Approximate fixed points of nonexpansive functions in product spaces (extended abstract)

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1 Introduction

This paper presents applications of another case study in the project of *proof mining*, by which we mean the logical analysis of mathematical proofs with the aim of extracting new numerically relevant information hidden in the proofs.

More specifically, we are concerned with the general theme of what is known about the existence of approximate fixed points for nonexpansive mappings in product spaces.

Let (X, ρ) be a metric space, and $C \subseteq X$ a non-empty subset. A mapping $T : C \to C$ is called *nonexpansive* if for all $x, y \in C$,

$$\rho(T(x), T(y)) \leq \rho(x, y).$$

The metric space (X, ρ) is said to have the *approximate fixed point property (AFPP)* for nonexpansive mappings if any nonexpansive mapping $T : X \to X$ has an approximate fixed point sequence; that is, a sequence $(u_n)_{n\in\mathbb{N}}$ in X for which $\lim_n d(u_n, T(u_n)) = 0$. It is easy to see that this is equivalent with

$$r_X(T) := \inf\{d(x, T(x)) \mid x \in X\} = 0.$$

If (X, ρ) and (Y, d) are metric spaces, then the metric d_∞ on $X \times Y$ is defined in the usual way:

$$d_\infty((x, u), (y, v)) = \max\{\rho(x, y), d(u, v)\}$$

for $(x, u), (y, v) \in X \times Y$. We denote by $(X \times Y)_\infty$ the metric space thus obtained.

A basic question now becomes:

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If (X,ρ), (Y,d) have the AFPP for nonexpansive mappings, then when does $(X\times Y)_\infty$ have the AFPP for nonexpansive mappings?

The following theorem was proved first by Espínola and Kirk [2] for normed spaces and then extended by Kirk to the more general class of hyperbolic spaces.

Theorem 1. *[6] Assume that X is a hyperbolic space, $C\subseteq X$ is a non-empty, convex, closed and* bounded *subset of X. If (M,d) is a metric space with AFPP for nonexpansive mappings, then*

$$H:=(C\times M)_\infty$$

has the AFPP for nonexpansive mappings.

In this paper, we present generalizations of this result to the case of *unbounded* convex subsets of *hyperbolic* spaces. We recall first some definitions and previous results.

A *hyperbolic space*[1] is a triple (X,ρ,W) where (X,ρ) is metric space and $W:X\times X\times[0,1]\to X$ such that

$$\begin{array}{ll}(W1) & \rho(z,W(x,y,\lambda))\le(1-\lambda)\rho(z,x)+\lambda\rho(z,y),\\ (W2) & \rho(W(x,y,\lambda),W(x,y,\tilde{\lambda}))=|\lambda-\tilde{\lambda}|\cdot\rho(x,y),\\ (W3) & W(x,y,\lambda)=W(y,x,1-\lambda),\\ (W4) & \rho(W(x,z,\lambda),W(y,w,\lambda))\le(1-\lambda)\rho(x,y)+\lambda\rho(z,w).\end{array}$$

If $x,y\in X$, and $\lambda\in[0,1]$ then we use the notation $(1-\lambda)x\oplus\lambda y$ for $W(x,y,\lambda)$.

The class of hyperbolic spaces contains all normed linear spaces and convex subsets thereof, but also the open unit ball in complex Hilbert spaces with the hyperbolic metric as well as Hadamard manifolds and CAT(0)-spaces in the sense of Gromov.

If $C\subseteq X$ is a non-empty convex subset of a hyperbolic space (X,ρ,W), and $T:C\to C$ is nonexpansive, then for any sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $[0,1]$, we can define the *Krasnoselski-Mann iteration* starting from $x\in C$ by

$$x_0:=x,\quad x_{n+1}:=(1-\lambda_n)x_n\oplus\lambda_nT(x_n).$$

An important result in the fixed point theory for nonexpansive mappings is the following theorem, due to Borwein, Reich, and Shafrir (generalizing earlier results of Ishikawa [5] and Goebel/Kirk [4]).

Theorem 2. *[1] If $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence in $[0,1]$ which is divergent in sum and bounded away from 1 then for all $x\in C$,*

$$\lim_n\rho(x_n,T(x_n))=r_C(T).$$

[1] If only axiom $(W1)$ is assumed this structure is a convex metric space in the sense of Takahashi [11]. If $(W1)$-$(W3)$ are assumed, the notion is equivalent to Kirk's spaces of hyperbolic type ([4]). Axiom $(W4)$ is used e.g. in [10]. See [7] for discussion of this and related notions.

