Strategy-proof assignment with a vanishing budget surplus

Hervé Moulin*
Department of Economics, Rice University

April 2007

1 The problem

A group of n agents must assign p *identical* desirable objects, where $p < n$. Cash transfers are used to compensate the losers (who get no object) from the winner's pocket, and to align incentives and efficiency.

Preferences are quasi-linear in money, described by a non negative *valuation* (willingness to pay) for an object. A Vickrey-Clarke-Groves (VCG) mechanism induces truthful revelation of individual valuations and implements an efficient assignment, i.e., assigns the objects to the p agents with the highest valuations. Cash transfers don't balance out at some profiles of valuations, therefore to run the mechanism the participants must find a *residual claimant* who will 'burn' the surplus of money.

We only consider VCG mechanisms that are *feasible* (self-sufficient): money may flow out but not in. The relative surplus loss at a profile of valuations is the ratio of the budget surplus (the money burnt) to the efficient surplus; the overall performance of a mechanism is the *worst* such ratio over all possible profiles of valuations; we call this number the *efficiency loss* of the mechanism.

*Thanks to Sandeep Baliga, Geoffroy de Clippel, Olivier Gossner, Jason Hartline, and Jay Sethuraman for their comments. This research is supported by the NSF under grant SES-0414543.

Dagstuhl Seminar Proceedings 07271
Computational Social Systems and the Internet
http://drops.dagstuhl.de/opus/volltexte/2007/1160

We compute for all n and p the smallest possible efficiency loss $\widehat{L}(n,p)$ among all feasible VCG mechanisms, and a canonical VCG mechanism achieving $\widehat{L}(n,p)$. We also compute the minimal efficiency loss $L^*(n,p)$ and an optimal mechanism, under the additional constraint of *Voluntary Participation* (individual rationality) requiring that no participant ends up with a net loss. We call such mechanisms *voluntary*. Our results illuminate some important limitations of the voluntary participation constraint.

If we assign a single object, $p=1$, our optimal voluntary mechanism is also optimal among unvoluntary ones. Moreover it achieves asymptotic efficiency at exponential speed in n: $\widehat{L}(n,1) = L^*(n,1) = \frac{n-1}{2^{n-1}-1}$. Start from the Vickrey auction, where the residual claimant sells the object at the second highest price. In addition to the Vickrey transfers, all agents, including the winner of the object, receive a cash *rebate* – a share of the auctioneer's revenue – that is independent of their own bid, hence preserves the incentive to bid truthfully. The challenge is to choose the rebates small enough that feasibility is preserved, but large enough that they absorb most of the auctioneer's revenue.

If we must assign two or more objects, the voluntary participation constraint has bite: $\widehat{L}(n,p) < L^*(n,p)$ for all $p \geq 2$. The most dramatic illustration is for $p = n-1$, where no voluntary mechanism improves upon the Vickrey auction and $L^*(n,n-1) = 1$: in any voluntary VCG mechanism, at some profiles the entire surplus goes to the residual claimant. But if we can use unvoluntary mechanisms we get $\widehat{L}(n,n-1) = \frac{n-1}{n2^{n-2}-1} \simeq \frac{1}{2^{n-2}}$.

It turns out that for unvoluntary mechanisms, asymptotic efficiency holds uniformly in p and the worst choice of p is the integer $\frac{n}{2}$ or $\frac{n-1}{2}$, denoted $\{\frac{n}{2}\}$:

$$\max_{1\leq p\leq n-1} \widehat{L}(n,p) = \widehat{L}(n,\{\frac{n}{2}\}) \simeq \frac{0.8}{\sqrt{n}}$$

The critical value of the ratio $\frac{p}{n}$, that we call the *scarcity index* of the assignment problem, is $\frac{1}{2}$. Loosely speaking, if $p < \frac{n}{2}$ the two optimal efficiency losses $\widehat{L}(n,p)$ and $L^*(n,p)$ vanish exponentially fast in n. On the other hand if $p > \frac{n}{2}$ no voluntary mechanism is asymptotically efficient ($L^*(n,p)$ remains bounded away from zero), whereas among unvoluntary mechanisms efficiency can still be achieved exponentially fast.

