Verification of Solutions for Almost Linear Complementarity Problems

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January 27, 2005

Abstract

The paper establishes a computational enclosure of the solution of a nonlinear complementarity problem $x \geq 0, l(x) \geq 0, x^T l(x) = 0$, where $l(x) = Mx + \Phi(x)$ is a so-called almost linear mapping with an H-matrix M with positive diagonal elements and an increasing diagonal mapping Φ. The procedure also delivers a simple proof for the uniqueness of the solution.

Mathematics Subject Classification (2000): 90C33, 65G30, 65K10
Keywords: Almost linear complementarity problem, existence of solutions, uniqueness of solution, enclosure of solutions, interval analysis, H-matrix.

1 Introduction

Let there be given a (nonlinear) mapping

$$l : D \subset \mathbb{R}^n \to \mathbb{R}^n.$$

We consider the problem of finding a vector x such that

$$\left.\begin{array}{rcl} x & \geq & 0 \\ l(x) & \geq & 0 \\ x^T l(x) & = & 0 \end{array}\right\} \tag{1}$$

(or to show that no such x exists). This problem is called nonlinear complementarity problem (NCP) and has many applications. See [6] and [7], for example. It is easy to show that (1) is equivalent to solving the nonlinear system of equations

$$\min\{x, l(x)\} = 0. \tag{2}$$

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Dagstuhl Seminar Proceedings 05391
Algebraic and Numerical Algorithms and Computer-assisted Proofs
http://drops.dagstuhl.de/opus/volltexte/2006/443

If Δ is a nonsingular diagonal matrix with positive diagonal elements, then (2) is equivalent to the system

$$\min\{x, \Delta l(x)\} = 0, \tag{3}$$

which in turn is equivalent to finding a fixed point of the mapping p defined by

$$p\,(x) = \max\{0, x - \Delta l(x)\}. \tag{4}$$

In (2), (3) and (4) the minimum and the maximum are taken componentwise, respectively.

Let $h(x) = x - \Delta l(x)$. Let the derivative $l'(x)$ exist and denote by $l'([x])$ its interval arithmetic evaluation for the interval vector $[x]$. Assume now that $x^* \in [x]$. Then

$$h(x^*) - h(x) = J(x^*, x)(x^* - x) \in h'([x])([x] - x)$$

for an arbitrary fixed $x \in [x]$. $J(x^*, x)$ denotes the Jacobian of h taken row-wise at some intermediate point. Since

$$h'(x) = I - \Delta l'(x),$$

we obtain

$$\begin{aligned} h(x^*) &\in h(x) + h'([x])([x] - x) \\ &= x - \Delta l(x) + (I - \Delta l'([x]))([x] - x). \end{aligned} \tag{5}$$

For a real interval $[a] = [\underline{a}, \overline{a}]$ we define

$$\max\{0, [a]\} = \begin{cases} 0, & \overline{a} < 0 \\ [0, \overline{a}], & 0 \in [a] \\ [a] & \underline{a} > 0 \end{cases}\ . \tag{6}$$

For an interval vector $[a] = ([a_i]), \max\{0, [a]\}$ is defined componentwise. It holds that

$$[a] \subseteq [b] \Rightarrow \max\{0, [a]\} \subseteq \max\{0, [b]\}. \tag{7}$$

Assume now that $x^* \in [x]$ is a fixed point of p. Then using (5) and (7), it follows that

$$\left.\begin{aligned} x^* = p(x^*) \ &= \max\{0, x^* - \Delta l(x^*)\} \\ &\subseteq \max\{0, x - \Delta l(x) + (I - \Delta l'([x]))([x] - x)\} \\ &=: \Gamma(x, [x], \Delta), \end{aligned}\right\} \tag{8}$$

where $x \in [x]$ is arbitrary, but fixed.

The following theorem indeed guarantees the existence of a fixed point x^* in $\Gamma(x, [x], \Delta)$.

Theorem 1 Let $[x]$ be an interval vector and let $l'([x])$ denote the interval arithmetic evaluation of the derivative $l'(x)$. If for some diagonal matrix Δ with positive elements in the diagonal and some $x \in [x]$

$$\Gamma(x, [x], \Delta) \subseteq [x] \tag{9}$$

holds, then there exists a fixed point $x^* \in [x]$ of the mapping p defined by (4). By the preceding remarks, x^* is also contained in $\Gamma(x, [x], \Delta)$.

The proof of theorem 1 has been given in [5].

If (9) holds, we can try to improve the enclosure of x^* by the following method

$$\left.\begin{array}{rl} [x^0] := & [x] \\ [x^{k+1}] := & \Gamma(x^k, [x^k], \Delta) \cap [x^k],\ k = 0, 1, 2, \ldots, \\ & \text{where } x^k = m([x^k]) \text{ is the center of } [x^k] \end{array}\right\} . \tag{10}$$

Note that in this iterative method we keep Δ and $l'([x]) = l'([x^0])$, which are used in the definition of Γ, fixed. See, however, remark 2 after the proof of theorem 2 below and the numerical examples.

It is easy to show that if $[x^0]$ contains a fixed point x^* of p, then all iterates contain x^*, and therefore, that (10) is well defined and is converging to an interval vector $[x^*]$ which contains all fixed points of p contained in $[x^0]$. See theorem 2, below.

Given the problem (1), which means, given the mapping l, it remains the question of how to choose Δ and to find an interval vector $[x]$ for which (9) holds. Furthermore, given $[x]$ and assuming that (9) holds, under which conditions will (10) converge to an interval vector with diameter equal to the zero vector? In this case the limit is a solution of (1) and there exists no other solution of (1) in $[x]$. In [5] these questions have been discussed for the special case that l is a so-called affine mapping

$$l(x) = Mx + q, \tag{11}$$

where the square matrix $M \in \mathbb{R}^{n \times n}$ and the vector $q \in \mathbb{R}^n$ are given. If l has the form (11), then (1) is called linear complementarity problem (LCP) in the literature. The purpose of this paper is to extend the results from [5] to so-called almost linear mappings l. See [13].

