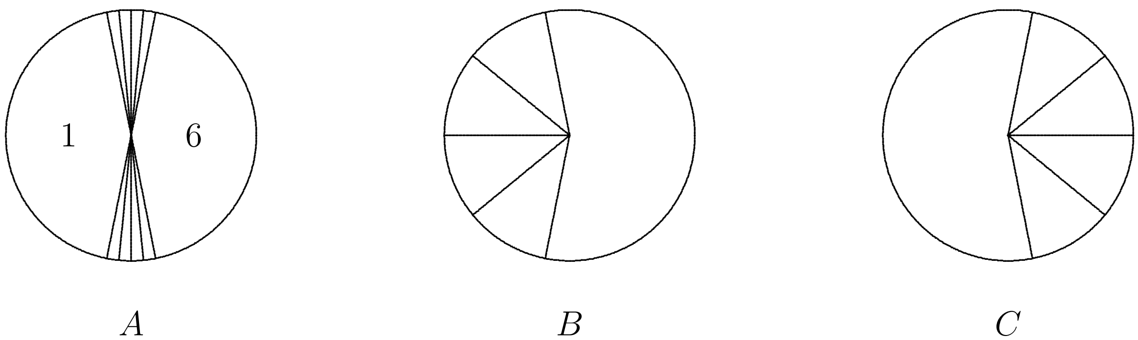
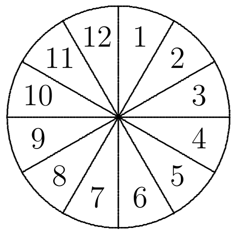


Figure 10: There is no acceptable allocation in the proportion $3 : 1 : 1$.

The above example does not depend on one player receiving more than half the value of the pie. Consider the following example, in which each of the three players is entitled to nearly one-third the value of the pie.

Example 2. *Suppose that the entitlements for 3 players are in the proportion* $\left(\frac{1}{3} + x\right) : \left(\frac{1}{3} + x\right) : \left(\frac{1}{3} - 2x\right)$ *where* $0 < x < \frac{1}{24}$. *Assume that the players' measures are uniform over each sector and valued as in the accompanying table in Figure 11. It follows that each physical* $1/3$ *of the pie is valued at* $1/3$ *by A and B. Hence, A and B together must get (physically) more than* $2/3$ *of the pie. However, C must receive more than* $1/4$ *the value of the pie, because* $\frac{1}{3} - 2x > \frac{1}{4}$ *when* $x < \frac{1}{24}$. *Because C's measure is uniform over the entire pie, this amounts to player C receiving at least* $1/4$ *of the physical pie. If A and B each receive* $1/3$ *(of the physical pie) and C receives* $1/4$ *(of the physical pie), then* $1/12$ *of the pie remains. To change this to an acceptable allocation, A and B must both receive x of the pie according to their respective measure and C must get* $\frac{1}{12} - 2x$*, according to her measure, from the* $1/12$ *physical piece of pie. But there is not a* $1/12$ *physical piece of pie that is valued at more than x by both A and B. To be precise, either A or B must receive a piece valued at less than* $\dfrac{x}{3}$*, according to his or her respective measure.*



	1	2	3	4	5	6	7	8	9	10	11	12
A	$\frac{1}{3}-x$	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{1}{3}-x$	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{1}{3}-x$	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{x}{3}$
B	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{1}{3}-x$	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{1}{3}-x$	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{1}{3}-x$	$\frac{x}{3}$
C	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

Figure 11: There does not exist an acceptable allocation of pie in the proportion $\left(\frac{1}{3}+x\right):\left(\frac{1}{3}+x\right):\left(\frac{1}{3}-2x\right)$ for players A, B, and C.

3 Open Questions about Procedures

Much of the focus on pie cutting has been on existence results. However, there are still a number of open questions about whether or not there exist procedures to yield the existence results. Although Barbanel and Brams (2007) prove the existence of an envy-free, undominated, and equitable allocation of pie for 2 players, does there exist a moving-knife procedure to produce such an allocation? Brams, Jones, and Klamler (2007) prove that there exist an envy-free, undominated, and proportional allocation of pie for 2 players with unequal entitlements. Their procedure is computational in nature and requires the players to be able to articulate their measures. This begs the question of whether there is a 2-player moving-knife procedure to produce an envy-free, undominated, and proportional allocation of pie? Barbanel and Brams (2007) provide an envy-free procedure for 3 players with equal entitlements. Do there exist moving-knife procedures for $n > 3$ that yield envy-free allocations?

References

J.B. Barbanel and S.J. Brams (2007) Cutting a pie is not a piece of cake. Preprint.

Boas, Jr. RP (1960) *A Primer of Real Functions.* Washington, D.C.: Mathematical Association of America.

Gale D (1993) Mathematical entertainments. *Mathematical Intelligencer* 15: 48-52.

Robertson J and Webb W (1998) *Cake-cutting algorithms: Be fair if you can.* Natick, MA: AK Peters.

Thomson W (2007) Children Crying at Birthday Parties. Why? *Economic Theory* 31: 501-521.