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Competitive Online Searching for a Ray in the Plane*

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Abstract. We consider the problem of a searcher that looks, for example, for a lost flashlight in a dusty environment. The search agent finds the flashlight as soon as it crosses the ray emanating from the flashlight, and in order to pick it up, the searcher has to move to the origin of the light beam.
First, we give a search strategy for a special case of the ray search—the window shopper problem—, where the ray we are looking for is perpendicular to a known ray. Our strategy achieves a competitive factor of ≈1.059, which is optimal. Then, we consider the search for a ray with an arbitrary position in the plane. We present an online strategy that achieves a factor of ≈22.513, and give a lower bound of ≈17.079.

Keywords: Online motion planning, competitive ratio, searching, ray search

1 Introduction

Searching for a goal in an unknown environment is a basic task in robot motion planning and well-studied in many settings. For example, Gal. [9] and independently Baeza-Yates et al. [2] considered the task of finding a point on an infinite line using a searcher that starts in the origin and neither knows the distance nor the direction towards the goal. They introduced the so called *doubling strategy*: The agent moves alternately to the left and to the right, doubling its search depth in every iteration step. Searching on the line was generalized to searching on m concurrent rays starting from the searcher's origin, see [9, 2, 1].

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Many variants of the problem were discussed since then, for example m-ray searching with restricted goal distance (Hipke et al. [11], Langetepe [20], López-Ortiz and Schuierer [26, 21]), m-ray searching with additional turn costs (Demaine et al. [5]), parallel m-ray searching (Kao et al. [16], Hammar et al. [10], López-Ortiz and Schuierer [22]), randomized searching (Schuierer [27], Kao et al. [17]), searching in polygons (Schuierer [25], Klein [18]), or searching with error-prone agents (Kamphans and Langetepe [15, 14]). Furthermore, some of the problems were again rediscovered by Jaillet et al. [13].

The quality of a strategy that deals with incomplete information—an *online strategy*—is usually measured by the cost of the online solution compared to the optimal solution. More precisely, let $|S|$ denote the cost of an online strategy, S, and $|S_{\text{Opt}}|$ the cost of the optimal solution, then we call S *C-competitive*, if there exists a constant A such that $|S| \leq C \cdot |S_{\text{Opt}}| + A$ holds for every input to S. In our case, the costs incurred by a search strategy is given by the length of the path covered by the searcher, and the optimal solution is the length of the shortest path from the searcher's origin to the goal. The competitive framework was introduced by Sleator and Tarjan [28] and used for many settings; see, for example, the survey by Fiat and Woeginger [7]. For a general overview of online motion planning problems and its analysis see the surveys [3, 23, 24, 12]. Another measure is the *search ratio*, see Koutsoupias et al. [19] and Fleischer et al. [8].

In this paper, we consider the search for the origin t of a ray R in the plane, see Figure 1. The searcher has no vision, but recognizes the ray and the ray's origin as soon as the searcher hits the ray. Similar problems were discussed by Alpern and Gal [1]. The position of the ray is not known in advance and we move along a search path Π starting at a given point s. Finally Π will hit the ray R at point p and the origin t is detected. The cost of the strategy is given by the length of the path from s to p (i.e., $|\Pi_s^p|$), plus the distance $|pt|$ from p to t. The performance of the path Π for the ray R ist given by the competitive ratio $\frac{|\Pi_s^p|+|pt|}{|st|}$; that is, we compare the length of the path to the shortest path form s to t. We would like to find a search path Π that guarantees a competitive ratio not greater than C for all possible rays R in the plane. In turn, C should be as small as possible.

First, in Section 2 we consider a simplified version of this problem: The origin s of the ray, R, we are looking for is located on another ray, R', perpendicular to R. The searcher's start point a and R are located on the same side of R'. Moreover, R' is known. We call this problem the *window shopper problem*, because we can imagine R' as a shopping window. A buyer walks along the windows—perhaps looking for a present—and walks towards the window as soon as the item is spoted. We present a search strategy for this problem that achieves an optimal competitive factor of $1.059\ldots$.

Furthermore in Section 3, we consider the general case as shown in Figure 1 and present a search strategy that achieves a factor of $22.513\ldots$ In Section 4 we give a lower bound of $17.079\ldots$ Surprisingly, the lower bound construction is also applicable to a search problem discussed by Alpern and Gal [1], leaving a gap between $17.38\ldots$ and $17.079\ldots$

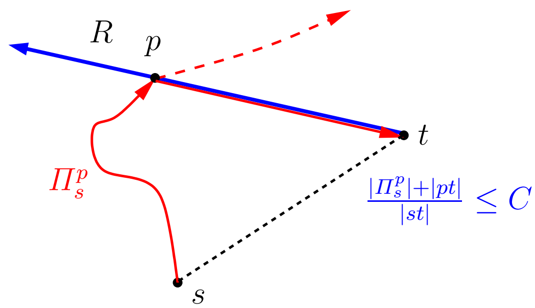

Fig. 1. Searching for the origin t of a ray R.

2 The Window Shopper Problem

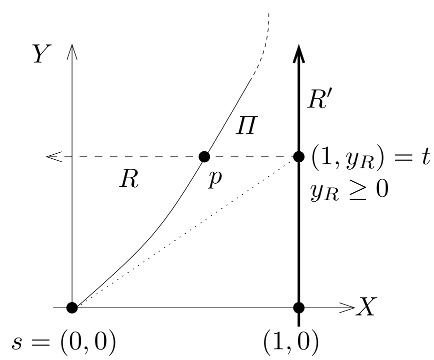

Fig. 2. A strategy for the window shopper problem.

In this section, we consider the problem of finding a gift s along a shopping window. The agent starts somewhere and looks toward the window. We assume that the item t gets into sight if the ray R, from t to the seachers position p, is perpendicular to the window. Then the searcher moves toward t.