A mapping $l : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is called almost linear if l can be written in the form

$$l(x) = Mx + \Phi(x), \tag{12}$$

where M is a real matrix and Φ is a diagonal mapping. A nonlinear mapping $\Phi : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is diagonal if, for $i = 1, 2, \ldots, n$, the i-th component φ_i is a function of only the i-th variable x_i.

If l is an almost linear mapping with an increasing mapping Φ and an H-matrix M with positive diagonal elements, we will show that (10) is always convergent to a solution of (1) if we start with an interval vector $[x^0] := [x]$ for which (9) holds. We can construct such an $[x]$ by solving a linear system of equations. Finally, we give a simple proof that (1) has a unique solution under our assumptions.

Finding an interval vector $[x]$ which contains a solution of a given problem is usually called verification of a solution. In this sense theorem 1 is a verification result for solutions of the (NCP). Different approaches for the verification of solutions of complementarity problems have been discussed in [1] and [3]. Since no special assumptions concerning the mapping l have been made in these papers, the verification procedures are much more complex than those which we will derive for almost linear mappings. Finally, we mention that in [4], [14] and [15] the case was considered, that the given (LCP) has interval data.

2 Preliminaries and Notation

Subsequently some basic facts from interval analysis are used. See [2], [9] or [12], for example. We denote by $a, b, \ldots$ real numbers and real vectors by $a = (a_i), b = (b_i), \ldots$, respectively. Compact intervals of real numbers are denoted by $[a] = [\underline{a}, \overline{a}], [b] = [\underline{b}, \overline{b}], \ldots$. Similarly $[a] = ([a_i]), [b] = ([b_i]), \ldots$ denote vectors with interval components. Occasionally we write an interval vector as $[a] = [\underline{a}, \overline{a}]$, where for $[a] = ([a_i])$ the real vector $\underline{a}$ has the lower bounds of the $[a_i]$ as its components and analogously for $\overline{a}$. $[A] = ([a_{ij}]), [B] = ([b_{ij}]), \ldots$, denote interval matrices. Similarly as for interval vectors, these are sometimes written as $[A] = [\underline{A}\,, \overline{A}], [B] = [\underline{B}\,, \overline{B}], \ldots$. For an interval $[a] = [\underline{a}\,, \overline{a}]$ we define the absolute value $|[a]|$ by

$$|[a]| = \max\{|\,\underline{a}\,|\,, |\,\overline{a}\,|\}. \tag{13}$$

Similarly

$$d([a]) = \overline{a} - \underline{a} \tag{14}$$

denotes the diameter of $[a]$. If the intersection of two intervals is not empty then

$$d([x] \cap [y]) \leq d([x]), \tag{15}$$

$$d([x] \cap [y]) \leq d([y]) \tag{16}$$

and therefore

$$d([x] \cap [y]) \leq \max\{d([x]), d([y])\}. \tag{17}$$

For interval vectors and interval matrices the absolute value and the diameter are defined via the components. If we use the componentwise partial ordering then (15), (16) and (17) hold analogously.

For completeness we recall that a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is called monotone increasing (or increasing, or isotone) if from $x \leq y$ it follows that $f(x) \leq f(y)$, where we use the componentwise defined partial ordering in $\mathbb{R}^n$.

A real matrix $A = D - B$, where D denotes the diagonal part, is called an M(inkowski)-matrix if it is nonsingular with $B \geq 0$ and $A^{-1} \geq 0$. The diagonal elements of an M-matrix are positive. If $A = D - B$ is an M-matrix and if $\tilde{D}$ is a nonnegative diagonal matrix, then $\tilde{A} = A + \tilde{D}$ is also an M-matrix. See [13] and [16].

A real (or complex) matrix $A = D - B$ is called an H(adamard)-matrix if the so-called comparison matrix

$$< A >:= |D| - |B|$$

is an M-matrix. The diagonal elements of an H-matrix are different from zero.

Given a matrix A, we call a splitting

$$A = R - S$$

of A regular, if $S \geq 0, R$ is nonsingular and $R^{-1} \geq 0$. See [13] and [16]. Assume that $A = R - S$ is a regular splitting of A. Then $\rho(R^{-1}S) < 1$ (ρ denotes the spectral radius), iff A is nonsingular and $A^{-1} \geq 0$. See [13] and [16].

3 Results

Given (1), where l is assumed to be almost linear. Let M be an H-matrix with positive diagonal elements. We split M into

$$M = D - B,$$

where D denotes the diagonal part and $-B$ the off-diagonal part of M, respectively. If Φ is differentiable, then

$$l'(x) = M + \Phi'(x)\,. \tag{18}$$

Furthermore, assuming that $\Phi = (\varphi_i)$ is monotone increasing, this is equivalent to $\varphi_i, i = 1, 2, \ldots, n$, being monotone increasing. Let $[x] = ([x_i])$ be an interval vector. If the derivative Φ' of Φ exists and if Φ' has an interval arithmetic evaluation $\Phi'([x])$ for $[x]$, then $\Phi'([x])$ is a diagonal interval matrix. We use the notation

$$\Phi'([x]) := [\Phi'_1\,,\ \Phi'_2]. \tag{19}$$

Φ'_1 and Φ'_2 are real diagonal matrices with $\Phi'_1 \leq \Phi'_2$. Since Φ is monotone increasing by assumption, we can assume $\Phi'_1 \geq 0$ subsequently without loss of generality. (Please note that the interval arithmetic evaluation gives an inclusion of the range, which is an overestimation, in general). The interval arithmetic evaluation of l' can be written as

$$l'([x]) = M + [\Phi'_1\,,\ \Phi'_2]. \tag{20}$$

We define the diagonal matrix

$$\Delta := (D + \Phi'_2)^{-1} = (D + |\,\Phi'([x])|)^{-1} \tag{21}$$

and consider the iteration method (10) under the assumption (9).

