Iterated Belief Change and the Levi Identity

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Abstract

Most works on iterated belief change have focussed on iterated belief revision, namely, on how to compute $(K^*_x)^*_y$. However, historically, belief revision has been defined in terms of belief expansion and belief contraction that have been viewed as primary operations. Accordingly, what we should be looking at are constructions like: $(K^+_x)^+_y$, $(K^-_x)^+_y$, $(K^+_x)^-_y$ and $(K^-_x)^-_y$. The first two constructions are relatively innocuous. The last two are, however, more problematic. We look at these sequential operations. In the process, we use the Levi Identity as the guiding principle behind state changes (as opposed to belief set changes).

Keywords: Iterated belief change, lexicographic revision, iterated belief contraction.

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1 Introduction

How new evidence impigns upon the knowledge of a rational agent has been the subject of vigorous discussion in the last couple of decades. Alchourrón, Gärdenfors and Makinson [2], who initiated discussion on this issue in the non-probabilistic framework provided the basic formal foundation for this discussion. Several variations and extensions of the basic framework have since been investigated by different researchers in the area including belief update, multiple belief change, iterated belief change, and belief merging. The subject of this paper is largely to do with the problem of iterated belief change.

Belief change has been viewed as any form of change in an agent's beliefs. Three forms of belief change have been investigated in the literature: expansion – simple addition of new beliefs, even if it means the agent's beliefs contradict each other; contraction – removal of a belief from one's belief corpus; and revision – addition of new beliefs while ensuring that the resulting belief corpus is consistent. The result of expanding, contracting or revising a belief corpus K by a sentence x is respectively represented as the corpus K^+_x, K^-_x and K^*_x. Properties of these operations are captured by well known rationality postulates, and constructive approaches to these operators are available in the literature. K^+_x is simply defined as $Cn(K \cup \{x\})$ where Cn is the consequence operation of the background logic. The connection between these operators is captured by the famous Levi Identity: $K^*_x = (K^-_{\neg x})^+_x$. So, belief revision can always be taken to be a secondary notion constructed via the primitive operations of belief expansion and belief contraction.

By "Iteratd Belief Change" we refer to the problem of dealing with sequential changes in belief. On the face of it, then, iterated belief change should deal with how we can construct the corpus $(K^\Box_x)^\diamond_y$ given belief corpus K, sentences x and y and belief change operations $\Box$ and $\diamond$. Literature in the area have largely dealt with iterated belief revision: constructing $(K^*_x)^*_y$. Given the Levi Identity, it would appear that we could do away with revision, in favour of expansion and contraction. If so, then what we should be discussing instead are construction of corpora such as $(K^+_x)^+_y$,$(K^-_x)^+_y$, $(K^+_x)^-_y$ and $(K^-_x)^-_y$. The first two of these construction, where the second operation is expansion, are unproblematic (given a contraction operation), since expansion is a very simple operation. It is the last two of these constructs that pose rather difficult problems. The aim of this paper is to address these two forms of iterated belief change.

Let us look at these problems in some what more detail. Expansion operation is not state sensitive – K^+_x is completely determined by K and x. But contraction operation is. The set K^-_x is not fully determined by K and x: depending on what belief state K is associated with, the value of K^-_x would be different. In particular, belief contraction inherently involves a choice among multiple candidate beliefs for removal, and the preference information that determines this choice is in the belief state but is extraneous to the belief set K.

Assume that a belief set K, two sentences x and y, and an appropriate contraction operation $-$ are given. Since $+$ is not state sensitive, $(K^+_x)^+_y$ is simply $Cn(K \cup \{x, y\})$. Similarly, $(K^-_x)^+_y$ is simply $Cn(K^-_x \cup \{y\})$, which is easily determined given that we know how the contraction operation $-$ behaves. But since $-$ is state sensitive, the construction of $(K^+_x)^-_y$ and of $(K^-_x)^-_y$ can not be subjected to such simple treatment. Assuming that K is different from K^+_x (respectively K^-_x), they are part of different belief states, and hence the contraction operation appropriate for removing beliefs from K is not appropriate for removing beliefs from K^+_x (respectively from K^-_x). This paper is therefore primarily about characterising the belief sets $(K^+_x)^-_y$ and of $(K^-_x)^-_y$.