This problem can be modelled as follows. W.l.o.g. we assume that the line of sight (i.e., the ray, R, we are looking for), is parallel to the X-axis, starts in $(1, y_R)$ for $y_R \geq 0$, and emanates toward the left side of the perpendicular ray R' (the *window*) which starts in $(1, 0)$. The searcher starts in the origin $s = (0, 0)$; see Figure 2. The goal (i.e., the ray's origin t) is discovered as soon as the searcher reaches its height, y_R. After the searcher has discovered to goal, it moves directly to the goal. Note that the shortest distance from s to R' can be fixed to 1 because scaling has no influence on the competitive ratio.

We would like to find a search path, Π, so that for any goal, t, the ratio $\frac{|\Pi_s^p|+|pt|}{|st|} \leq C$ holds, where C is the smallest achievable ratio for all search paths.

Theorem 1. *There is a strategy Π with an optimal competitive factor of* $1.059\ldots$ *for searching the origin of a ray, R, that emanates from a known ray R' perpendicular to R.*

Proof. We solve two tasks.

1. We will design a search path Π that consists of the following three parts (or conditions), see Figure 3(i).
   - Π_1: A straight line segment from $(0, 0)$ to some point (a, b) where the competitive ratio strictly increases from $C = 1$ to $C_{\max}$ for goals from $(1, 0)$ to $(1, b)$.
   - Π_2: A strictly monotone curve f from (a, b) to some point $(1, D)$ on R' where the competitive ratio is exactly $C_{\max}$ for all goals from $(1, b)$ to $(1, D)$.
   - Π_3: A ray starting form $(1; D)$ to $(1, \infty)$ where the competitive ratio strictly decreases from $C_{\max}$ to 1 for goals from $(1, D)$ to $(1, \infty)$.

   Furthermore, we prove that the full path Π is convex. The competitive ratio of Π is $C_{\max}$.
2. We will show that such a path is optimal and the best achievable ratio is $C_{\max}$.

We start with the second task. Let us assume that we have designed a search path Π with the given properties and let us assume that there is an optimal search path K with $K \neq \Pi$, see Figure 3(ii).

The path K might hit the ray B from $(1, b)$ to $(-\infty, b)$ at a point p_1 to the left of (a, b). Then the ratio $\frac{|K_s^{p_1}|+|p_1(1,b)|}{|s(1,b)|}$ is bigger than $C_{\max} = \frac{|s(a,b)|+|(a,b)(1,b)|}{|s(1,b)|}$. On the other hand K might move to the right of (a, b) and hits Π_2 at a point p_2 between B and the ray D from $(1, D)$ to $(-\infty, D)$. In this case the length of $K_s^{p_2}$ has to be bigger than $\Pi_s^{p_2}$ because Π is fully convex. Thus, the ratio $\frac{|K_s^{p_2}|+|p_2(1,p_{2_y})|}{|s(1,p_{2_y})|}$ is bigger than $C_{\max} = \frac{|\Pi_s^{p_2}|+|p_2(1,p_{2_y})|}{|s(1,p_{2_y})|}$, where p_{2_y} denotes the Y-coordinate of p_2. This also holds if K hits R' first and p_2 equals $(1, D)$; see the dotted path in Figure 3(ii).

This means that K has to follow Π from s up to some point beyond B and might leave Π_2 then. In this case K has at least the ratio $C_{\max}$ and Π is optimal, too.

It remains to show that we can design a path with the given properties. The motivation for the construction comes from the following intuition. In the very beginning the ratio starts from 1 and has to increase for a while, this is true for any strategy. Additionally, any reasonable strategy should be monotone in x and y. Moving backwards or away from the window will allow shortcuts with a smaller ratio. Therefore it is reasonable that we will get closer and closer to the window R' and the factor should decrease to 1. So, finally, we can hit R' because at the end the ratio will not be the worst case. Furthermore, in many application strategies are designed by the fact that they achieves exactly the same factor for a set of goals. Altogether, we would like to design a strategy Π by the properties formulated above, and as we already know such a strategy is optimal.

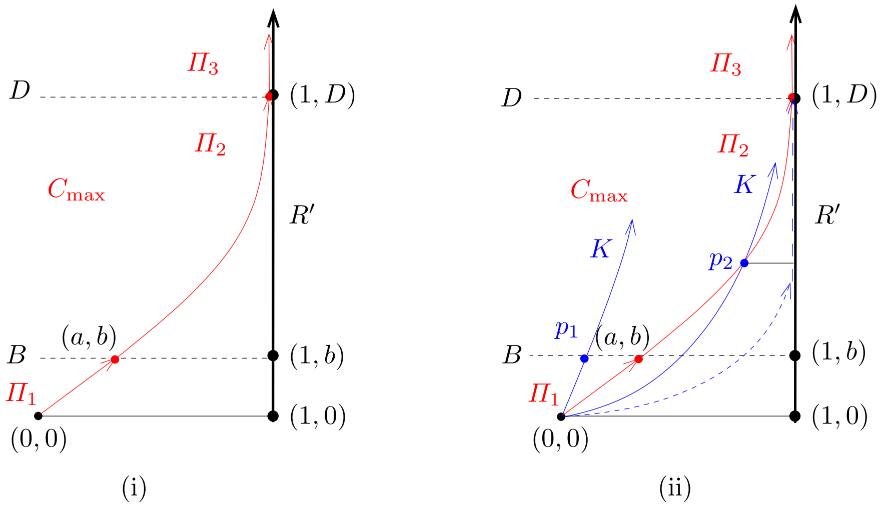

Fig. 3. An arbitrary search path K is not better than Π.