Theorem 2 Consider the problem (1) where l is an almost linear mapping with an H-matrix $M = D - B$ with positive diagonal elements in the diagonal part D. The mapping Φ is assumed to be increasing. Furthermore suppose that the derivative Φ' exists and that it has an interval arithmetic evaluation $\Phi'([x])$. If there exists an interval vector $[x]$ for which (9) holds, where Δ is defined by (21), then (1) has a solution $x^* \in \Gamma(x, [x], \Delta) \subseteq [x]$. The method (10) is well-defined, all iterates $[x^k]$ contain the solution x^* of (1) and $\lim_{k\to\infty}[x^k] = x^*$.

Proof If (9) holds, then by theorem 1, there exists a solution x^* of (1) which is contained in $\Gamma(x, [x^0], \Delta)$, and therefore $x^* \in [x^1]$, where $[x^1]$ is defined by (10). By the remarks preceding theorem 1 it follows that $x^* \in \Gamma(x^1, [x^1], \Delta)$ and therefore that

$$x^* \in \Gamma(x^1, [x^1], \Delta) \cap [x^1] =: [x^2].$$

By mathematical induction we have $x^* \in [x^k]$, $k \geq 0$, and therefore the method (10) is well-defined. Consider now the diameters of the sequence

$\{[x^k]\}$:

$$\left.\begin{aligned} d([x^{k+1}]) &\leq d\,(\Gamma(x^k,[x^k],\Delta)) \\ &\leq d\,(x^k - \Delta l(x^k) + (I - \Delta\, l'([x^k]))([x^k] - x^k)) \\ &= d\,((I - \Delta\, l'([x^k]))([x^k] - x^k)) \\ &= |\, I - \Delta\, l'([x^k])|\, d\,([x^k]) \\ &= |\, \Delta(\Delta^{-1} - l'([x^k]))|\, d([x^k]) \\ &= |(D + \Phi_2')^{-1}\,(D + \Phi_2' - D + B - [\Phi_1', \Phi_2'])|\, d([x^k]) \\ &= |(D + \Phi_2')^{-1}\,([0, \Phi_2' - \Phi_1'] + B)|\, d([x^k]) \\ &\leq (D + \Phi_2')^{-1}\,(\Phi_2' - \Phi_1' + |B|)\, d([x^k]). \end{aligned}\right\} \quad (22)$$

(Here we have used the representation (20) of $l'([x^k])$. Furthermore $[0, \Phi_2' - \Phi_1']$ denotes a fixed diagonal interval matrix, which is computed from $\Phi'([x^0]) = [\Phi_1', \Phi_2']$. See the remarks after (10). The lower bounds of the diagonal entries are all equal to zero, the upper bounds are equal to the diameters of the interval arithmetic evaluation $\varphi_i([x_i^0])$ of the derivative of $\varphi_i(x_i), i = 1, 2, \ldots, n$).
Consider now the real matrix

$$\begin{aligned} \tilde{M} &= D + \Phi_1' - |B| = D + \Phi_2' - (\Phi_2' - \Phi_1' + |B|) \\ &= R - S, \end{aligned}$$

where

$$R = D + \Phi_2', \quad S = \Phi_2' - \Phi_1' + |B|.$$

D is a diagonal matrix with positive diagonal elements. Since Φ is increasing it follows that $\Phi_2' \geq 0$. Therefore $\Delta = (D + \Phi_2')^{-1} \geq 0$, and since $\Phi_2' \geq \Phi_1'$, we have also $S \geq 0$. Therefore the splitting $\tilde{M} = R - S$ of $\tilde{M}$ is a regular splitting. See section 2. Furthermore, since the matrix is by assumption an H-matrix, it holds that $(D - |B|)^{-1} \geq 0$. By the remark in section 2 we also have $\tilde{M}^{-1} = (D + \Phi_1' - |B|)^{-1} \geq 0$. Therefore

$$\rho(R^{-1}S) = \rho((D + \Phi_2')^{-1}(\Phi_2' - \Phi_1' + |B|)) < 1.$$

By mathematical induction we obtain from (22) that

$$d([x^{k+1}]) \leq (R^{-1}S)^{k+1}\, d([x^0]),$$

from which it follows that $\lim_{k\to\infty}[x^k] = x^*$, since

$$x^* \in [x^k] \text{ for all } k \geq 0. \qquad \square$$

Remark 1 It is easy to see that $\lim_{k\to\infty}[x^k] = x^*$ also holds if we only assume the existence of a solution $x^* \in [x^0]$. (The assumption $\Gamma(x^0, [x^0], \Delta) \subseteq$

$[x^0]$ was only made to guarantee the existence of a solution $x^* \in [x^0]$.) $\square$

Remark 2 One could think of computing $\Phi'([x])$ for each k and using the new diagonal matrix in the next iteration step.

Let $\Phi'([x^k]) = [\Phi'_{1,k}, \Phi'_{2,k}]$ be the interval arithmetic evaluation of $\Phi'(x)$ for the interval vector $[x^k]$.

Let

$$\tilde{M}_k = D + \Phi'_{1,k} - |B| = D + \Phi'_{2,k} - (\Phi'_{2,k} - \Phi'_{1,k} + |B|) \\ = R_k - S_k,$$

where

$$R_k = D + \Phi'_{2,k}\,,\; S_k = \Phi'_{2,k} - \Phi'_{1,k} + |B|.$$

Since by the inclusion monotonicity of interval arithmetic we have $\Phi'_{1,k} \leq \Phi'_{1,k+1}$, it follows that

$$0 \leq \tilde{M}_{k+1}^{-1} \leq \tilde{M}_k^{-1}.$$

Furthermore, by the same reasoning $\Phi'_{2,k+1} \leq \Phi'_{2,k}$, and therefore

$$\Phi'_{2,k+1} - \Phi'_{1,k+1} \leq \Phi'_{2,k} - \Phi'_{1,k},$$

which implies

$$0 \leq S_{k+1} \leq S_k.$$

Hence

$$0 \leq \tilde{M}_{k+1}^{-1} S_{k+1} \leq \tilde{M}_k^{-1} S_k.$$

By the Perron-Frobenius theorem on nonnegative matrices we have

$$\rho(\tilde{M}_{k+1}^{-1} S_{k+1}) \leq \rho(\tilde{M}_k^{-1} S_k).$$

Since

$$\rho(R_k^{-1} S_k) = \frac{\rho(\tilde{M}_k^{-1} S_k)}{1 + \rho(\tilde{M}_k^{-1} S_k)}\,, \quad \text{(see [16])}$$

and since $f(x) = x/(1+x)$ is increasing for $x > 0$, it follows that

$$\rho(R_{k+1}^{-1} S_{k+1}) \leq \rho(R_k^{-1} S_k) < 1.$$

If can easily be seen that the inequality

$$d([x^{k+1}]) \leq (R^{-1}S)^{k+1} d([x^0])$$

in the proof of the last theorem can be replaced by the estimation

$$d([x^{k+1}]) \leq (\prod_{j=0}^{k} R_j^{-1} S_j)\; d([x^0])$$

where, by the preceding discussion, the spectral radii of the matrices $R_j^{-1}S_j$ are decreasing with increasing j.