The mathematical backbone of our results in the one object case is the approximation of the function $\max_i\{x_i\}$ for $x \geq 0$, by an additively decomposable function of the form $\sum_i g_i(x_{-i})$. We require an approximation from above and measure the approximation error by the maximal ratio $\frac{\sum_i g_i(x_{-i})}{\max_i\{x_i\}}$. We find the optimal approximation. For the p objects case, we solve two similar approximation problems (with or without the voluntary participation constraint) for the function $\sum_{k=1}^{p} x^{*k}$ where x^{*k} is the k-th highest among the n variables x_i.

We state the results without proof, and refer the reader to [11] for a complete treatment, including the fair division aspects of the mechanisms we propose.

2 Relation to the literature

The worst case analysis is commonplace in the Operations Research and Computer Science literatures where it is often referred to as *competitive analysis* (e.g., [8], [14]). Although less familiar, it is not without precedent in the micro-economic literature on VCG mechanisms: [10] and [4] use it to discuss the pivotal mechanism in the public good provision problem; [12] does the same in a cost sharing problem.

The idea of refunding part of the budget surplus of the pivotal mechanism while respecting incentives goes back to [9], and was developed by [2] for the public good provision problem. It was recently used in the assignment problem by Cavallo [3] and Guo and Conitzer [5], who also apply the worst case analysis. They use a different performance index, namely the ratio of the budget surplus to the revenue of the Vickrey auctioneer: this revenue is not a good proxy for the potential welfare gains in the assignment economy because it can be arbitrarily smaller than the efficient surplus, therefore the interpretation of their performance index is unclear. Reference [3] proposes a rebate that only depends upon the $p+2$ highest bids and belongs to the family of mechanisms identified in our Proposition 2; [5] discovers independently the linear optimal mechanism of Theorem 1, and offer an alternative characterization in the class of anonymous VCG mechanisms. In turn, the difference in the performance index eliminates the non voluntary mechanisms of our Theorem 2, with their superior efficiency properties (Theorem 3). On the other hand [5] extends the linear optimal mechanism beyond the assignment problem to multi-unit demands.

Two more papers applying VCG mechanisms to the assignment of p identical *bads* (tasks) follow a different yet related route. Porter, Shoam and Tennenholtz ([13]) propose an original test of equity called k-fairness (discussed in subsection 5.3; see also [1]) that leads to their 3-Fair mechanism. In the one object case, this is the mechanism introduced in [3] (see Proposition 2). However when $p \geq 2$, the 3-Fair mechanism is unvoluntary and is not comparable to the mechanisms in our Theorem 2.

3 The model

We have p **identical** objects and a set N of n agents, who each need at most one object. Assume a rationing situation, i.e., $n > p \geq 1$ (if $n \leq p$ everyone gets an object and goes home). Monetary transfers are feasible and agent i's privately known value for an object (relative to no object) is a_i. We assume $a_i \geq 0$: objects are weakly desirable. Given a profile of valuations $a \in \mathbb{R}_+^N$, the vector $a^* \in \mathbb{R}_+^n$ is its permutation where coordinates are arranged decreasingly

$$a^{*1} \geq a^{*2} \geq \cdots \geq a^{*n} \tag{1}$$

For any $i \in N$, the $(N \diagdown \{i\})$-profile a_{-i} obtains by deleting the i-th coordinate, and a^*_{-i} denotes its permutation by weakly decreasing coordinates. The efficient surplus given p objects and the profile of valuations a is

$$v_p(a) = a^{*1} + \cdots + a^{*p} \tag{2}$$

Similarly $v_p(a_{-i}) = a^{*1}_{-i} + \cdots + a^{*p}_{-i}$ is the efficient surplus in the absence of agent i.

A general VCG mechanism ([7]) is defined by n arbitrary real valued functions h_i on $\mathbb{R}_+^{N \diagdown \{i\}}$. The function h_i determines agent i's net utility U_i

$$U_i(a) = v_p(a) - h_i(a_{-i}) \text{ for all } a \in \mathbb{R}_+^N \tag{3}$$

At a profile a, the mechanism assigns an object to an efficient subset of p agents. Monetary compensations are adjusted so that the net utility of every agent, whether or not she gets an object, is given by (3). Recall that the tie breaking rule (if there is more than one efficient subset, i.e., $a^{*p} = a^{*(p+1)}$) is arbitrary, it affects neither welfare nor incentives.