With the first two conditions for Π_1 and Π_2 we fix a and b. We consider the line segment from the origin $(0, 0)$ to (a, b) with $a, b > 0$ to be parametrized by (ta, tb) for $t \in [0, 1]$. The competitive factor is given by

$$C(t) = \frac{t\sqrt{a^2 + b^2} + 1 - ta}{\sqrt{1 + t^2 b^2}}, \quad t \in [0, 1] .$$

We want $C(t)$ to be a monotone and increasing function. From $C'(t) \geq 0 \forall t \in [0, 1]$ we conclude

$$\begin{aligned}
C'(t) &= \frac{(\sqrt{a^2 + b^2} - a)(1 + t^2 b^2) - (t(\sqrt{a^2 + b^2} - a) + 1)tb^2}{\sqrt{1 + t^2 b^2}(1 + t^2 b^2)} \geq 0 \quad \forall t \in [0.1] \\
&\Leftrightarrow \sqrt{a^2 + b^2} - a \geq tb^2 \quad \forall t \in [0.1] \\
&\Leftrightarrow \sqrt{a^2 + b^2} - a \geq b^2 \\
&\Leftrightarrow a^2 + b^2 \geq b^4 + 2ab^2 + a^2 \\
&\Leftrightarrow 1 - 2a \geq b^2 .
\end{aligned}$$

Hence, $a \leq \frac{1-b^2}{2}$ follows. From now on we set $a := \frac{1-b^2}{2}$. For $t = 1$ and $a := \frac{1-b^2}{2}$ we obtain a competitive factor of

$$\begin{aligned}
\frac{\sqrt{a^2 + b^2} + 1 - a}{\sqrt{1 + b^2}} &= \frac{\sqrt{(\frac{1-b^2}{2})^2 + b^2} + 1 - \frac{1-b^2}{2}}{\sqrt{1 + b^2}} = \frac{\sqrt{\frac{1-2b^2+b^4+4b^2}{4}} + \frac{1}{2} + \frac{b^2}{2}}{\sqrt{1 + b^2}} \\
&= \frac{\sqrt{(\frac{1+b^2}{2})^2} + \frac{1}{2}(1 + b^2)}{\sqrt{1 + b^2}} = \sqrt{1 + b^2} \; =: \; C . \qquad (1)
\end{aligned}$$

We can consider the line segment Π_1 also as a function of $x \in [0, a]$. Now, C is the worst case competitive factor for $x \in [0, a]$ and goals t between $[1, 0]$ and $[1, b]$.

For Π_2 we construct a curve $f(x)$ for $x \in [a, 1]$ that runs from $[a, b]$ to some point $[1, D]$ and achieves the ratio $C = \sqrt{1 + b^2}$ for all goals t between $[1, b]$ and $[1, D]$. This means that the length of the path of the searcher (i.e., the line segment up to (a, b), the part of the curve f up to the height y_R, and the final line segment to the goal $(1, y_R)$) equals C times the Euclidean distance from the origin $(0, 0)$ to the goal $(1, y_R)$. Thus, f can be defined by the differential equation

$$\sqrt{a^2 + b^2} + 1 - x + \int_a^x \sqrt{1 + f'(t)^2}\, dt = C \cdot \sqrt{1 + f(x)^2}. \tag{2}$$

We would like to rearrange Equation (2) in order to apply standard methods for solving differential equations. Derivating Equation (2) and squaring twice gives

$$\begin{aligned}
&\sqrt{1 + f'(x)^2} - 1 = \frac{C}{2} \cdot \frac{1}{\sqrt{1 + f(x)^2}} \cdot 2f(x)f'(x) \\
&\Leftrightarrow 1 + f'(x)^2 - 2\sqrt{1 + f'(x)^2} + 1 = C^2\, \frac{f(x)^2 f'(x)^2}{1 + f(x)^2} \\
&\Leftrightarrow f'(x)^2 \left[1 - C^2\, \frac{f(x)^2}{1 + f(x)^2}\right] + 2 = 2\sqrt{1 + f'(x)^2} \\
&\Leftrightarrow f'(x)^4 \left[1 - C^2 \frac{f(x)^2}{1 + f(x)^2}\right]^2 + 4f'(x)^2 \left[1 - C^2 \frac{f(x)^2}{1 + f(x)^2}\right] = 4f'(x)^2\,.
\end{aligned}$$

The curve f was assumed to be strictly monotone, which means $f'(x) \neq 0$. Therefore we have

$$\begin{aligned}
&\Leftrightarrow f'(x)^2 \left[1 - C^2 \frac{f(x)^2}{1 + f(x)^2}\right]^2 = 4C^2 \frac{f(x)^2}{1 + f(x)^2} \\
&\Leftrightarrow f'(x)^2 = \left[\frac{1 + f(x)^2}{1 + (1 - C^2) f(x)^2}\right]^2\, 4C^2\, \frac{f(x)^2}{1 + f(x)^2} \\
&\Leftrightarrow f'(x)^2 = 4C^2 \frac{(1 + f(x)^2) f(x)^2}{(1 + (1 - C^2) f(x)^2)^2} \\
&\Leftrightarrow f'(x) = 2C \frac{\sqrt{1 + f(x)^2}\, f(x)}{1 + (1 - C^2) f(x)^2}\,.
\end{aligned} \tag{3}$$

Note that the point $(a, b) = (\frac{1-b^2}{2}, b)$ lies on f and C equals $\sqrt{1 + b^2}$. Altogether, we have to solve the differential equation

$$y' = 1 \cdot 2\sqrt{1 + b^2} \frac{\sqrt{1 + y^2}\, y}{1 - b^2 y^2} = 1 \cdot g(y) \tag{4}$$

for $y = f(x)$ with starting point $(\frac{1-b^2}{2}, b)$.

Equation (4) is a first order differential equation $y' = h(x)g(y)$ with separated variables and point (k, l) on y. A general solution is given by

$$\int_l^y \frac{dt}{g(t)} = \int_k^x h(z)dz\,;$$

see Walter [29]. Thus, we have to solve

$$\int_b^y \frac{1-b^2t^2}{2\sqrt{1+b^2}\sqrt{1+t^2}t}dt = \int_{(1-b^2)/2}^x 1 \cdot dz = x - (1-b^2)/2$$

By simple analysis, we obtain

$$x = -\frac{b^2\sqrt{1+y^2} + \operatorname{arctanh}\left(\frac{1}{\sqrt{1+y^2}}\right) - \operatorname{arctanh}\left(\frac{1}{\sqrt{1+b^2}}\right) - \sqrt{1+b^2}}{2\sqrt{1+b^2}}$$

which is the solution for the inverse function $x = f^{-1}(y)$. By simple analysis we get

$$x' = \frac{1}{g(y)} = -\frac{(b^2y^2-1)}{2\sqrt{1+y^2}y\sqrt{(1+b^2)}} \geq 0 \text{ for } y \in [0, 1/b]$$

and

$$x'' = -\frac{(b^2y^2+2y^2+1)}{2(1+y^2)^{3/2}\sqrt{1+b^2}y^2} \leq 0 \text{ for } y \geq 0, .$$

Scince $x = f^{-1}(y)$ is concave in the given interval, $y = f(x)$ is convex. Additionally, f^{-1} attains a maximum for $y = \frac{1}{b}$.