It is easy to see that

$$\lim_{j\to\infty} \rho(R_j^{-1}S_j) = \rho((R^*)^{-1}S^*),$$

where $\tilde{M}^* = D + \Phi'(x^*) - |B|$ and $R^* = D + \Phi'(x^*),\ S^* = |B|$. $\square$

In the next theorem we show how to find an interval vector $[x]$, for which (9) holds. We need the following result, which was proved in [5].

Lemma 1 Let $a, b, c \in \mathbb{R}^n, a \leq b$, and $c \geq 0$.
Then

$$\max\{0, [a, b]\} \subseteq [-c, c] \tag{23}$$

iff $b \leq c$. $\square$

For the almost linear mapping $l(x) = Mx + \Phi(x)$ we obtain for $x = m([x]) = 0$ (which means $[x] = -[x]$)

$$\begin{aligned} x - \Delta l(x) + (I - \Delta(M + [\Phi_1', \Phi_2']))([x] - x) = \\ = -\Delta\Phi(0) + |\Delta(\Delta^{-1} - (M + [\Phi_1', \Phi_2']))|[-\frac{d[x]}{2}, \frac{d[x]}{2}] \\ = -\Delta\Phi(0) + \Delta(\Phi_2' - \Phi_1' + |B|)[-\frac{d([x])}{2}, \frac{d([x])}{2}]. \end{aligned}$$

By (23) we therefore have

$$\Gamma(0, [-\frac{d([x])}{2}, \frac{d([x])]}{2}], \Delta]) \subseteq [-\frac{d([x])}{2}, \frac{d([x])}{2}]$$

iff

$$-\Delta\,\Phi(0) + \Delta(\Phi_2' - \Phi_1' + |B|)\,\frac{d([x])}{2} \leq \frac{d([x])}{2},$$

which is equivalent to

$$(D + \Phi_1' - |B|)\,\frac{d([x])}{2} \geq -\Phi(0). \tag{24}$$

From this inequality we get the proof of the following result.

Theorem 3 Let $l(x) = Mx + \Phi(x)$ be an almost linear mapping with an H-matrix with positive diagonal elements. Assume that Φ is increasing and Φ' has an interval arithmetic evaluation for interval vectors $[x]$ with $[x] = -[x]$. Let r be the solution of the linear system

$$(D - |B|)\,r = u,$$

where $u = (u_i) \geq 0$ is defined as follows. Let $\Phi(0) = (\varphi_i(0))$. Then

$$u_i = \max\{0, -\varphi_i(0)\}.$$

If $[x] = \alpha[-r, r]$, $\alpha \geq 1$, then

$$\Gamma(0, [x], \Delta) \subseteq [x].$$

Proof From the definition of u it follows that

$$u \geq -\Phi(0). \tag{25}$$

We have

$$d([x]) = 2\alpha r = 2\alpha(D - |B|)^{-1}u \geq 0,$$

and since

$$D + \Phi_1' - |B| \geq D - |B|,$$

we obtain

$$\begin{aligned}(D + \Phi_1' - |B|)\frac{d([x])}{2} &= \alpha(D + \Phi_1' - |B|)(D - |B|)^{-1}u \\ &\geq \alpha(D - |B|)(D - |B|)^{-1}u \\ &= \alpha u \geq u \geq -\Phi(0),\end{aligned}$$

where we have used $\alpha \geq 1$ and (25). □

From a practical point of view we want to choose the components of $d([x])$ as small as possible, and therefore $\alpha = 1$ in the preceding theorem. However, if $r > 0$ then the proof of the preceding theorem shows that we can make all components of $d([x])$ arbitrarily large by choosing α large enough. Together with theorem 2 this gives in the proof of the following result in the case $r > 0$.

Theorem 4 If $l(x) = Mx + \Phi(x)$ is an almost linear mapping with an increasing mapping Φ and an H-matrix M with positive diagonal elements, then (1) has a unique solution. □

It remains to be shown that this result also holds if some components of r are equal to zero.
Let u be defined as before and define $\tilde{u} = (\tilde{u}_i)$ by

$$\tilde{u}_i = \begin{cases} 1 & \text{if } u_i = 0 \\ u_i & \text{if } u_i \neq 0. \end{cases}$$

It follows that $\tilde{u} \geq u$ and $\tilde{u} > 0$. The solution $\tilde{r}$ of the linear system

$$(D - |B|)\tilde{r} = \tilde{u}$$

is strictly positive, $\tilde{r} > 0$ since $(D - |B|)^{-1} \geq 0$. Then setting

$$[x] = \alpha[-\tilde{r}, \tilde{r}], \quad \alpha \geq 1,$$

it follows as before that

$$(D + \Phi_1' - |B|)\frac{d([x])}{2} \geq \alpha\tilde{u} \geq u \geq -\Phi(0).$$

Since $\tilde{r}$ is strictly positive, we can conclude as before that (1) has a unique solution.

Remark 3 The uniqueness statement of theorem 4 is not new. It is contained as a special case in [11]. See also [10].

