ENCLOSURE FOR THE BIHARMONIC EQUATION

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Abstract In this paper we give an enclosure for the solution of the biharmonic problem and also for its gradient and Laplacian in the L_2-norm, respectively.

1. Introduction

The linear biharmonic problem to be studied here is the following: Let $\Omega \subset \mathbb{R}^2$ be a simply connected, bounded Lipschitz domain, $r \in L_2(\Omega)$, and we consider the boundary value problem

$$\begin{aligned} \Delta^2 u = \; & r \qquad \text{on } \Omega, \\ u = \frac{\partial u}{\partial \nu} = \; & 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

with ν denoting the outward pointing unit normal field to $\partial\Omega$. There are several solution approaches. One can investigate classical solutions, i.e., functions $u \in C^4(\Omega) \cap C^1(\overline{\Omega})$ solving our problem in the classical sense. A more general solution approach is the one of strong solutions, i.e., one looks for a function $u \in H^4(\Omega) \cap H_0^2(\Omega)$ that solves our problem. We can get weaker conditions on the solution, if we consider the weak formulation of the problem, i.e., we look for $u \in H_0^2(\Omega)$ that fulfills

$$\int_\Omega \Delta u \Delta \varphi = \int_\Omega r\varphi \quad \text{for all } \varphi \in H_0^2(\Omega). \tag{1}$$

Such a function u is called a weak solution.

As one of our aims is to compute numerical approximations to the solution, we would need, in the finite elements context, C^4-, C^3- and C^1-elements, respectively, to approximate these kinds of solution functions. This would be numerically too "expensive".

Therefore we reformulate the problem as a system of second order:

$$\begin{aligned} -\Delta u &= v, \\ -\Delta v &= r. \end{aligned}$$

Strong solutions of this system satisfy $u \in H_0^2(\Omega)$, $v \in H^2(\Omega)$. For numerical approximations we still would need C^1-elements. To avoid this, we rewrite the system in weak formulation, i.e.,

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla \varphi &= \int_\Omega v \cdot \varphi \quad \text{for all } \varphi \in H^1(\Omega), \\ \int_\Omega \nabla v \cdot \nabla \psi &= \int_\Omega r \cdot \psi \quad \text{for all } \psi \in H_0^1(\Omega), \end{aligned} \tag{2}$$

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and consider weak solutions of this problem, i.e., $u \in H_0^1(\Omega)$, $v \in H^1(\Omega)$. Then by constructing finite element approximations one needs only C^0-elements. This is numerically much cheaper than the above mentioned approximations.

Our ***aim*** is therefore the following: First we construct $H^1(\Omega)$-approximations to u, v, ∇u, ∇v (it means C^0-elements in the finite element method). Then using only these approximations we give an enclosure for the true solution $u^* \in H_0^2(\Omega)$ of the weak problem (1), for its gradient and Laplacian, respectively, in the L_2-norm.

2. Results

Suppose that numerical approximations $\tilde{u} \in H_0^1(\Omega)$ to u, $\tilde{\sigma} \in (H_0^1(\Omega))^2$ to ∇u, $\tilde{v} \in H^1(\Omega)$ to v and $\tilde{\rho} \in H(\text{div}, \Omega)$ to ∇v, with u, v denoting the solutions to problem (2), have been computed.

At this point we can choose between two possibilities:
A) we construct $\tilde{\sigma}$ with the additional condition curl $\tilde{\sigma} = 0$. It means more numerical effort. Alternatively,
B) we construct $\tilde{\sigma}$ without additional condition. This implies additional analytical requirements, as we will see later.

In what follows we consider the two approaches separately. First we construct in both cases an auxiliary function $\hat{u} \in H_0^2(\Omega)$. We denote by t the unit tangential field to $\partial\Omega$, with (t, ν) positively oriented.

Approach A: As curl $\tilde{\sigma} = 0$, there exists $\hat{u} \in H^2(\Omega)$ such that $\nabla \hat{u} = \tilde{\sigma}$. Because of $\tilde{\sigma} \in (H_0^1(\Omega))^2$ it holds that $\tilde{\sigma} \cdot t = 0$ on $\partial\Omega$ which means $\frac{\partial \hat{u}}{\partial t} = 0$. Since Ω is simply connected, this yields that $\hat{u}$ is constant on $\partial\Omega$. W.l.o.g one can set $\hat{u} = 0$ on $\partial\Omega$. Furthermore, from $\tilde{\sigma} \cdot \nu = 0$ follows $\frac{\partial \hat{u}}{\partial \nu} = 0$. We have therefore proved that $\hat{u} \in H_0^2(\Omega)$.

Approach B: First we need a lemma. The proof is simple.