Altogether we have a situation for the inverse function $x = f^{-1}(y)$ for $y \in \left[0, \frac{1}{b}\right]$ as shown in Figure 4(i).

Now, we have to find a value for b so that $f^{-1}(\frac{1}{b})$ equals 1, so that f^{-1} behaves as depicted in Figure 4(ii). That is, we have to find a solution for

$$1 = -\frac{b^2\sqrt{1+\frac{1}{b^2}} + \operatorname{arctanh}\left(\frac{1}{\sqrt{1+\frac{1}{b^2}}}\right) - \operatorname{arctanh}\left(\frac{1}{\sqrt{1+b^2}}\right) - \sqrt{1+b^2}}{2\sqrt{1+b^2}}\,. \tag{5}$$

This fixes b and, in turn, D to $\frac{1}{b}$. Note that in this case $y = f(x)$ has the desired properties for $x \in [a, 1] = \left[\frac{1-b^2}{2}, 1\right]$.

We have already seen that $y = f(x)$ is convex for $x \in [a, 1]$. Additionally, the line segment from $(0, 0)$ to (a, b) is convex. To show that the conjunction of both elements is also convex, we have to show that the tangent to f at (a, b) equals a prologation of the line segment; see Figure 4. In other words we have to show ${f^{-1}}'(b) = \frac{a}{b} = \frac{1-b^2}{2b}$. This is equivalent to $\frac{1}{g(b)} = \frac{1-b^2}{2b}$ which is obviously true.

Solving Equation (5) numerically, we get $b = 0.34\ldots$ This gives $D = \frac{1}{b} = 2.859\ldots$, $a = \frac{1-b^2}{2} = 0.43\ldots$ and a worst-case ratio $C = \sqrt{1+b^2} = 1.05948\ldots$ The corresponding curve f^{-1} is shown in Figure 4(ii).

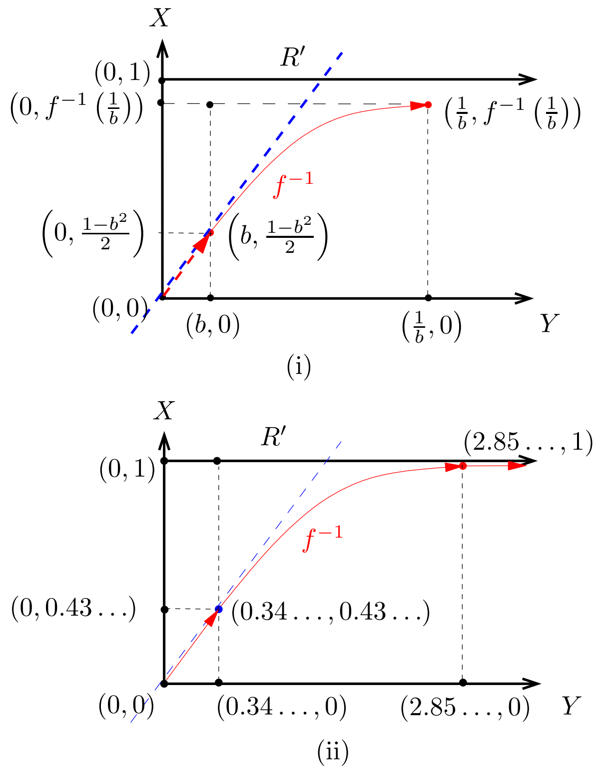

Fig. 4. The inverse situation of the window shopper problem. The curve f^{-1} should hit the line $X = 1$.

Altogether, we combine Π_1, the line segment, Π_2, the constructed curve f, and the ray from $(1, D)$ to $(1, \infty)$, Π_3, and obtain a convex curve with the given properties and an optimal competitive factor of $C = \sqrt{1 + b^2} = 1.05948\ldots$ $\square$

3 Searching for a Ray in the Plane

In this section, we consider the general problem of searching for the origin of a ray in the plane. We assume that the distance to the goal is at least 1 and use the competitive ratio as a quality measure for or search. In our case, the competitive ratio is given by the length of the searcher's path compared to the Euclidean distance from the start point to the ray's origin. For a fixed scenario (i.e., a start point, s, and a given ray, R, emanating from point t), the cost of the search for the ray is given by the ratio

$$C_\Pi := \frac{|\Pi| + |pt|}{|st|}, \tag{6}$$

where Π denotes the searcher's path from s to R, and p the point where the searcher finds the ray R; see Figure 5.

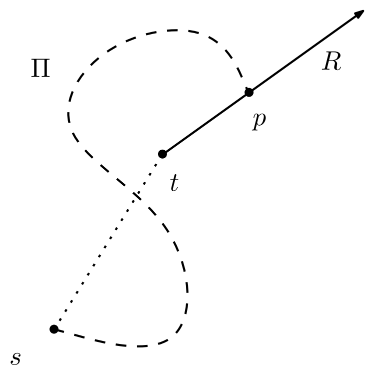

Fig. 5. Searching for a ray R.

3.1 A Competitive Search Strategy

Now, we are interested in a path for the searcher that achieves a good ratio. To find an upper bound for the costs of a search strategy, we see the search as a two-person game: First, a searcher chooses a search path. Then, based on the seachers decisions, a hider chooses its hiding point t, and the ray, R, emanating from t such that the ratio C_{Π} is maximized. The intention of the searcher is to minimize the maximum that can be achieved by the hider.