4 Numerical Experiments

We consider examples

$$l(x) = Mx + \Phi(x)$$

where M is an H-matrix with positive diagonal elements and with an increasing diagonal mapping $\Phi = (\varphi_i)$. Let $M = D - B$ where D denotes the diagonal part and $-B$ the off diagonal part of M. According to theorem 3 the interval vector

$$[x^0] = [-r, r],$$

where r is the solution of the linear system

$$(D - |B|)r = \max\{0, -\Phi(0)\}, \tag{26}$$

contains the solution x^* of problem (1). In all numerical examples we have computed r (and therefore $[x^0]$) by solving (26). By the definition of $\Gamma(0, [x^0], \Delta) =: [\gamma_1, \gamma_2]$, (see (8) and (6)), its lower bound is nonnegative, $\gamma_1 \geq 0$, and therefore, $x^* \in \Gamma(0, [x^0], \Delta) \subseteq [0, r]$. Hence, instead of $[-r, r]$ we can use $[x^0] = [0, r]$ and, according to remark 1, following the proof of theorem 2, the convergence of the iterative method (10) is guaranteed.

In our numerical examples we compare the original method (10) for $[x^0] = [0, r]$ with two additional modifications of (10). In the original method we compute the diagonal matrix $[\Phi_1', \Phi_2']$ for $[x^0]$ and this matrix is kept fixed for all k in order to save work. Therefore we have for all $k \geq 0$

$$\Delta = (D + \Phi_2')^{-1},$$

$$l'([x^k]) = M + [\Phi_1', \Phi_2'],$$

and (10) reads

(I) $\qquad [x^{k+1}] = \Gamma(x^k, [x^k], \Delta) \cap [x^k], k = 0, 1, 2 \dots .$

In the first modification we only improve the upper bound Φ'_2, which means that we use the enclosure $[\Phi'_1, \Phi'_{2,k}]$, where $\Phi'_{2,k}$ is an upper bound of Φ' on the interval $[x^k]$. Therefore, we have

$$\Delta_k = (D + \Phi'_{2,k})^{-1} ,$$

$$l'([x^k]) = M + [\Phi'_1, \Phi'_{2,k}] ,$$

and the modified method reads

(II) $\qquad [x^{k+1}] = \Gamma(x^k, [x^k], \Delta_k) \cap [x^k], \ k = 0, 1, 2, \dots .$

In the second modification we also improve the lower bound Φ'_1 in each step. Therefore, we have

$$\Delta_k = (D + \Phi'_{2,k})^{-1} ,$$

$$l'([x^k]) = M + [\Phi'_{1,k}, \Phi'_{2,k}] ,$$

and the modified method can again be written as

(III) $\qquad [x^{k+1}] = \Gamma(x^k, [x^k], \Delta_k) \cap [x^k], \ k = 0, 1, 2, \dots .$

We test the methods (I) - (III) for three examples with the above implementation details, via Matlab 6.5 on a PC. The methods will not terminate until the radius r^k of the computed interval is less than ϵ componentwise, or the number of iterations is over 20000. The experiments are performed for the choices $\epsilon = 1e-5$ and $1e-10$. Numerical results reported are as follows:

N: the number of iterations;
δ_1: $\|r^k\|_\infty$;
δ_2: $\|\min\{x^k, Mx^k + \Phi(x^k)\}\|_\infty$;
cpu: the cpu time(seconds) needed for the iteration.

Example 1 Let $c \in \mathbb{R}^n$ be constant, $\Phi(x) = (e^{x_1}, e^{x_2}, \cdots, e^{x_n})^T + c$,

$$M = \frac{1}{h^2} \begin{pmatrix} H & -I & & \\ -I & H & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & H \end{pmatrix} \in \mathbb{R}^{n^2 \times n^2},$$

where $h = 1/(n+1)$,

$$H = \begin{pmatrix} 4 & -1 & & \\ -1 & 4 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Set $x^* = (0,1,0,1,\cdots,1)^T \in \mathbb{R}^n$ and choose $c = (c_i)^T \in \mathbb{R}^n$ as in [3]:

$$c_i = -\begin{cases} (Mx^*)_i + e^{x_i^*}, & \text{if } x_i^* > 0 \\ (Mx^*)_i + e^{x_i^*} - \xi_i, & \text{otherwise,} \end{cases}$$

where ξ_i is a random nonnegative number. The corresponding problem (1) is the result of the application of centered five points difference method to the equilibrium problem given in [8]. We generate the nonnegative random numbers in $[0,1]$. The matrix M is an H-matrix with positive diagonal elements. Φ is a monotone increasing diagonal mapping.
By the definition of $\Phi(x)$ we can set $\Phi_1' = I, \Phi_2' = \text{diag}(e^{r_i})$, where $r = (r_i)$ is the solution of (26).

Table A for example 1 ($\epsilon = 1e-5$)

n		N	δ_1	δ_2	cpu
3	(I)	21	1.7479e-007	4.9116e-005	4.7000e-002
	(II)	20	3.2243e-007	8.3099e-005	4.7000e-002
	(III)	19	9.5896e-006	3.8612e-003	3.1000e-002
5	(I)	64	3.7588e-006	1.0153e-002	2.8100e-001
	(II)	63	4.9967e-006	2.2765e-003	2.9700e-001
	(III)	62	4.2047e-006	1.8347e-003	2.9700e-001
6	(I)	88	8.6689e-006	2.3886e-002	5.1600e-001
	(II)	87	6.5721e-006	1.7844e-002	6.2500e-001
	(III)	87	6.2550e-006	1.6929e-002	6.2500e-001
8	(I)	163	7.5861e-006	6.0866e-002	2.2660e+000
	(II)	161	7.8401e-006	6.2421e-002	2.9370e+000
	(III)	161	7.5784e-006	6.0752e-002	2.9850e+000
10	(I)	278	7.0333e-006	1.4841e-001	6.5940e+000
	(II)	259	6.4748e-006	1.3299e-001	9.2810e+000
	(III)	259	6.3179e-006	1.2912e-001	9.2970e+000