In [8], we obtained the following quantitative version of theorem 2 (which subsequently turned out to be an instance of a general logical metatheorem, see [3]):

Theorem 3. *[8] Let (X,ρ,W) be a hyperbolic space, $C \subseteq X$ a non-empty convex subset and $T : C \to C$ a nonexpansive mapping. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[0,1)$, and $K \in \mathbb{N}, \alpha : \mathbb{N} \to \mathbb{N}$ be such that $\lambda_n \leq 1 - \frac{1}{K}$ and $n \leq \sum\limits_{i=0}^{\alpha(n)} \lambda_i$. Let $x, x^* \in C$ and $b > 0$ be such that*

$$\rho(x, T(x)) \leq b \wedge \rho(x, x^*) \leq b.$$

Then the following holds

$$\forall \varepsilon > 0 \forall n \geq h(\varepsilon, b, K, \alpha)\big(\rho(x_n, T(x_n)) < \rho(x^*, T(x^*)) + \varepsilon\big),$$

where the bound $h(\varepsilon, b, K, \alpha)$ can be explicitly computed.

Using theorem 3 the following quantitative version of a theorem due to Ishikawa [5] (for the normed case) and Goebel and Kirk [4] (for the hyperbolic case) can be obtained:

Theorem 4. *[8] Let $(X,\rho,W), C, K, \alpha, (\lambda_n)$ be as in the previous theorem and assume that (x_n) is bounded by $b \in \mathbb{N}$. Then the following holds*

$$\forall \varepsilon > 0 \forall n \geq \tilde{h}(\varepsilon, b, K, \alpha)\ \big(\rho(x_n, T(x_n)) < \varepsilon\big),$$

where the bound $\tilde{h}(\varepsilon, b, K, \alpha)$ can be explicitly computed.

The main significance of the bounds in the previous theorems is that they depend on x, x^*, T, C, X only via b. In particular, if in theorem 4, C is assumed to have a bounded diameter, then the convergence $\rho(x_n, T(x_n)) \to 0$ is uniform in x and T. This result was first obtained (ineffectively) in [4] and used in [6] to prove theorem 1 discussed above. In the next section we indicate how our stronger uniformity results can be used to generalize theorem 1.

2 Main results

In the sequel, (X,ρ,W) is a hyperbolic space and $C \subseteq X$ a non-empty convex subset. We assume that (M,d) is a metric space which has AFPP for nonexpansive mappings. Let $H := (C \times M)_\infty$ and $T : H \to H$ nonexpansive.

For each $u \in M$, let us define

$$T_u : C \to C, \quad T_u(x) = (P_1 \circ T)(x,u),$$

where $P_1 : C \times M \to C$ is the projection. Then T_u is nonexpansive for all $u \in M$, so we can associate with T_u the Krasnoselski-Mann iteration x_n^u starting with $x \in C$.

Applying theorem 3 to the family $(T_u)_{u\in M}$ of nonexpansive mappings, we can prove the following theorem.

Theorem 5. *Assume that*

$$\sup_{u\in M} r_C(T_u) < \infty,$$

and that there are $x \in C$ and $\varphi : \mathbb{R}_+ \to \mathbb{N}$ such that

$$\forall\varepsilon > 0 \forall v \in M \exists x^* \in C\big(\rho(x, x^*) \le \varphi(\varepsilon) \wedge \rho(x^*, T_v(x^*)) \le \sup_{u\in M} r_C(T_u) + \varepsilon\big).$$

Then

$$r_H(T) \le \sup_{u\in M} r_C(T_u).$$

Since each bounded C satisfies the hypotheses of the above theorem, we get as an immediate consequence theorem 1.
We also get

Corollary 1. *H has AFPP for all nonexpansive functions $T : H \to H$ satisfying:*

$$(*) \begin{cases} \text{there are } x \in C \text{ and } \varphi : \mathbb{R}_+ \to \mathbb{N} \text{ such that} \\ \forall\varepsilon > 0 \forall u \in M \exists x^* \in C\big(\rho(x, x^*) \le \varphi(\varepsilon) \wedge \rho(x^*, T_u(x^*)) \le \varepsilon\big). \end{cases}$$

The next theorem is obtained using theorem 4 above:

Theorem 6. *Let $x \in C$. Assume that there is a $d > 0$ such that*

$$\forall u \in M \forall n, m \in \mathbb{N}(\rho(x^u_n, x^u_m) \le d). \tag{1}$$

Then T has an approximate fixed point sequence.

We finish by pointing out that all the above results can be generalized to families $(C_u)_{u\in M}$ of non-empty *unbounded* convex subsets of the hyperbolic space (X, ρ, W).

All this will be carried out in detail together with many further generalizations in a forthcoming paper [9].

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