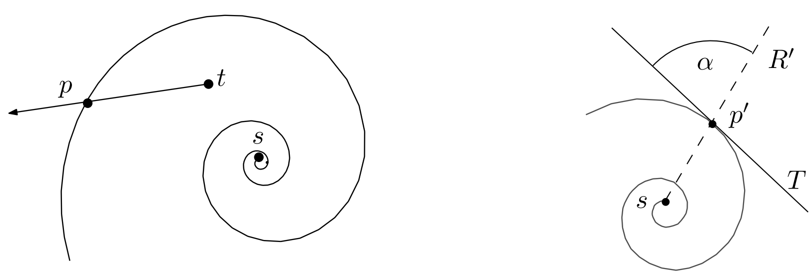

Fig. 6. (i) A spiral and a ray, (ii) the tangent angle α.

It seems to be a good strategy to search for a ray by walking a logarithmic spiral that starts in s; see Figure 6(i). A logarithmic spiral is given (in polar coordinates) by

$$r(\theta) = a\mathrm{e}^{b\theta},\ a > 0, b > 0, -\infty < \theta < \infty\,.$$

An important property of a logarithmic spiral is that every ray, R', emanating from the spiral's origin s intersects the spiral with the same tangent angle, α [4];

see Figure 6(ii). For a given spiral, α fulfills

$$b = \cot \alpha\,.$$

The Worst Case Position for the Ray Given a logarithmic spiral, a hider may now choose a position for the target ray that maximizes the ratio C_Π depending on the spiral parameters a and b. W.l.o.g. we assume that the searcher follows a counterclockwise spiral.

It is easy to see that the worst case is achieved if the searcher slightly misses the target ray, R, and has to walk another full loop until it meets R again:

Lemma 1. *Given a point, t, and a logarithmic spiral, the ray that emanates from t and maximizes the ratio C_Π is a tangent to the spiral.*

Proof. Consider the set of rays emanating from the point t, and their first intersection with the spiral; see Figure 7. The ray R_4 achieves the highest ratio among all rays that emanate from t: We can increase the ratio C_Π of any other ray by rotating it counterclockwise around t until the ray is almost a tangent to the spiral.[4] Note that p' in Figure 7 is not actually an intersection, but the searcher moving on the spiral slightly misses the ray R_4 in p', but detects the ray in p_4. However, p' is arbitrarily close to the spiral; thus, we consider p' to be a point on the spiral. We call p' *tangent point.* □

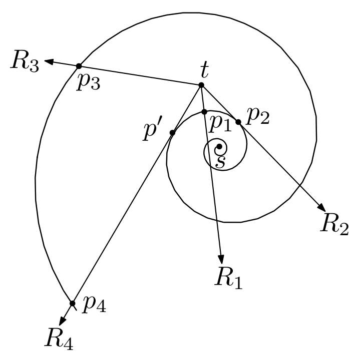

Fig. 7. Different positions of rays.

Position of the starting point of the ray Now, the hider is still free to choose the position of the ray's origin, t, to maximize the search costs $C_\Pi(t)$.

[4] Of course there are two possible tangents; we choose the tangent whose intersection with the spiral is farther away from the ray's origin t.

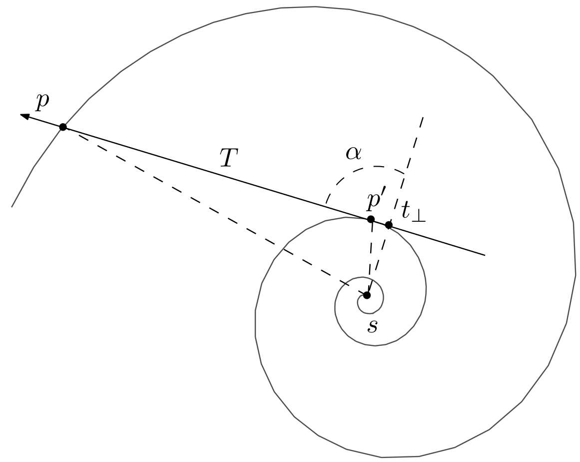

Fig. 8. The tangent T to the spiral in point p'.

We fix a tangent, T, and examine different positions of the ray's origin on T. Let p' be the tangent point for T on the spiral as defined in Lemma 1; see Figure 8.

If we place t in a position between p and p' on T, the resulting ray is no tangent to the spiral. Thus, we consider possible positions for t only on the opposite side.

To find the worst case position for t (i.e., the position that maximizes $C_\Pi(t)$), we can place t in p' and move it along T away from p observing $C_\Pi(t)$; see Figure 8. Let $t_\perp$ be the point on T such that $st_\perp$ is perpendicular to T. It is easy to see that $C_\Pi(t)$ increases if we move t from p' towards $t_\perp$ because we simultaneously increase the numerator and decrease the denominator of $C_\Pi(t)$; see Equation 6. If we move t farther than $t_\perp$, we increase both the numerator and the denominator of $C_\Pi(t)$, so this case requires a more careful analysis. In the following, we give a value for $C_\Pi(t_\perp)$. Then, we examine whether there is a point t right to $t_\perp$ that yields a higher value for $C_\Pi(t)$.

$C_\pi(t)$ depends on the given spiral (i.e., the parameters a and b) and on the ray. By Lemma 1 the ray that maximizes $C_\Pi(t)$ is a tangent to the spiral, so the tangent point, p', is given by $|sp'| = a\mathrm{e}^{b\theta_{p'}}$ for some $\theta_{p'}$.

For convenience, we assume that our searcher starts in the origin and p' is a point on the X-axis, see Figure 9(i). Thus, the searcher makes a number of full turns on the spiral from s to p' and we have

$$\theta_{p'} = k \cdot 2\pi \text{ for some } k \in \mathbb{Z}^+ . \tag{7}$$

Now, we want to compute the distance $|pp'|$. The point p can be computed using the angle $\beta := \angle psp'$; see Figure 9(i). It turns out that β depends only on the spiral parameter b!