Lemma 2.1. *Let us denote the unit disc of $\mathbb{R}^2$ by $\mathbb{D}$. Let $h \colon \mathbb{D} \to \mathbb{R}$ with $\int_{\mathbb{D}} h \, dxdy = 0$ be harmonic, $g \colon \mathbb{D} \to \mathbb{R}$ the harmonic conjugate of h, such that $\int_{\mathbb{D}} g \, dxdy = 0$. Then for $C = 1$ it holds that*

$$\|g\|_2 \leq C \cdot \|h\|_2. \tag{3}$$

It would be nice to have such an estimate on general simply connected domains as well. So far, however, we could not prove (3) in the general case, whence we can use Approach B for the unit disc only. But our *conjecture* is, that if Ω is any bounded simply connected domain in $\mathbb{R}^2$, then there exists a (computable) constant C, such that (3) holds.

Now with the help of Lemma 2.1 we can define our auxiliary function: let $\hat{u} \in H_0^2(\Omega)$ be defined via

$$\begin{aligned} \Delta^2 \hat{u} &= \Delta \text{div } \tilde{\sigma}, \\ \text{i.e.,} \qquad < \Delta \hat{u}, \Delta \varphi > &= < \text{div } \tilde{\sigma}, \Delta \varphi > \quad \forall \varphi \in H_0^2(\Omega). \end{aligned}$$

Denote $\hat{\sigma} = \nabla \hat{u}$.

The functions $\hat{u}$ and $\hat{\sigma}$ will not be constructed by numerical means. To enable us to use them in our calculations, it must be proved that $\hat{\sigma}$ is in some sense not far away from the approximation function $\tilde{\sigma}$. This is the statement of the following lemma.

Lemma 2.2. *For the approximation function $\tilde{\sigma}$ and the auxiliary function $\hat{\sigma}$ defined above the following estimates hold:*

$$\begin{aligned} \|div\ (\hat{\sigma}-\tilde{\sigma})\|_2 &\leq C\cdot\|curl\ \tilde{\sigma}\|_2\,, &(4)\\ \|\hat{\sigma}-\tilde{\sigma}\|_2 &\leq C_1\cdot\sqrt{C^2+1}\cdot\|curl\ \tilde{\sigma}\|_2\,, &(5)\end{aligned}$$

with C from Lemma 2.1 *and the embedding constant C_1 for the embedding $H_0^1(\Omega)\hookrightarrow L_2(\Omega)$.*

Proof of (4). We use the Helmholtz-decomposition for the function $\tilde{\sigma}$: There exist $\omega\in H_0^1(\Omega)$ and $\psi\in H^2(\Omega)$ such that

$$\tilde{\sigma}=\nabla\omega+\begin{pmatrix}-\psi_y\\ \psi_x\end{pmatrix}.$$

As $\tilde{\sigma}\in(H_0^1(\Omega))^2$ and $\omega\in H_0^1(\Omega)$ it follows that $\tilde{\sigma}\cdot t=\nabla\omega\cdot t=0$ on $\partial\Omega$. This means $\binom{-\psi_y}{\psi_x}\cdot t=0$ on $\partial\Omega$, i.e., $\frac{\partial\psi}{\partial\nu}=0$ on $\partial\Omega$. Furthermore,

$$\begin{aligned}\Delta\psi &= \frac{\partial}{\partial x}(\psi_x)+\frac{\partial}{\partial y}(\psi_y)\\ &= \frac{\partial}{\partial x}(\tilde{\sigma}-\nabla\omega)_2-\frac{\partial}{\partial y}(\tilde{\sigma}-\nabla\omega)_1\\ &= \frac{\partial\tilde{\sigma}_2}{\partial x}-\frac{\partial\tilde{\sigma}_1}{\partial y}\\ &= \text{curl }\tilde{\sigma}.\end{aligned}$$

From the definition of $\hat{u}$ the function $h=\Delta\hat{u}-\text{div }\tilde{\sigma}$ is harmonic. We give an upper estimate for the L_2-norm of h, which yields the result for $\hat{\sigma}$ we would like to obtain. We denote the harmonic conjugate of h by g, with normalization $\int_\Omega g\,dxdy=0$. Then,

$$\begin{aligned}\|h\|_2^2 &=< \Delta\hat{u},h> - <\text{div }\tilde{\sigma},h>\\ &= \underbrace{<\hat{u},\Delta h>}_{=0\text{ as }\Delta h=0}+<\tilde{\sigma},\nabla h>\\ &= <\nabla\omega,\nabla h>+<\begin{pmatrix}-\psi_y\\ \psi_x\end{pmatrix},\nabla h>\\ &= -<\omega,\Delta h>+<\begin{pmatrix}-\psi_y\\ \psi_x\end{pmatrix},\begin{pmatrix}g_y\\ -g_x\end{pmatrix}>\\ &= -<\nabla\psi,\nabla g>\\ &= <\Delta\psi,g>\\ &= <\text{curl }\tilde{\sigma},g>\\ &\leq \|\text{curl }\tilde{\sigma}\|_2\|g\|_2\\ &\leq C\cdot\|\text{curl }\tilde{\sigma}\|_2\|h\|_2\,,\end{aligned}$$

with C from Lemma 2.1. Thus,

$$C \cdot \|\text{curl } \tilde{\sigma}\|_2 \geq \|h\|_2 = \|\Delta\hat{u} - \text{div } \tilde{\sigma}\|_2 = \|\text{div } (\hat{\sigma} - \tilde{\sigma})\|_2,$$

which is the inequality we wanted to get.