We denote by Δ the budget surplus of mechanism (3):

$$\Delta(a) = v_p(a) - \sum_{i \in N} U_i(a) = \sum_{i \in N} h_i(a_{-i}) - (n-1)v_p(a)$$

We restrict attention to *feasible* mechanisms where the money only flows out:

$$\textit{Feasibility (F): } \Delta(a) \geq 0 \Leftrightarrow \sum_{i \in N} h_i(a_{-i}) \geq (n-1)v_p(a) \text{ for all } a \quad (4)$$

The performance of a mechanism is measured by the following index, that we call its *worst efficiency loss,* or simply its *efficiency loss*:

$$L(n,p) = \max \frac{\text{outflow of money}}{\text{efficient surplus}} = \max_{a \in \mathbb{R}_+^N \diagdown \{0\}} \frac{\Delta(a)}{v_p(a)} \quad (5)$$

If $\Delta(0) > 0$ it is natural to set $L(n,p) = \infty$, and to modify the definition (5) accordingly.

We call a mechanism *voluntary* if no one ever suffers a net loss as a result of participating. This requirement is compelling when our model is interpreted as a fair division problem**:**

$$\begin{aligned} \textit{Voluntary Participation (VP)} \quad &: \quad U_i(a) \geq 0 \text{ for all } a, \text{ all } i \quad (6) \\ &\Leftrightarrow \quad h_i(a_{-i}) \leq v_p(a_{-i}) \text{ for all } a_{-i}, \text{ all } i \end{aligned}$$

Note that F and VP imply $0 \leq L(n,p) \leq 1$. Indeed inequality (6) gives $h_i(a_{-i}) \leq v_p(a_{-i}) \leq v_p(a)$ for all i, which sums to $\Delta(a) \leq v_p(a)$.

A benchmark mechanism is the *Vickrey auction* (also called *pivotal mechanism* see [7]), in which the residual claimant "owns" the objects and sells them at the $(p+1)$st highest price. Thus $h_i^{vick}(a_{-i}) = v_p(a_{-i})$ and

$$U_i^{vick}(a) = v_p(a) - v_p(a_{-i}) \text{ for all } i \text{ and } a \quad (7)$$

The Vickrey auction is feasible and (barely) induces voluntary participation: an inefficient agent breaks even, an efficient one gets $U_i^{vick}(a) = a_i - a^{*(p+1)}$. The residual claimant captures the whole surplus if $a_1^* = \cdots = a_{p+1}^*$, implying $L^{vick}(n,p) = 1$. The Vickrey auction has the largest efficiency loss among all feasible and voluntary VCG mechanisms.

In view of definition (7), inequality (6)) reads $U_i^{vick}(a) \leq U_i(a)$ for all i and a: a VCG mechanism is voluntary *if and only if* it is Pareto superior to the Vickrey auction.

Whether or not the mechanism under consideration is voluntary, it will be convenient to write the functions $h_i(a_{-i})$ in (3) as $h_i(a_{-i}) = v_p(a_{-i}) - r_p(i; a_{-i})$ where $r_p(i; a_{-i})$ is a *rebate function.* Hence the general form of VCG mechanisms in our model:

$$U_i(a) = v_p(a) - v_p(a_{-i}) + r_p(i; a_{-i}) = U_i^{vick}(a) + r_p(i; a_{-i}) \text{ for all } a \in \mathbb{R}_+^N \quad (8)$$

When Voluntary Participation holds, and only then, we interpret $r_p(i; a_{-i})$ as agent i's share of the seller's revenue in the Vickrey auction.

4 Results

4.1 Optimal feasible mechanisms: voluntary and unvoluntary

Let $\binom{s}{k}$ be the binomial coefficient "take k among s". For any integers such that $t \leq t' \leq s$ we define

$$B_s^{t,t'} = \sum_{k=t}^{t'} \binom{s}{k}, \quad B_s^{t\rightarrow} = B_s^{t,s}, \quad B_s^{\rightarrow t} = B_s^{0,t} \tag{9}$$

Theorem 1 (*under F and VP)*
Among all feasible and voluntary VCG mechanisms (8), the smallest efficiency loss (5) is

$$L^*(n,p) = \frac{\binom{n-1}{p}}{B_{n-1}^{p\rightarrow}} \tag{10}$$

The following linear rebate functions define an optimal mechanism

$$r_p^*(a_{-i}) = \sum_{k=p+1}^{n-1} (-1)^{k-p-1} \frac{pL^*(n,p)}{kL^*(n,k)} a_{-i}^{*k} \text{ if } p \leq n-2;\ r_{n-1}^*(a_{-i}) = 0 \tag{11}$$