Table B for example 1 ($\epsilon = 1e - 10$)

n		N	δ_1	δ_2	cpu
3	(I)	23	4.3838e-011	6.6316e-009	4.7000e-002
	(II)	22	5.8697e-012	3.7483e-010	6.3000e-002
	(III)	20	1.2415e-012	5.6843e-014	3.1000e-002
5	(I)	69	2.0216e-011	5.1912e-008	2.9700e-001
	(II)	66	5.0349e-013	4.5475e-013	3.1200e-001
	(III)	65	2.2204e-015	5.0022e-012	2.9700e-001
6	(I)	102	6.2339e-011	1.8679e-007	5.9400e-001
	(II)	101	8.9348e-011	2.7046e-007	7.3400e-001
	(III)	101	8.3784e-011	2.5478e-007	7.3500e-001
8	(I)	179	5.1870e-011	4.7094e-007	2.4370e+000
	(II)	176	7.8307e-011	7.0722e-007	3.2190e+000
	(III)	175	7.5263e-011	6.9269e-007	3.2810e+000
10	(I)	298	7.8705e-011	1.7902e-006	7.0940e+000
	(II)	275	8.0302e-011	1.7890e-006	9.8280e+000
	(III)	275	7.6904e-011	1.7322e-006	9.8590e+000

Example 2 Let

$$M = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 2 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathbb{R}^{n\times n}$$

and let $\Phi(x) = q + s(x)$, where $s(x) = (s_i(x_i))$ with $s_i(x_i) = (x_i + 1)^3 - i$, $i = 1, 2, \cdots, n$. Furthermore, let $x^* = (x_i^*)^T$ with

$$x_i^* = \begin{cases} 0, & \text{if } i\text{mod}7 = 0 \\ i, & \text{otherwise.} \end{cases}$$

q is chosen such that

$$q_i = \begin{cases} i - (Mx^*)_i - s_i(x_i^*), & \text{if } \ i\text{mod}7 = 0 \\ -(Mx^*)_i - s_i(x_i^*), & \text{otherwise.} \end{cases}$$

M is an H-matrix with positive diagonal elements and Φ is a monotone increasing diagonal mapping. It is easy to verify that x^* is the exact solution of the problem. By the definition of Φ we can set $\Phi_1' = 3I, \Phi_2' = diag(3(r_i + 1)^2)$, where $r = (r_i)$ is the solution of (26).

Extensive numerical experiments show that, if the row index i is bigger than $n/2$, the radius of the components of the initial enclosure is very large, which is mainly due to the fact that the elements of $(D - |B|)^{-1}$ increase strongly along with its column index. The larger the dimension of the problem is, the more obvious this phenomenon is. Studying the case of $n = 5, 10, 20, 50, 100$, we list the maxima and the minima of the radii of the components of the initial enclosure, which are denoted by the abbreviations: rmax and rmin, respectively.

n	5	10	20	50	100
rmax	1.5008e+04	2.3317e+07	1.0105e+13	2.4212e+28	1.6210e+53
rmin	2.2000e+02	1.3400+e03	9.2800e+03	1.3270e+05	1.0304e+06

Table A for example 2 ($\epsilon = 1e - 5$)

n		N	δ_1	δ_2	cpu
5	(I)	20000	1.8274e+002	1.8274e+002	2.2359e+001
	(II)	528	9.7239e-006	0.0000e-000	6.8800e-001
	(III)	190	5.2661e-006	7.0218e-008	2.3400e-001
10	(I)	20000	2.8545e+005	2.8545e+005	3.4141e+001
	(II)	1949	9.8942e-006	0.0000e-000	3.8120e+000
	(III)	363	2.6325e-007	2.7536e-009	7.0300e-001
20	(I)	20000	1.2371e+011	1.2371e+011	5.9594e+001
	(II)	9246	9.9713e-006	0.0000e-000	3.3547e+001
	(III)	668	1.9975e-007	2.0453e-009	3.2660e+000
50	(I)	20000	2.9641e+026	2.9641e+026	1.6347e+002
	(II)	20000	4.0368e+001	4.0010e+001	2.1833e+002
	(III)	2594	5.1084e-007	1.1155e-009	3.4812e+001
100	(I)	20000	1.9845e+051	1.9845e+051	4.7777e+002
	(II)	20000	9.5978e+001	9.5007e+001	7.3158e+002
	(III)	9630	3.8565e-007	8.2764e-010	3.5580e+002

Table B for example 2 ($\epsilon = 1e - 10$)

n		N	δ_1	δ_2	cpu
	(I)	20000	1.8274e+002	1.8274e+002	2.2359e+001
5	(II)	836	9.6792e-011	0.0000e-000	1.0950e+000
	(III)	191	5.1196e-011	3.4817e-013	2.3400e-001
	(I)	20000	2.8545e+005	2.8545e+005	3.4141e+001
10	(II)	2990	9.9249e-011	0.0000e-000	5.8430e+000
	(III)	364	1.2795e-013	1.4211e-014	7.0300e-001
	(I)	20000	1.2371e+011	1.2371e+011	5.9594e+001
20	(II)	20000	4.4928e-010	0.0000e-000	7.1218e+001
	(III)	669	7.3830e-014	5.6843e-014	3.2820e+000
	(I)	20000	2.9641e+026	2.9641e+026	1.6347e+002
50	(II)	20000	4.0368e+001	4.0010e+001	2.1833e+002
	(III)	2595	4.8178e-013	0.0000e-000	3.4828e+001
	(I)	20000	1.9845e+051	1.9845e+051	4.7777e+002
100	(II)	20000	9.5978e+001	9.5007e+001	7.3158e+002
	(III)	9631	2.7489e-013	0.0000e-000	3.5583e+002