In previous works [6] we have argued that the belief revision operation $*$ itself may be taken to represent the belief state, with the attendant view that alongwith changes in belief, the belief revision operation $*$ also changes. It has been argued that [] this change in the operation can be conveniently hidden under simple notation: $*$ denotes the primoddial belief state, and $*(i)$ for input sequence (i) denotes the belief set that results from the input sequence (i). If so, then the contraction operation $-$ can as well be taken to be an alternative representation of the belief state. There is some asymmetry between the representation of a state by a revision operation $*$ and a contraction operation $-$ however. In the former case, since we are interested in revision, given initial state $*$, subsequent states can be represented simply by $*(i)$ where (i) is the input sequence. But in the latter case an input sequence itself does not specify the required change to the initial state $-$. For instance, to take a simple case, given some input sentence x, we cannot tell what state $-(x)$ really refers to since the sentence x could be an argument for contraction or expansion!

Although classical belief change accounts (such as AGM [1, 2]) provide a systematic way to deal with belief change, they are designed for dealing with one-shot belief change rather than iterated belief change. While providing a cogent account of how a rational agent should change his/her beliefs in light of a piece of evidence, they fail to give a systematic account for belief change as an iterative process of how an epistemic agent should deal with a sequence of evidence. A dynamic approach to iterated belief revision has been proposed by Nayak, *et.al* [6] to resolve this problem. The purpose of this paper is to complement this dynamic approach by supplying it with the iterated expansion and contraction operations.

In Section 1, we briefly discuss the AGM Framework, its problems and how others have attempted to resolve the problem and then we review the dynamic approach proposed in [6] which provides an iterative account to deal with belief revision. The paper will then complement this approach by the dynamic expansion and contraction operators. Section 2 begins with the analogues of the Levi identity which establishes the relationship between the three belief change operators: revision, contraction and expansion. The dynamic expansion operator will be introduced briefly next, followed by a sematic approach to the construction of a dynamic contraction operator. Section 3 provides a partial syntatic account for this dynamic contraction operator. Finally, we conclude with a brief summary and discussion.

2 Background

The theory of belief change purports to model how a current theory or body of beliefs, K, can be rationally modified in order to accommodate a new observation x. A piece of observation, such as x is represented as a sentence in a propositional language L, and a theory, such as K, is assumed to be a set of sentenecs in L, closed under a supraclassical consequence operation, Cn. Since the new piece of information x may contravene some current beliefs in K, chances are, some beliefs in K will be discarded before x is eased into it. Accordingly, three forms be belief change are recognised in the belief change framework:

1. CONTRACTION: K^-_x is the result of discarding some unwanted information x from the theory K
2. EXPANSION: K^+_x is the result of simple-mindedly incorporating some information x into the theory K
3. REVISION: K^*_x is the result of incorporating some information x into the theory K in a manner so as to avoid internal contradiction in K^*_x.

The intuitive connection among these operators is captured by the following two identities named, respectively, after Isaac Levi and William Harper:

1. LEVI IDENTITY: $K^*_x = (K^-_{\neg x})^+_x$
2. HARPER IDENTITY: $K^-_x = K^*_{\neg x} \cap K$.

There is another, third, identity that, though well known, has not merited special nomenclature:

1. THIRD IDENTITY:

$$K^+_x = \begin{cases} K^*_x & \text{if } \neg x \notin K \\ K_\perp & \text{otherwise} \end{cases}$$

The three belief change operations are traditionally introduced with three sets of *rationality postulates*. These postulates, along with motivation and interpretation for them, may be found in [1]. The expansion operation is very easily constructed:

- $K^+_x = Cn(K \cup \{x\})$

Contraction and Revision operations are relatively more sophisticated operations since they deal with choice. The three identities mentioned above show that the three operations are to a large extent inter-definable. However, right from the start, the contraction and expansion operations have been taken to be more fundamental operations than the revision operation, and accordingly, the Levi Identity has typically been used to define revision.