Proof of (5). Let us denote $\sigma = \hat{\sigma} - \tilde{\sigma}$. Then $\sigma \in (H_0^1(\Omega))^2$, thus $\|\sigma_i\|_2 \leq C_1 \|\nabla\sigma_i\|_2$ $(i = 1, 2)$, where C_1 is the embedding constant for the embedding $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$. Then

$$\|\sigma_1\|_2^2 + \|\sigma_2\|_2^2 \leq C_1^2 \left(\|\nabla\sigma_1\|_2^2 + \|\nabla\sigma_2\|_2^2\right)$$

$$= C_1^2 \left(\left\|\frac{\partial\sigma_1}{\partial x}\right\|_2^2 + \left\|\frac{\partial\sigma_1}{\partial y}\right\|_2^2 + \left\|\frac{\partial\sigma_2}{\partial x}\right\|_2^2 + \left\|\frac{\partial\sigma_2}{\partial y}\right\|_2^2 \right)$$

$$= C_1^2 \left(\|\text{div } \sigma\|^2 + \|\text{curl } \sigma\|^2 - 2\int_\Omega \frac{\partial\sigma_1}{\partial x}\frac{\partial\sigma_2}{\partial y} - \frac{\partial\sigma_2}{\partial x}\frac{\partial\sigma_1}{\partial y}\, dxdy \right)$$

$$= C_1^2 \left(\|\text{div } (\hat{\sigma} - \tilde{\sigma})\|^2 + \|\text{curl } \tilde{\sigma}\|^2\right) \leq C_1^2 \left(C^2 \|\text{curl } \tilde{\sigma}\|^2 + \|\text{curl } \tilde{\sigma}\|^2\right) .$$

□

The main result of this paper is the following theorem. We enclose the true solution, its gradient and its Laplacian of problem (1) in the L_2-norm, respectively.

Theorem 2.3. *Denoting the solution of (1) by $u^* \in H_0^2(\Omega)$ it holds that*

$$\begin{aligned} \|\Delta u^* - \mathit{div}\ \tilde{\sigma}\|_2 &\leq F(\tilde{u}, \tilde{\sigma}, \tilde{v}, \tilde{\rho}), \\ \|\nabla u^* - \tilde{\sigma}\|_2 &\leq G(\tilde{u}, \tilde{\sigma}, \tilde{v}, \tilde{\rho}), \\ \|u^* - \tilde{u}\|_2 &\leq H(\tilde{u}, \tilde{\sigma}, \tilde{v}, \tilde{\rho}), \end{aligned}$$

with computable (and "small") $F(\tilde{u}, \tilde{\sigma}, \tilde{v}, \tilde{\rho})$, $G(\tilde{u}, \tilde{\sigma}, \tilde{v}, \tilde{\rho})$, $H(\tilde{u}, \tilde{\sigma}, \tilde{v}, \tilde{\rho})$.

Proof. First we estimate the term $\|\Delta u^* - \Delta\hat{u}\|_2$ from above with computable terms that we expect to be small:

$$\|\Delta u^* - \Delta\hat{u}\|_2 = \sup_{\varphi \in H_0^2(\Omega)} \frac{< \Delta(u^* - \hat{u}), \Delta\varphi >}{\|\Delta\varphi\|_2} = \sup_{\varphi \in H_0^2(\Omega)} \frac{(\Delta^2(u^* - \hat{u}))[\varphi]}{\|\varphi\|_{H_0^2(\Omega)}}$$

$$= \|\Delta^2 u^* - \Delta^2\hat{u}\|_{H^{-2}(\Omega)} = \|r - \Delta\text{div } \tilde{\sigma}\|_{H^{-2}(\Omega)}$$

$$\leq \|r + \text{div } \tilde{\rho}\|_{H^{-2}(\Omega)} + \|\Delta\tilde{v} - \text{div } \tilde{\rho}\|_{H^{-2}(\Omega)} + \|\Delta\tilde{v} + \Delta\text{div } \tilde{\sigma}\|_{H^{-2}(\Omega)}$$

$$\leq C_2 \cdot \|r + \text{div } \tilde{\rho}\|_2 + C_3 \cdot \|\nabla\tilde{v} - \tilde{\rho}\|_2 + \|\tilde{v} + \text{div } \tilde{\sigma}\|_2 ,$$

with embedding constants C_2 and C_3 for the embeddings $H_0^2(\Omega) \hookrightarrow L_2(\Omega)$ and $H_0^2(\Omega) \hookrightarrow H_0^1(\Omega)$, respectively. Now it is enough to estimate the terms $\|\Delta u^* - \text{div } \tilde{\sigma}\|_2$ and $\|\nabla u^* - \tilde{\sigma}\|_2$ from above in terms of $\|\Delta u^* - \Delta\hat{u}\|_2$ and $\|\text{curl } \tilde{\sigma}\|_2$, as the latter is also computable and we expect it to be small.