The corresponding budget surplus is

$$\Delta^*(a) = pL^*(n,p)\{\sum_{k=p+1}^{n} (-1)^{k-p-1} a^{*k}\} \tag{12}$$

Theorem 2 (*under F only)*
Among all feasible VCG mechanisms (8) the smallest efficiency loss $\widehat{L}(n,p)$ *(5) is*

$$\widehat{L}(n,1) = L^*(n,1)\ ;\ \widehat{L}(n,p) = \frac{\binom{n-1}{p}}{B_{n-1}^{p\rightarrow} + \frac{n}{p} B_{n-2}^{\rightarrow(p-2)}} \text{ if } 2 \leq p \leq n-1 \tag{13}$$

The following linear rebate functions define an optimal mechanism

$$\widehat{r}_1(a_{-i}) = r_1^*(a_{-i})$$

$$\widehat{r}_p(a_{-i}) = \widehat{L}(n,p)\{\sum_{k=1}^{p-1} \gamma_k a_{-i}^{*k}\}+(1-\frac{\widehat{L}(n,p)}{L^*(n,p)})a_{-i}^{*p}+\sum_{k=p+1}^{n-1}(-1)^{k-p-1}\frac{p\widehat{L}(n,p)}{kL^*(n,k)}a_{-i}^{*k} \tag{14}$$

$$\gamma_k = -\frac{n}{n-1}\frac{B_{n-2}^{(n-k)\rightarrow}}{\binom{n-2}{n-k-1}}-\frac{1}{n-1} \text{ if } p-k \text{ is odd; } \gamma_k = \frac{n}{n-1}\frac{B_{n-2}^{(n-k)\rightarrow}}{\binom{n-2}{n-k-1}} \text{ if } p-k \text{ is even}$$

(the right summation in (14) is zero if $p = n-1$).The budget surplus is

$$\widehat{\Delta}(a) = \widehat{L}(n,p)\{\sum_{k=1,3,\cdots}^{\leq p-1}(p-k)(a^{*(p-k)}-a^{*(p-k+1)})+p\sum_{k=p+1}^{n}(-1)^{k-p-1}a^{*k}\} \tag{15}$$

The rebate functions (11) and (14) are not the only choices of $r_p(a_{-i})$ achieving, respectively, $L^*(n,p)$ and $\widehat{L}(n,p)$. They are the only choices if we restrict attention to symmetric rebate functions, linear in the a_{-i}^{*k}.

Compare the two theorems first when $p = 1$, then for $p = (n-1)$. For $p = 1$ Voluntary Participation comes free: the optimal linear rebates under F define a voluntary mechanism, therefore the optimal efficiency loss under F is also the optimal loss under F and VP:

$$\widehat{L}(n,1) = L^*(n,1) = \frac{n-1}{2^{n-1}-1} \simeq \frac{2n}{2^n}$$

The situation for $p = n-1$ is much different. Then we cannot improve upon the Vickrey auction by a voluntary and feasible VCG mechanism ($L^*(n,n-1) = 1$ and $r_{n-1}^*(a_{-i}) = 0$), whereas the optimal feasible (unvoluntary) mechanism achieves an efficiency loss even smaller than in the one object case:

$$\widehat{L}(n,n-1) = \frac{n-1}{n2^{n-2}-1} \simeq \frac{4}{2^n}$$

We illustrate the optimal linear mechanism for $p = 1$ and small values of n. For $n = 2$ the Vickrey auction cannot be improved. For $n = 3,4,5,6$ equation (11) gives the optimal rebates $r_1^*(a_{-i}) = \widehat{r}_1(a_{-i})$:

$$L^*(3,1) = \frac{2}{3} \text{ and } r_1^*(a_{-i}) = \frac{1}{3}a_{-i}^{*2}$$

$$L^*(4,1) = \frac{3}{7} \text{ and } r_1^*(a_{-i}) = \frac{2}{7}a_{-i}^{*2} - \frac{1}{7}a_{-i}^{*3}$$
$$L^*(5,1) = \frac{4}{15} \text{ and } r_1^*(a_{-i}) = \frac{11}{45}a_{-i}^{*2} - \frac{1}{9}a_{-i}^{*3} + \frac{1}{15}a_{-i}^{*4}$$
$$L^*(6,1) = \frac{5}{31} \text{ and } r_1^*(a_{-i}) = \frac{13}{62}a_{-i}^{*2} - \frac{8}{93}a_{-i}^{*3} + \frac{3}{62}a_{-i}^{*4} - \frac{1}{31}a_{-i}^{*5}$$
$$L^*(7,1) = \frac{2}{21} \text{ and } r_1^*(a_{-i}) = \frac{19}{105}a_{-i}^{*2} - \frac{1}{15}a_{-i}^{*3} + \frac{11}{315}a_{-i}^{*4} - \frac{1}{45}a_{-i}^{*5} + \frac{1}{63}a_{-i}^{*6}$$