Notice from the data in table A and table B that the numbers of the iterations of method (III) needed to obtain the precision $\epsilon = 1e-10$ is more than that to obtain the precision $\epsilon = 1e-5$ just by one, in other words, one more iteration sharpens the enclosure from $\epsilon = 1e-5$ to $\epsilon = 1e-10$. Studying the case $n = 5$ we find that after 190 iterations of method (III) an enclosure with the radius $r^{(190)} = (5.2661e-6, 0, 0, 0, 0)^T$ is obtained. We record the matrix $\Delta_{190}(B + \Phi'_{2,190} - \Phi'_{1,190})$:

$$\begin{pmatrix} 9.7219e-6 & 1.5385e-1 & 1.5385e-1 & 1.5385e-1 & 1.5385e-1 \\ 0 & 0 & 7.1429e-2 & 7.1429e-2 & 7.1429e-2 \\ 0 & 0 & 0 & 4.0816e-2 & 4.0816e-2 \\ 0 & 0 & 0 & 0 & 2.6316e-2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is verified that

$$\Delta_{190}(B + \Phi'_{2,190} - \Phi'_{1,190})r^{(190)} = (5.1196e-11, 0, 0, 0, 0)^T,$$

which means one more iteration reduces the maximum radius of the enclosure by a factor of approximately $1e-5$. In Fig.1 at the end of the paper we have depicted the spectral radius $\rho(\Delta_k(B+\Phi'_{2,k} - \Phi'_{1,k})) = p(k)$ in dependence of k. The similar phenomenon appears for the remaining cases.

Example 3a Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g(t, u) = 2(u - 4t + 1)^3$, $\Phi(x) = (\varphi_i(x_i))$ with $\varphi_i(x_i) = g(t_i, x_i)$, $t_j = i/(n + 1)$, $i = 1, \cdots, n$, and

$$M = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

The function $l(x) = Mx + \Phi(x)$ is obtained via discretizing a two-point boundary value problem, see [13]. $l(x)$ is an almost linear mapping with an H-matrix M and an increasing diagonal mapping Φ. In the numerical experiments the phenomenon, similar to the example 2, is observed. We also list the maxima and the minima of the radii of the components of the initial enclosure, for the cases $n = 5, 10, 20, 50, 100$.

n	5	10	20	50	100
rmax	1.5508e+03	6.0806e+05	4.6975e+10	1.1479e+25	8.7401e+48
rmin	2.5407e+01	3.6648e+01	4.4354e+01	4.9874e+01	5.1889e+01

Table A for example 3a ($\epsilon = 1e - 5$)

n		N	δ_1	δ_2	cpu
5	(I)	20000	5.2385e-001	3.5397e-002	1.6797e+001
	(II)	300	9.4733e-006	2.2204e-016	3.1200e-001
	(III)	205	8.9808e-006	1.0582e-007	2.1900e-001
10	(I)	20000	3.3758e+000	3.3758e+000	2.4078e+001
	(II)	624	9.7657e-006	2.2204e-016	9.0600e-001
	(III)	426	9.6042e-006	3.3927e-008	6.0900e-001
20	(I)	20000	7.0287e+000	7.0287e+000	3.9782e+001
	(II)	1545	9.9252e-006	4.4409e-016	3.8750e+000
	(III)	1011	9.8453e-006	4.0524e-009	2.5620e+000
50	(I)	20000	3.1985e+001	3.1985e+001	1.0955e+002
	(II)	6825	9.9838e-006	5.0654e-016	5.3641e+001
	(III)	4257	9.9806e-006	1.0134e-010	3.4250e+001
100	(I)	20000	1.2005e+002	1.2005e+002	3.4120e+002
	(II)	20000	1.7331e-004	1.1102e-015	5.8323e+002
	(III)	14932	9.9848e-006	4.1072e-012	4.4539e+002

Table B for example 3a ($\epsilon = 1e-10$)

n		N	δ_1	δ_2	cpu
	(I)	20000	5.2385e-001	3.5397e-002	1.6797e+001
5	(II)	427	9.2691e-011	2.2204e-016	4.2200e-001
	(III)	236	8.1467e-011	9.5988e-013	2.3400e-001
	(I)	20000	3.3758e+000	3.3758e+000	2.4078e+001
10	(II)	904	9.9050e-011	2.2204e-016	1.2970e+000
	(III)	510	9.3701e-011	3.3135e-013	7.5000e-001
	(I)	20000	7.0287e+000	7.0287e+000	3.9782e+001
20	(II)	2350	9.9243e-011	4.4409e-016	5.8280e+000
	(III)	1273	9.6921e-011	3.9693e-014	3.2030e+000
	(I)	20000	3.1985e+001	3.1985e+001	1.0955e+002
50	(II)	10994	9.9784e-011	5.0654e-016	8.5297e+001
	(III)	5578	9.9222e-011	1.6015e-015	4.5125e+001
	(I)	20000	1.2005e+002	1.2005e+002	3.4120e+002
100	(II)	20000	1.7331e-004	1.1102e-015	5.8323e+002
	(III)	19671	9.9919e-011	1.1102e-015	5.8147e+002

Notice that the methods (I) - (III) converge slowly for the above problem. In fact the numerical performance depends not only on the matrix M but also on the mapping Φ. We use the spectral radii of the corresponding matrices

$$\begin{aligned} u(k) &= \rho((D+\Phi_2')^{-1}(|B|+\Phi_2'-\Phi_1')) \\ v(k) &= \rho((D+\Phi_{2,k}')^{-1}(|B|+\Phi_{2,k}'-\Phi_1')) \\ p(k) &= \rho((D+\Phi_{2,k}')^{-1}(|B|+\Phi_{2,k}'-\Phi_{1,k}')) \end{aligned}$$

to explain the convergence rate. It holds

$$u(1) = v(1) = p(1).$$

We compute $u(1)$ (which is equal to $u(k)$ for all k) in the case $n = 5$ and obtain $u(1) = 9.99921e-1$. In Fig. 2 at the end of the paper we have depicted the decrease of the spectral radii $v(k)$ and $p(k)$. The spectral radii of the matrices at the points, where the methods (I) - (III) terminate are as follows:

n	5	10	20	50	100
u	9.9992e-1	≈ 1	≈ 1	≈ 1	≈ 1
v	9.1318e-1	9.6656e-1	9.8977e-1	9.9819e-1	9.9953e-1
p	6.8761e-1	8.7325e-1	9.5718e-1	9.9132e-1	9.9757e-1

Studying the matrices $(D+\Phi_{2,k}')^{-1}(|B|+\Phi_{2,k}'-\Phi_{1,k}')$, with a larger $\Phi_{2,k}'$ and a smaller $\Phi_{2,k}'-\Phi_{1,k}'$, we can get the smaller $p(k) = \rho((D+\Phi_{2,k}')^{-1}(|B|+\Phi_{2,k}'-\Phi_{1,k}'))$, and a more rapid convergence will be expected. The claim holds also for the method (II).

