Enhanced Contractions and (In)dependence Preliminary report

Alexander Bochman
Computer Science Department,
Holon Institute of Technology, Israel
bochmana@hit.ac.il

October 25, 2007

1 Introduction

The aim of this study consists in exploring some new ways of representing and discerning information embodied in our epistemic states. It is based on a presupposition that epistemic state is much more than just a set of propositions. It is a structured entity in which propositions are organized in a tight net of dependencies and justifications.

In previous publications I have suggested to formalize this notion of an epistemic state as follows.

Definition. An *epistemic state* is a triple $\mathbb{E} = (\mathcal{S}, l, \prec)$,where $\mathcal{S}$ is a set of *belief states*, $\prec$ a preference relation on $\mathcal{S}$, l a valuation function: for every $s \in \mathcal{S}$, $I(s)$ is a deductively closed *belief set* of s.

Definition. A is *believed* in an epistemic state $\mathbb{E} \equiv A$ holds in all preferred states of $\mathbb{E}$.

Traditional models in belief change theory presuppose, however, that an epistemic state is *determinate*, that is, it always has a unique most preferred belief state. But though many epistemic states are determinate, there are also important cases where they are not. Moreover, it has been shown in [Boc01] that the restriction to determinate epistemic states leads to inevitable difficulties in describing the general process of belief change. That is why we are interested in this study in developing new formal means for dealing with non-determinate states.

2 Multiple contraction

The first belief function on epistemic states we will consider generalizes the usual belief contraction to an operation that may have multiple conclusions:

Dagstuhl Seminar Proceedings 07351
Formal Models of Belief Change in Rational Agents
http://drops.dagstuhl.de/opus/volltexte/2007/1204

Definition. $a \dashv b$ is *valid* in an epistemic state if every preferred belief state that does not contain any proposition from a includes at least one proposition from b.

As can be seen from the above definition, if an epistemic state is not determinate, then $A \dashv B, C$ implies that $B \vee C$ will belong to the contracted belief set $\mathbb{E} - A$, *but not vice versa*. Accordingly, multiple contraction is a more expressive operation that allows us to make fine-grained distinctions concerning the relations among propositions in an epistemic state.

It turns out that multiple contraction operation can be axiomatized as follows:

Tautology $a \dashv \mathbf{t}$

Monotony If $a \dashv b$, then $a \dashv b, c$.

Weakening If $B \vDash C$ and $a \dashv b, B$, then $A \dashv b, C$.

And If $a \dashv b, A$ and $a \dashv b, B$, then $a \dashv b, B \wedge C$.

Success If $a, A \dashv b, A$, then $a, A \dashv b$.

Export If $a, A \dashv b$, then $a \dashv A, b$.

Import If $a \dashv A \to a$ and $a \dashv b$, then $a, A \dashv b$, where $A \to a = \{A \to A_i \mid A_i \in a\}$.

3 Choice contraction

Taking seriously the idea that an epistemic state may have multiple preferred belief states, it is only a short step to concluding that propositions holding in at least one preferred belief state function as *default assumptions* in our reasoning. This justifies the following definition.

Definition. A is an *assumption* (an expectation, default) in an epistemic state $\mathbb{E} \equiv A$ holds in at least one preferred state from $\mathbb{E}$.
$\langle\mathbb{E}\rangle$ – the assumption set of $\mathbb{E}$ = the union of all preferred belief sets.

It seems clear that the contraction operation on epistemic states (as defined in [Boc01]) produces also a certain change in its assumptions. Formally, it is defined as follows.

$$\langle\mathbb{E}\rangle - A = \langle\mathbb{E} - A\rangle$$

$$B \in \langle\mathbb{E} - A\rangle \ \equiv \ B \text{ holds in at least one preferred state not containing } A.$$

It turns out that this operation is also axiomatizable. Moreover, its axiomatization is much similar to the axiomatization of ordinary contractions.

Success $A \notin \langle\mathbb{E} - A\rangle$

Weak Closure If $B \models C$ and $B \in \langle \mathbb{E} - A \rangle$, then $C \in \langle \mathbb{E} - A \rangle$.

Extensionality If $\models A \leftrightarrow B$, then $\langle \mathbb{E} - A \rangle = \langle \mathbb{E} - B \rangle$.

Distributivity $\langle \mathbb{E} - A \wedge B \rangle \subseteq \langle \mathbb{E} - A \rangle \cup \langle \mathbb{E} - B \rangle$.

Conjunction If $A \notin \langle \mathbb{E} - B \rangle$, then $\langle \mathbb{E} - A \wedge B \rangle = \langle \mathbb{E} - B \rangle$.

Basically, the distinction between ordinary and choice contractions boils down to weakening the claim that the contracted set should be deductively closed. Instead, we have only Weak Closure for choice contractions.

4 Dependence

Belief contraction functions reflect in some sense the notion of independence of propositions belonging to an epistemic state. Indeed, if B belongs to the belief set resulting from contracting an epistemic state with respect to A, then B can be seen as independent of A. The connection between contraction and (in)dependence of propositions has already been used in [CH96].

It turns out however, that ordinary belief contraction, though adequate for independence, allows us to define only a very weak notion of dependence among propositions. A better notion of dependence can be defined using the above notion of choice contraction.

Dependence relation $B \rhd A \equiv \quad A \not\sim\!\mid B$ - B *strongly depends* on A in the epistemic state.

Definition. $B \rhd A$ is *valid* in $\mathbb{E}$ if B does not hold in any preferred state that does not contain A.

Using the above axiomatization of the choice contraction, we immediately obtain the following axiomatization of strong dependence:

Reflexivity	$A \rhd A$
Left Strengthening	If $B \models C$ and $C \rhd A$, then $B \rhd A$.
Extensionality	If $\models A \leftrightarrow C$ and $B \rhd C$, then $B \rhd A$.
Cut	If $A \rhd B$ and $C \rhd A \wedge B$, then $C \rhd A$.
Cumulativity	If $A \rhd B$ and $A \wedge B \rhd C$, then $A \rhd B \wedge C$.

Below we list some interesting properties of this new notion of dependence.

1. (And) If $A \rhd B$ and $A \rhd C$, then $A \rhd B \wedge C$.
2. If $A \rhd B$, then $C \wedge A \rhd C \wedge B$;
3. If $A \rhd B$ and $B \rhd C$, then $A \rhd B \wedge C$.
4. If $A \rhd A \wedge B$ and $B \rhd B \wedge C$, then $A \rhd A \wedge C$.
5. If $A \rhd B$ and $B \rhd A$, then $C \rhd A$ iff $C \rhd B$.

References

[Boc01] A. Bochman. *A Logical Theory of Nonomonotonic Inference and Belief Change.* Springer, 2001.

[CH96] L. Fariñas Del Cerro and A. Herzig. Belief change and dependence. In Y. Shoham, editor, *Proc. 6th Conf. On Theoretical Aspects of Rationality and Knowledge, TARK'96*, pages 147–162. Morgan Kaufmann, 1996.