Equation (11) and the fact that $L^*(n,k)$ increases in k (Theorem 3 below) imply $r_1^*(a_{-i}) = \sum_{k=2}^{n-1} \beta_k a_{-i}^{*k}$ where, the β_k start with $\beta_1 > 0$, alternate in sign and $|\beta_k|$ decreases in k. For an arbitrary $p, 1 \leq p \leq n-2$, the general form of the optimal rebates is similarly $r_p^*(a_{-i}) = \sum_{k=p+1}^{n-1} \beta_k a_{-i}^{*k}$, where the coefficients start positive, alternate in sign and decreases in absolute value.

Turning to the case $p = n-1$, where $r_{n-1}^*(a_{-i}) = 0$, equation (14) (where the second summation disappears) gives on the other hand:

$$\widehat{L}(3,2) = \frac{2}{5} \text{ and } \widehat{r}_2(a_{-i}) = -\frac{1}{5}a_{-i}^{*1} + \frac{3}{5}a_{-i}^{*2}$$
$$\widehat{L}(4,3) = \frac{1}{5} \text{ and } \widehat{r}_3(a_{-i}) = -\frac{1}{5}a_{-i}^{*2} + \frac{4}{5}a_{-i}^{*3}$$
$$\widehat{L}(5,4) = \frac{4}{39} \text{ and } \widehat{r}_4(a_{-i}) = -\frac{1}{39}a_{-i}^{*1} + \frac{5}{117}a_{-i}^{*2} - \frac{23}{117}a_{-i}^{*3} + \frac{35}{39}a_{-i}^{*4}$$
$$\widehat{L}(6,5) = \frac{1}{19} \text{ and } \widehat{r}_5(a_{-i}) = -\frac{1}{38}a_{-i}^{*2} + \frac{1}{19}a_{-i}^{*3} - \frac{7}{38}a_{-i}^{*4} + \frac{18}{19}a_{-i}^{*5}$$
$$\widehat{L}(7,6) = \frac{6}{223} \text{ and}$$

$$\widehat{r}_6(a_{-i}) = -\frac{1}{223}a_{-i}^{*1} + \frac{7}{1115}a_{-i}^{*2} - \frac{26}{1115}a_{-i}^{*3} + \frac{56}{1115}a_{-i}^{*4} - \frac{187}{1115}a_{-i}^{*5} + \frac{217}{223}a_{-i}^{*6}$$

The pattern is now $\widehat{r}_{n-1}(a_{-i}) = \sum_{k=1}^{n-1} \gamma_k a_{-i}^{*k}$ where the γ_k alternate in sign, $|\gamma_k|$ increase in absolute value, and γ_{n-1} is positive and slightly below 1; if n is even $\gamma_1 = 0$ and $\gamma_2 < 0$ and if n is odd $\gamma_1 = -\frac{\widehat{L}(n,n-1)}{n-1}$.

Comparing (13) and (10) we see that if $2 \leq p \leq n-1$, allowing unvoluntary mechanisms strictly decreases the optimal efficiency loss: $\widehat{L}(n,p) < L^*(n,p)$. In the rebates $\widehat{r}_p(a_{-i})$ ((14)) the first p terms make a sum $\sum_{k=1}^{p} \gamma'_k a_{-i}^{*k}$

similar to that for $\widehat{r}_{n-1}(a_{-i})$ (coefficients increasing in absolute value, alternating in sign and $\gamma_p > 0$) while the next $(n-1-p)$ terms are just $\frac{\widehat{L}(n,p)}{L^*(n,p)} r_p^*(a_{-i})$. For instance

$$\widehat{L}(6,3) = \frac{5}{13} \text{ and } \widehat{r}_3(a_{-i}) = -\frac{3}{26}a_{-i}^{*2} + \frac{5}{13}a_{-i}^{*3} + \frac{9}{26}a_{-i}^{*4} - \frac{3}{13}a_{-i}^{*5}$$

where the first two terms are $\sum_{k=1}^{p} \gamma'_k a^{*k}$ and the last two are $\frac{8}{13} r_3^*(a_{-i})$.