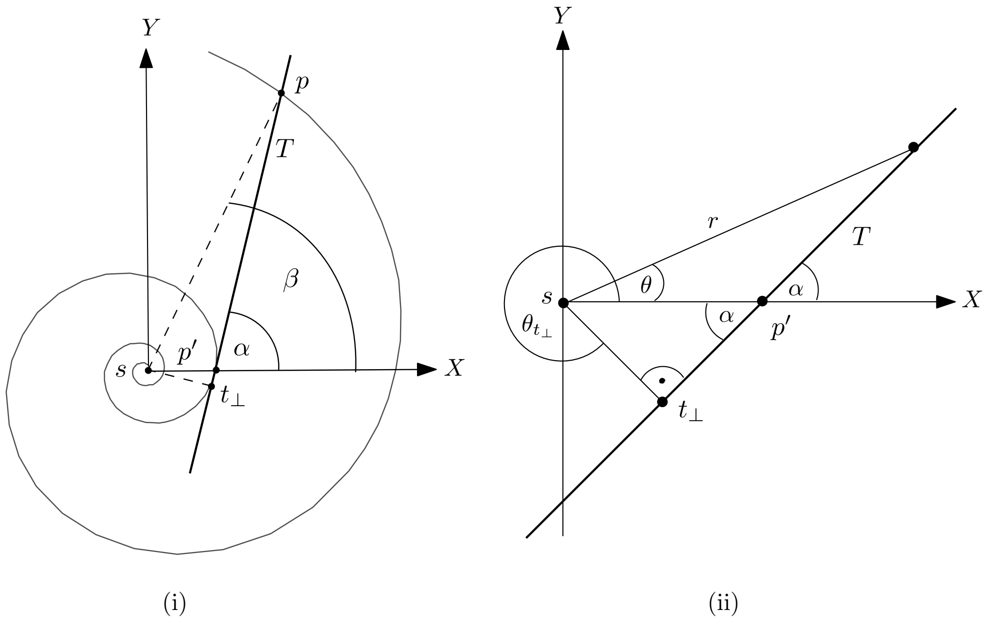

Fig. 9. (i) The angle β, (ii) Tangent T.

Lemma 2. *The angle $\beta(b) := \angle psp'$ is given by the solution to*

$$\frac{\sin\alpha}{\sin(\alpha - \beta(b))} = \mathrm{e}^{b(2\pi+\beta)}\,.$$

Proof. A line running through (r_0, θ_0) and perpendicular to the line $\theta = \theta_0$ is given in polar coordinates by $r(\theta) = \frac{r_0}{\cos(\theta-\theta_0)}$. In our case, $t_\perp$ is perpendicular to our tangent T. Further, $|st_\perp| = |sp'|\sin\alpha = a\mathrm{e}^{b\theta_{p'}}\sin\alpha$ and $2\pi - \theta_{t_\perp} = \frac{\pi}{2} - \alpha$ holds; see Figure 9(ii). Thus, our tangent T is given by

$$r(\theta) = \frac{a\mathrm{e}^{b\theta_{p'}}\sin\alpha}{\sin(\alpha-\theta)}\,.$$

The point p is on the tangent as well as on the spiral; thus, we have

$$r(\theta_p) = \frac{a\mathrm{e}^{b\theta_{p'}}\sin\alpha}{\sin(\alpha-\theta_p)} = a\mathrm{e}^{b\theta_p}\,.$$

From s to p, the seacher moves on the spiral first to p', then a full turn, and last the arc given by $\beta(b)$, so we have $\theta_p = \theta_{p'} + 2\pi + \beta(b) = (k+1)\,2\pi + \beta(b)$. This yields

$$\frac{\mathrm{e}^{b\theta_{p'}}\sin\alpha}{\sin(\alpha-\beta(b))} = \mathrm{e}^{b(\theta_{p'}+2\pi+\beta(b))} \iff \frac{\sin\alpha}{\sin(\alpha-\beta(b))} = \mathrm{e}^{b(2\pi+\beta(b))}\,.$$

Remark that $\beta(b)$ is independent from $\theta_{p'}$; that is, the angle β is the same for every point p on the given spiral! □

Now, we can compute $|pp'|$ using $\beta(b)$, which allows us to prove the following theorem:

Theorem 2. *Given a spiral and a tangent, T, to the spiral, the ratio $C_{\Pi}(t_{\perp})$ depends only on the spiral parameter b and is given by*

$$C_{t_{\perp}}(b) = \frac{\mathrm{e}^{b(2\pi+\beta(b))}}{\sin\alpha \cdot \cos\alpha} + \frac{\mathrm{e}^{b(2\pi+\beta(b))} \cdot \sin(\beta(b))}{\sin^2\alpha} + b\,.$$

Its minimum value is $22.49084026\ldots$ *for* $b = 0.11371\ldots$.

Proof. Consider the triangle $\triangle spp'$; see Figure 9(ii). Because p is a point on the spiral we have $|sp| = a\mathrm{e}^{b\theta_p}$ for some θ_p. As in the proof of Lemma 2, we have $\theta_p = \theta_{p'} + 2\pi + \beta(b)$. Further, we have $\angle sp'p = \pi - \alpha$. Applying the law of sines yields

$$\begin{aligned}
& \frac{|sp|}{\sin(\pi-\alpha)} = \frac{|sp|}{\sin\alpha} = \frac{|pp'|}{\sin\beta(b)} \\
\iff\ & |pp'| = \frac{a\mathrm{e}^{b\theta_p}\sin\beta(b)}{\sin\alpha} = \frac{a\mathrm{e}^{b(\theta_{p'}+2\pi+\beta(b))}\sin\beta(b)}{\sin\alpha}.
\end{aligned}$$

Because the triangle $\triangle sp't$ is right angled we have $|p't_{\perp}| = |sp'|\cos\alpha = a\mathrm{e}^{b\theta_{p'}}\cos\alpha$; thus, the distance $|pt_{\perp}| = |pp'| + |p't_{\perp}|$ is given as

$$|pt_{\perp}| = \frac{a\mathrm{e}^{b(\theta_{p'}+2\pi+\beta(b))}\sin\beta(b)}{\sin\alpha} + a\mathrm{e}^{b\theta_{p'}}\cos\alpha\,.$$