Example 3b To demonstrate this, we choose a function Φ with a large derivative φ_i' and a small second derivative φ_i'' for its components φ_i. Set $\Phi(x) = (\frac{2}{3}(x_i + 100)^{\frac{3}{2}})^T + c$, where $c = (c_i)^T$, and $c_1, c_2, \cdots, c_n$ are random numbers distributed in $[0, 1000]$. With the same matrix M given in case of example 3a, we get the following results for the cases $n = 5, 10, 20, 50, 100$.

Table A for example 3b ($\epsilon = 1e - 5$)

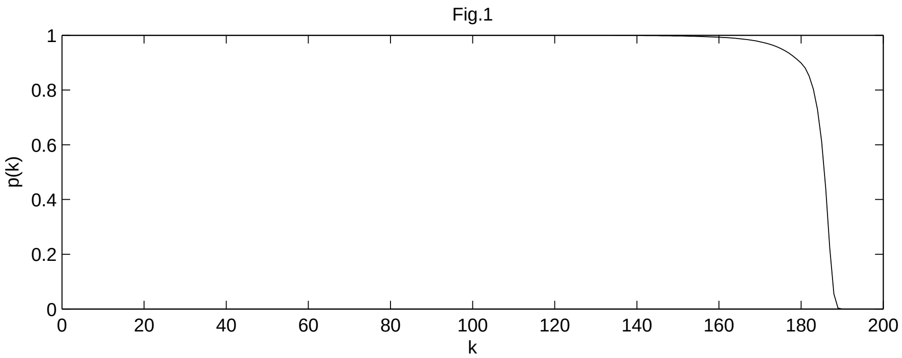
n		N	δ_1	δ_2	cpu
5	(I)	37	6.7092e-006	8.4681e-006	4.6000e-002
	(II)	14	2.6593e-006	4.0522e-007	3.2000e-002
	(III)	12	5.7832e-006	2.4246e-005	1.5000e-002
10	(I)	57	7.7647e-006	2.2974e-005	1.4100e-001
	(II)	15	8.1186e-006	1.9215e-006	4.7000e-002
	(III)	14	3.1606e-006	1.4659e-005	4.7000e-002
20	(I)	121	8.7738e-006	1.3360e-005	6.2500e-001
	(II)	18	9.7205e-006	7.1337e-007	1.0900e-001
	(III)	16	8.5548e-006	4.3016e-005	9.4000e-002
50	(I)	221	9.1999e-006	2.8874e-005	4.4680e+000
	(II)	22	6.3337e-006	3.0242e-006	5.1600e-001
	(III)	21	4.6540e-006	2.2213e-005	4.8400e-001
100	(I)	444	9.7295e-006	3.4103e-005	4.0437e+001
	(II)	27	2.5037e-006	1.0994e-006	2.9850e+000
	(III)	26	1.6306e-006	7.9528e-006	2.8280e+000

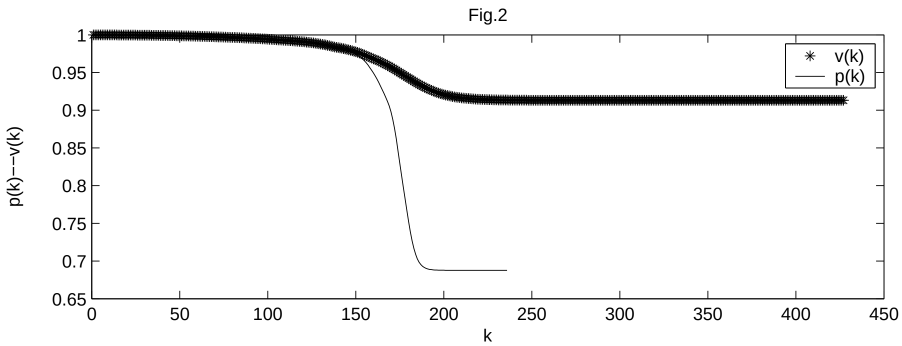
Table B for example 3b ($\epsilon = 1e - 10$)

n		N	δ_1	δ_2	cpu
5	(I)	60	8.8800e-011	3.4291e-011	7.8000e-002
	(II)	21	1.8932e-011	2.4158e-013	4.7000e-002
	(III)	18	2.5231e-011	1.0581e-010	3.1000e-002
10	(I)	93	7.2673e-011	1.0370e-010	2.1900e-001
	(II)	22	5.7824e-011	9.7700e-013	7.8000e-002
	(III)	20	1.3781e-011	6.3878e-011	6.3000e-002
20	(I)	194	9.4062e-011	4.1506e-011	1.0000e+000
	(II)	25	2.5342e-011	4.4764e-013	1.4000e-001
	(III)	22	3.0866e-011	1.8796e-010	1.4100e-001
50	(I)	354	9.2170e-011	1.4666e-010	7.1560e+000
	(II)	29	4.3856e-011	2.9292e-012	6.8800e-001
	(III)	27	3.2127e-011	1.5333e-010	6.2400e-001
100	(I)	709	9.5834e-011	1.8690e-010	6.4609e+001
	(II)	33	9.4371e-011	7.6934e-012	3.6880e+000
	(III)	31	8.0705e-011	3.9339e-010	3.4060e+000

The spectral radii of the matrices at the points, where the methods (I) - (III) terminate are also reported:

n	5	10	20	50	100
u	6.1461e-1	7.3043e-1	8.6434e-1	9.2532e-1	9.6050e-1
v	1.8919e-1	1.9339e-1	2.0248e-1	2.0248e-1	2.0248e-1
p	1.3771e-1	1.5590e-1	1.5994e-1	1.6430e-1	1.6547e-1





Acknowledgement The authors are thankful to U. Schäfer for reading the complete paper and for clarifying the original version of the proofs of theorem 3 and 4.

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