4.2 Asymptotic efficiency

The scarcity ratio $\frac{p}{n}$ essentially determines the asymptotic behavior of $\widehat{L}(n,p)$ and $L^*(n,p)$ for large n.

We use the notation $f(n) \simeq g(n)$ for $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$; the expression "$f(n)$ is exponential" means the existence of $K > 0$ and $\alpha < 1$ such that $f(n) \leq K\alpha^n$ for all n; finally $\{\frac{n}{2}\}$ is the integer $\frac{n}{2}$ or $\frac{n-1}{2}$.

Theorem 3

i) $L^*(n,p)$ *increases strictly in* p*, decreases strictly in* n*;* $\widehat{L}(n,p)$ *increases in* n *for* $p \leq n \leq 2p-1$*, decreases in* n *if* $2p \leq n$*;* $\widehat{L}(n,p)$ *increases in* p *for* $1 \leq p \leq \{\frac{n}{2}\}$*, decreases in* p *if* $\{\frac{n}{2}\} \leq p \leq n$*.*

ii) $\widehat{L}(n,p)$ *converges to zero uniformly in* p*:*

$$\max_{1\leq p\leq n} \widehat{L}(n,p) = \widehat{L}(n,\{\frac{n}{2}\}) \leq \frac{4}{3\sqrt{n}} \text{ for all } n$$

iii) For p *fixed,* $L^*(n,p)$ *and* $\widehat{L}(n,p)$ *are exponential in* n*:* $L^*(n,p) \simeq \widehat{L}(n,p) \simeq \frac{2}{p!}\frac{n^p}{2^n}$.

iv) For any sequence $p_n, n = 1,2,\cdots$ *such that for some* $K, \frac{n}{2} - K \leq p_n \leq \frac{n}{2} + K$ *for all* n*:*

$$L^*(n,p_n) \simeq 2\sqrt{\frac{2}{\pi n}} = \frac{1.59\cdots}{\sqrt{n}};\ \widehat{L}(n,p_n) \simeq \sqrt{\frac{2}{\pi n}} = \frac{0.80\cdots}{\sqrt{n}} \tag{16}$$

v) For any sequence $p_n, n = 1,2,\cdots$ *and any positive number* δ

$$\frac{p_n}{n} \leq \delta < \frac{1}{2} \Rightarrow L^*(n,p_n) \text{ and } \widehat{L}(n,p_n) \text{ are exponential in } n \tag{17}$$

$$\frac{p_n}{n} \geq \delta > \frac{1}{2} \Rightarrow \widehat{L}(n,p_n) \text{ is exponential in } n;\ L^*(n,p_n) \geq \frac{2\delta-1}{\delta} \text{ for all } n \tag{18}$$

Loosely speaking, $\widehat{L}(n,p)$ and $L^*(n,p)$ converge exponentially fast to zero in n if $\frac{p}{n} < \frac{1}{2}$, and as $\frac{1}{\sqrt{n}}$ if $\frac{p}{n} \simeq \frac{1}{2}$. Yet their behavior is very different if $\frac{p}{n} > \frac{1}{2}$: unvoluntary mechanisms still allow exponentially fast efficiency, while voluntary ones preclude asymptotic efficiency altogether.

5 Two open problems

5.1 Non VCG mechanisms

The family of strategyproof assignment mechanisms contains many non-VCG members that do not always assign the objects efficiently. If $U_i(a)$ denotes as before agent i's net utility, the worst efficiency loss of a feasible mechanism is

$$L(n,p) = \max_{a \in \mathbb{R}_+^N \diagdown \{0\}} \frac{v_p(a) - \sum_N U_i(a)}{v_p(a)}$$

The challenge is to compute the optimal value of $L(n,p)$ over all feasible (or all feasible and voluntary) strategyproof mechanisms.

