# Natural Halting Probabilities, Partial Randomness, and Zeta Functions Extended Abstract

Cristian S. Calude[1] and Michael A. Stay[2]

[1]Department of Computer Science
University of Auckland, New Zealand
Email: cristian@cs.auckland.ac.nz

[2]Department of Mathematics
University of California Riverside, USA
Email: mike@math.ucr.edu

We introduce the *zeta number, natural halting probability* and *natural complexity* of a Turing machine and we relate them to Chaitin's Omega number, halting probability, and program-size complexity. A classification of Turing machines according to their natural zeta numbers is proposed: divergent (zeta number is infinite), convergent (zeta number is finite), and tuatara (zeta number is less or equal to one). Every self-delimiting Turing machine is tuatara, but the converse is not true. Also, there exist universal convergent and tuatara machines.

The zeta number of a universal self-delimiting Turing machines is c.e. and random, and for each tuatara machine there effectively exists a self-delimiting Turing machine whose Chaitin halting probability equals its zeta number; if the tuatara machine is universal, then the self-delimiting Turing machine can also be taken to be universal.

For each self-delimiting Turing machine there is a tuatara machine whose zeta number is exactly the Chaitin halting probability of the self-delimiting Turing machine; it is an open problem whether the tuatara machine can be chosen to be a universal self-delimiting Turing machine in the case when the original machine is universal.

Let $s > 1$ be a computable real, $T$ a universal Turing machine, and $K_T$ be the plain complexity induced by $T$. A string $x$ is $1/s - K$-random if $K_T(x) \geq m/s - c$, for some $c \geq 0$. In analogy with the notion of Chaitin partial random real we introduce the notion of a "$1/s$-$K$-random real" (a real such that the prefixes of its binary expansion are $1/s - K$-random) as well as the notion of an "asymptotically $K$-random real" ($1/s$-$K$-random real, for every computable $s > 1$). The result due to Chaitin and Martin-Löf showing that the plain complexity $K$ cannot characterise random reals is no longer true for $1/s - K$-random (or Chaitin $1/s-$random reals), nor for asymptotically $K$-random reals. The zeta number of a universal self-delimiting Turing machine is asymptotically $K$-random, but the converse implication fails to be true: there exists a self-delimiting Turing machine whose zeta number is asymptotically $K$-random, but not random.

Some open problems conclude the paper.

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