The length of the arc Π on the spiral from s to p is given by $|\Pi| = \frac{\sqrt{1+b^2}}{b}\, r(\theta_p) = \frac{\sqrt{1+b^2}}{b}\, a\mathrm{e}^{b\theta_p}$ [4]. With $b = \cot\alpha$, we have

$$\frac{\sqrt{1+b^2}}{b} = \frac{\sqrt{1+\cot^2\alpha}}{\cot\alpha} = \sqrt{\frac{1}{\cot^2\alpha}\left(1+\cot^2\alpha\right)} = \sqrt{\frac{\sin^2\alpha\cdot\cos^2\alpha}{\cos^2\alpha}} = \frac{1}{\cos\alpha}\,,$$

Now, using $|st_{\perp}| = |sp'|\sin\alpha = a\mathrm{e}^{b\theta_{p'}}\sin\alpha$, we can compute the ratio $C_{t_{\perp}}(b)$:

$$\begin{aligned}
C_{t_{\perp}}(b) &= \frac{|\Pi| + |pt_{\perp}|}{|st_{\perp}|} \\
&= \frac{\frac{1}{\cos\alpha}\, a\mathrm{e}^{b(\theta_{p'}+2\pi+\beta(b))} + \frac{1}{\sin\alpha}\, a\mathrm{e}^{b(\theta_{p'}+2\pi+\beta(b))}\, \sin\beta(b) + a\mathrm{e}^{b\theta_{p'}}\cos\alpha}{a\mathrm{e}^{b\theta_{p'}}\sin\alpha} \\
&= \frac{\frac{1}{\cos\alpha}\,\mathrm{e}^{b(2\pi+\beta(b))} + \frac{1}{\sin\alpha}\,\mathrm{e}^{b(2\pi+\beta(b))}\,\sin\beta(b) + \cos\alpha}{\sin\alpha} \\
&= \left(\frac{1}{\sin\alpha\cdot\cos\alpha} + \frac{\sin\beta(b)}{\sin^2\alpha}\right)\mathrm{e}^{b(2\pi+\beta(b))} + \cot\alpha\,.
\end{aligned}$$

We observe that $C_{t_{\perp}}(b)$ is independent of $\theta_{p'}$; that is, the value of $C_{\Pi}(t_{\perp})$ is the same for every given tangent T. Now, the searcher is allowed to minimize the search costs by choosing an appropriate value for b. Evaluating $C_{t_\perp}(b)$ numerically yields a minimum value of $22.49084026\ldots$ for $b = 0.11371\ldots$. □

Remark that every tangent to a given spiral yields the same value for $C_\Pi(t_\perp)$! So far, we have found the best achievable ratio C_Π for the case that the hider chooses $t_\perp$. Further, we found a value for b that yields the optimal spiral for all tangents in this case.

In the following, we examine whether there is a point t right to $t_\perp$ that achieves a ratio that is worse than the ratio of $t_\perp$. As mentioned above, we move the point t along the tangent T. Let angle γ denote the angle $\angle tst_\perp$.

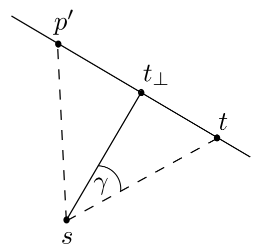

Fig. 10. The triangle $st_\perp t$

Theorem 3. *The best achievable value for C_π is $22.51306056\ldots$ and is achieved by the point t which is specified by $\gamma = 0.4443328023\ldots$ and $b = 0.1137\ldots$.*

Proof. Since the $\triangle st_\perp t$ is right angled, we have $\sin\gamma = \frac{|tt_\perp|}{|st|}$ and $|st| = \frac{|st_\perp|}{\cos\gamma}$. $C_\Pi(t)$ depends on γ and b:

$$\begin{aligned} C_t(b,\gamma) &= \frac{|\Pi| + |pt_\perp| + |t_\perp t|}{|st|} = \frac{|\Pi| + |pt_\perp|}{|st_\perp|}\cos\gamma + \frac{|t_\perp t|}{|st|} \\ &= C_{t_\perp}(b)\,\cos\gamma + \sin\gamma \end{aligned}$$

As $0 < \gamma < \frac{\pi}{2}$ holds, we have $C_{t_\perp}(b) < C_t(b,\gamma)$. Now, the hider can maximize $C_t(b,\gamma)$ using γ and the searcher can minimize $C_t(b,\gamma)$ by choosing an appropriate b. Since γ is independent from b, the searcher can use the results from Theorem 2 to minimize the ratio. Numerical analysis shows that $\gamma = 0.4443328023\ldots$ yields the minimum $C_\Pi(t) = 22.51306056\ldots$ for $C_\Pi(t_\perp) = 22.49084026\ldots$. This completes our proof. □

We can see that t lies in fact very close to $t_\perp$.

4 A Lower Bound for Searching a Ray

To get a lower bound on the competitive ratio for our problem, we discuss the following subproblem: We require that the ray, R, we are looking for is part of

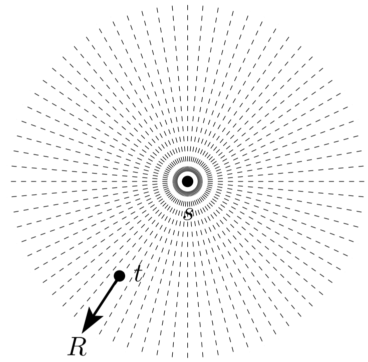

Fig. 11. A ray, R, that emanates from t and is part of a ray that emanates from s.

a rays that emanates from the searcher's start point, s (i.e., the start point, s, lies on the the extension of R to a straight line)

If we consider the full bundle of lines passing through s, the given problem is equivalent to the problem of searching for a point in the plane as presented by Alpern and Gal [1]. We assume that the searcher detects the goal if it is swept by the radius vector of its trajectory; that is, the searchers knows the position of the goal as soon as it hits the ray emanating from the goal. Alpern and Gal [1] showed that among all monotone and periodic strategies, a logarithmic spiral represented by polar coordinates $(\theta, \mathrm{e}^{b\theta})$ gives the best search strategy in this setting. A strategy S represented by its radius vector $X(\theta)$ is called periodic and monotone if θ is always increasing and X also satisfies $X(\theta + 2\pi) \geq X(\theta)$.