When the scarcity ratio $\frac{p}{n}$ is large enough, it is easy to construct non VCG mechanisms improving upon the efficiency losses $\widehat{L}(n,p)$ and $L^*(n,p)$. Consider the (non anonymous) mechanism picking an arbitrary agent, say agent 1, giving her an object and making her the residual claimant of a voluntary VCG mechanism (e.g., the Vickrey auction) assigning the remaining $p-1$ objects among agents other than 1. This ensures budget-balance and a worst efficiency loss of $\frac{1}{p}$ (the worst case is that the residual claimant has $a_1 = 0$ while the p efficient agents have the same positive valuation)[1]. For $p = \{\frac{n}{2}\}$ and n large enough this improves upon $\widehat{L}(n, \{\frac{n}{2}\}) \simeq \frac{0.6}{\sqrt{n}}$.

5.2 Heterogenous objects

In the general assignement problem we have a set N of n agents and a set P of p desirable objects. Agent i's valuation for object k is an arbitrary non negative number $a_i(k)$. The efficient surplus $v_p(a)$ maximizes $\sum_N a_i(k(i))$ over all feasible assignments. The definition (5) of the worst relative efficiency

[1]Thanks to Jason Hartline for this remark.

loss for a feasible VCG mechanism is now

$$\mathcal{L}(n,p) = \max_{a \in \mathbb{R}_+^{N \times P} \setminus \{0\}} \frac{\Delta(a)}{v_p(a)}$$

The agents may view all objects as identical, therefore the optimal value $\mathcal{L}^*(n,p)$ over all feasible and voluntary VCG mechanisms is at least $L^*(n,p)$. Clearly $\mathcal{L}^*(n,1) = L^*(n,1)$, moreover VP and F imply $\mathcal{L}(n,p) \leq 1$ as before. Therefore Theorem 1 determines $\mathcal{L}^*(n,p)$ in the following cases:

$$\mathcal{L}^*(n,1) = \frac{n-1}{2^{n-1}-1}; \mathcal{L}^*(n,p) = 1 \text{ if } p \geq n-1$$

(for $p \geq n$ we use the fact that the matrix of valuations a may be such that only $n-1$ objects are desired).

Computing $\mathcal{L}^*(n,p)$ for $2 \leq p \leq n-2$ appears to be difficult. The computation of the optimal efficiency loss $\widehat{\mathcal{L}}(n,p)$ over all feasible VCG mechanisms is equally open. A general characterization result by Guo and Conitzer [6] may be helpful to reach an answer.

References

[1] M. Atlamaz and D. Yengin, *Fair Groves mechanisms*, mimeo, Rochester University, 2006.

[2] M.J. Bailey, *The demand revealing process: to distribute the surplus,* Public Choice, 91:107-126, 1997.

[3] R. Cavallo, *Optimal decision-making with minimal waste: strategyproof redistribution of VCG payments,* International Conference on Autonomous Agents and Multi-agents Systems, (AAMAS) Hakodate, Japan, 2006.

[4] R. Deb and T.K. Seo, *Maximal surplus from the pivotal mechanism: a closed form solution,* Review of Economic Design, 3, 347-357, 1998.

[5] M. Guo and V. Conitzer, *Worst case optimal redistribution of VCG payments,* Conference on Electronic Commerce (EC), San Diego, June 2007.

[6] M. Guo and V. Conitzer, *Undominated VCG redistribution mechanisms,* mimeo Duke University, 2007.

[7] J. Green, and J.J. Laffont , *Incentives in public decision making,* Amsterdam: North-Holland, 1979.

[8] E. Koutsoupias and C. Papadimitriou *Worst case equilibria* Proceedings of the 16th Symposium on Theoretical aspects of computer science, 404-413, 1999.

[9] J.J. Laffont and E. Maskin, *A differential approach to ecpected utility maximizing mechanisms,* in, Laffont J.J. (eds), Aggregation and Revelation of Preferences, New York: North Holland, 1979.

[10] H. Moulin, *Characterizations of the Pivotal Mechanism*, Journal of Public Economics, 31, 53–78, 1986.

[11] H. Moulin, *Efficient, strategy-proof, and almost budget-balanced assignment,* mimeo, Rice University, 2007, http://www.ruf.rice.edu/~econ/faculty/Moulin/spassgnt.pdf.

[12] H.Moulin and S. Shenker, *Strategy-proof sharing of submodular costs: budget balance versus efficiency,* Economic Theory, 18, 3, 511-533, 2001.

[13] R. Porter, Y. Shoam and M. Tennenholtz, *Fair imposition,* Journal of Economic Theory, 118, 209-228, 2004.

[14] M. Tennenholtz, *Rational competitive analysis,* IJCAI-01, 2001.