In approach A)

$$\|\Delta u^* - \text{div } \tilde{\sigma}\|_2 \overset{\tilde{\sigma} = \nabla\hat{u}}{=} \|\Delta u^* - \Delta\hat{u}\|_2 ,$$

$$\|\nabla u^* - \tilde{\sigma}\|_2 \overset{\tilde{\sigma} = \nabla\hat{u}}{=} \|\nabla u^* - \nabla\hat{u}\|_2 \leq C_3 \cdot \|\Delta u^* - \Delta\hat{u}\|_2 .$$

In approach B)

$$\begin{aligned}\|\Delta u^* - \text{div } \tilde\sigma\|_2 &\le \|\Delta u^* - \Delta\hat u\|_2 + \|\Delta\hat u - \text{div } \tilde\sigma\|_2 \\ &\le \|\Delta u^* - \Delta\hat u\|_2 + \|\text{div } \hat\sigma - \text{div } \tilde\sigma\|_2 \\ &\le \|\Delta u^* - \Delta\hat u\|_2 + C\cdot\|\text{curl } \tilde\sigma\|_2\end{aligned}$$

and

$$\begin{aligned}\|\nabla u^* - \tilde\sigma\|_2 &\le \|\nabla u^* - \nabla\hat u\|_2 + \|\nabla\hat u - \tilde\sigma\|_2 \\ &\le C_3\cdot\|\Delta u^* - \Delta\hat u\|_2 + \|\hat\sigma - \tilde\sigma\|_2 \\ &\le C_3\cdot\|\Delta u^* - \Delta\hat u\|_2 + C_1\cdot\sqrt{C^2+1}\cdot\|\text{curl } \tilde\sigma\|_2\,.\end{aligned}$$

For the estimation of $\|u^* - \tilde u\|_2$ we can use, in *both approaches*, the above terms and the computable term $\|\tilde\sigma - \nabla\tilde u\|_2$:

$$\begin{aligned}\|u^* - \tilde u\|_2 &\le C_1\cdot\|\nabla u^* - \nabla\tilde u\|_2 \\ &\le C_1\cdot\big(\|\nabla u^* - \tilde\sigma\|_2 + \|\tilde\sigma - \nabla\tilde u\|_2\big)\,.\end{aligned}$$

□

Remark 2.4. To give upper bounds for the embedding constants C_1, C_2, C_3 one can use bounds for the smallest eigenvalues of appropriate eigenvalue problems.

Example 2.5. We consider the following example: Let $\Omega = (-1,1)\times(-1,1)$ and $r \equiv 1$. Now it holds that $C_1 = \frac{\sqrt2}{\pi}$, $C_2 \le \frac{2}{\pi^2}$, $C_3 \le \frac{\sqrt2}{\pi}$. We calculated the approximating functions $\tilde u, \tilde\sigma, \tilde v, \tilde\rho$ using linear triangular finite elements, without the additional condition curl $\tilde\sigma = 0$. The results of the computation of the terms occuring in the functions F, G, H from Theorem 2.3 are as follows. (The levels denote the number of the triangles in the triangulation; starting with 2 elements at level 1 every triangle is quarteled when the level is increased by 1.)

level	$\|\nabla\tilde v - \tilde\rho\|$	$\|\text{div } \tilde\rho + r\|$	$\|\text{div } \tilde\sigma + \tilde v\|$	$\|\text{curl } \tilde\sigma\|$	$\|\tilde\sigma - \nabla\tilde u\|$
3	0.3111	0.2794	0.0289	0.0260	0.0071
4	0.1303	0.1703	0.0146	0.0145	0.0040
5	0.0617	0.0954	0.0073	0.0075	0.0021
6	0.0333	0.0513	0.0037	0.0038	0.0011

We would need the constant C as well to calculate the values F, G, H from these terms. But as already mentioned after Lemma 2.1 we could only determine it for the unit disc so far. Its computation for the rectangular domain Ω is still an open problem.

One could also use approach A, which would require to compute an (exactly) curl-free approximation $\tilde\sigma$. This needs interval-arithmetical computations within the process of calculating $\tilde\sigma$, which is still left to do.

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