The factor of the best achievable monotone and periodic strategy is given by $\min_b \mathrm{e}^{2\pi} b\sqrt{1 + \frac{1}{b^2}} = 17.289\ldots$ and achieves its minimum for $b = 0.15540\ldots$, see Alpern and Gal [1]. Note, that the searcher does not have to reach the ray's origin in this setting.

Unfortunately, it was not shown that a periodic and monotone strategy is the best strategy for this problem. Alpern and Gal state that it *seems to be a complicated task* to show that the spiral optimizes the competitive factor. Thus, the given factor cannot be adapted to be a lower bound to our problem. Therefore, we consider a discrete bundle of n rays that emanate from the start and which are separated by an angle $\alpha = \frac{2\pi}{n}$, see Figure 12. We are searching for a goal on one of the n rays.[5] Again, the goal is detected if it is swept by the radius vector of the trajectory. Note that if n goes to infinity we are back to the original problem. But we can neither assume that we have to visit the rays in a periodic order nor that the depth of the visits increases in every step. Thus, we represent a search strategy, S, as follows: In the kth step, the searcher hits a ray—say ray i—at distance x_k from the origin, moves a distance $\beta_k x_k - x_k$

[5] Note that the searcher is not confined to walk on the rays, but can move arbitrarily in the plane; in contrast to the m-ray search problem.

along the ray i, and leaves the ray at distance $\beta_k x_k$ with $\beta_k \geq 1$. Then, it moves to the next ray within distance $\sqrt{(\beta_k x_k)^2 - 2\beta_k x_k x_{k+1} \cos\gamma + x_{k+1}^2}$, see Figure 12. Note that any search strategy for our problem can be described in this way.

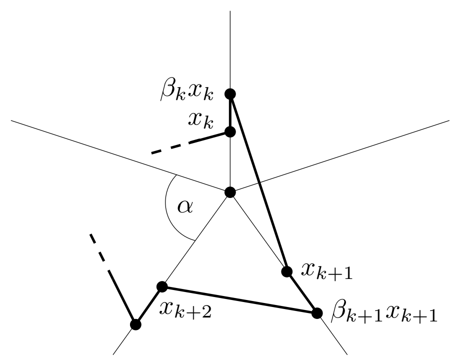

Fig. 12. A bundle of n rays and the representation of a strategy.

Let us assume that the ray i is visited the next time at index J_k. The worst case occurs if the searcher slightly misses the goal while visiting ray i up to distance x_k. Instead, it finds the goal at step x_{J_k} on ray i arbitrarily close to $\beta_k x_k$. Either we have $x_{J_k} > \beta_k x_k$; that is, the searcher discovers the goal in distance x_{J_k} on ray i and moves $x_{J_k} - \beta_k x_k$ to the goal, or we have $x_{J_k} < \beta_k x_k$. In the latter case, the searcher moves $\beta_k x_k - x_{J_k}$ from x_{J_k} and finds the goal by accident. In both cases, the searcher moves $|x_{J_k} - \beta_k x_k|$ in the last step. Altogether, the competitive factor, $C(S)$, is bigger than

$$\frac{|x_{J_k} - \beta_k x_k| + \sum_{i=1}^{J_k-1} \beta_i x_i - x_i + \sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1} \cos\gamma + x_{i+1}^2}}{\beta_k x_k}.$$

By simple trigonometry, the shortest distance from $\beta_i x_i$ to a neighboring ray is given by $\beta_i x_i \sin\frac{2\pi}{n}$. Fortunately, this distance is smaller than the distance $\sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1} \cos\gamma + x_{i+1}^2}$ to any other ray. Thus, we have

$$C(S) > \frac{\sum_{i=1}^{J_k-1} \beta_i x_i}{\beta_k x_k} \sin\frac{2\pi}{n}.$$

Altogether, we have to find a lower bound for $\frac{\sum_{i=1}^{J_k-1} f_i}{f_k}$, where J_k denotes the index of the next visit of the ray of x_k and $f_i = \beta_i x_i$ denotes the search depth in step i. Fortunately, this problem is the same problem is in the competitive analysis for the usual m-ray problem where the searcher can move only along the rays. It was shown by Gal [9] and Baeza-Yates et al. [2] that for this problem there is an optimal strategy that visits the rays with increasing depth and in a periodic order; that is, $J_k = k + n$ and $i = k$. The best achievable strategy is given by $f_i = (n/(n-1))^i$. Altogether, this results in a function

$$(n-1)\left(\frac{n}{n-1}\right)^n \sin\frac{2\pi}{n}$$

for n rays. We can make n arbitrarily big because our construction is valid for every n. Note that we also have a lower bound for the problem of searching a point in the plane; this lower bound is close to the factor that is achieved by a spiral search.

Theorem 4. *For the ray search problem there is no strategy that achieves a better factor than*

$$\lim_{n\to\infty}(n-1)\left(\frac{n}{n-1}\right)^n \sin\frac{2\pi}{n} = 17.079\ldots$$

Additionally, every strategy for searching a point in the plane achieves a competitive factor bigger then $17.079\ldots$ *(the optimal spiral achieves a factor of* $17.289\ldots$ *[9]).*

5 Summary

We considered the problem of searching a ray and its origin using the competitive framework.

If the ray starts on a known ray r' and is also perpendicular to r' we will find the origin within a path length of $1.059\ldots$ times the shortest path to the origin. This factor is optimal.

In general, a logarithmic spiral solves the task with a competitive factor of $22.51\ldots$ whereas a lower bound of $17.079\ldots$ is given.

The lower bound construction can also be used if it is not necessary to visit the origin and if the corresponding line of every ray goes through the starting point. For this subproblem a competitive strategy with factor $17.289\ldots$ was already known. We showed that there is no strategy with a factor better than $17.079\ldots$ in this setting.

Unfortunately, there are still some gaps between the lower and upper bounds of the factors which remain to be closed.

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