The AGM postulates deal with "one-shot" belief change. The only interesting inference about iterated belief revision that can be drawn from the AGM postulates is the following AGM-It (Nayak *et.al*, 2003, p. 4)

(AGM-It): If $\neg y \notin K^*_x$ then $(K^*_x)^*_y = K^*_{x \wedge y}$

that places no constraints on iterated belief change when $\neg y \in K^*_x$. Consider the following problem (Darwiche $\&$ Pearl, 1997 [4]). We learnt from independent sources that X is smart and and that X is rich. In other words, we would retain the belief that X is rich even if X is found not to be smart and vice versa. Imagine that we first obtained some information to the effect that X is not smart, so we still retain our belief that X is rich. But then, we receive another piece of evidence confirming that X is smart after all. Should we, or should we not, now believe that X is rich? Intuitively, we should continue to believe that X is rich, since the two (sequential) pieces of evidence have no bearing upon whether or not X is rich. However, surprisingly, as the AGM-It does not handle the case $\neg smart \in K^*_{\neg smart}$, the AGM system would allow us to believe that X is smart and not rich!

To alleviate this situation, several proposals have been advanced. Here we revisit the proposal by Nayak, *et.al* [6], namely, that the revision operation $*$ is best viewed as a dynamic operator, an operator that evolves as new pieces of information are accepted. Accordingly, the AGM framework is extended as follows:

(0^*)	$(K_\perp)^*_x = Cn(\{x\})$ for any sentence x	(Absurdity)
$(1^*$-$6^*)$	As in the AGM	
$(7^*$new)	If $x \wedge y \not\vdash \perp$ then $(K^*_x)^{*\|x}_y = K^*_{x \wedge y}$	(Conjunction)
$(8^*$new)	If $x \wedge y \vdash \perp$ but $\not\vdash \neg x$ then $(K^*_x)^{*\|x}_y = K^*_y$	(DP2′)

The last two postulates may be viewed as constraints on $|$ rather than on $*$, and in fact jointly defining the meta-revision operation $|$ itself.

It has been argued ([6]) that this dynamic revision operator, representing a belief state, is best viewed as a unary operation, say initially $*_0$. When a piece of evidence e_i (for $i > 0$) is received, the then current belief state $*_{i-1}$ gets updated to the new belief state $*_i$ as follows: $*_i = *_{i-1}|e_i$. A revision step is carried out in two sub-steps. Given current belief set $K_{i-1} = *_{i-1}(\top)$ and evidence e_i,

$$K_i := K_{e_i}^{*_{i-1}}$$
$$*_i := *_{i-1}|e_i$$

This belief (dynamic) revision operation is given a semantic characterisation in terms of an evolving plausibility ordering over the interpretations generated by the background language.

Definition 1 *Let Ω be the set of possible worlds (interpretations) of the background language $\mathcal{L}$ and $\sqsubseteq$ a total preorder (a connected, transitive and reflexive relation) over Ω. For any set $\Sigma \subseteq \Omega$ and world ω we will say ω is a $\sqsubseteq$-minimal member of Σ if and only if both $\omega \in \Sigma$ and $\omega \sqsubseteq \omega'$ for all $\omega' \in \Sigma$.*

By $\omega_1 \sqsubseteq \omega_2$ we will understand that ω_2 is not more plausible than ω_1. The expression $\omega_1 \equiv \omega_2$ will be used as a shorthand for ($\omega_1 \sqsubseteq \omega_2$ *and* $\omega_2 \sqsubseteq \omega_1$). The symbol $\sqsubset$ will denote the strict part of $\sqsubseteq$. For any set $S \subseteq \mathcal{L}$ we will denote by $[S]$ the set $\{\omega \in \Omega \mid \omega \models s \text{ for every } s \in S\}$. For readability, we will abbreviate $[\{s\}]$ by $[s]$. Intuitively, the preorder $\sqsubseteq$ will be the semantic analogue of the revision operation $*$, and represents the belief states of an agent. We will say that $K_\sqsubseteq$ is the belief set associated with the preorder $\sqsubseteq$. It is defined as the set of sentences satisfied by the $\sqsubseteq$-minimal worlds, i.e.

$$K_\sqsubseteq = \{x \in \mathcal{L} \mid \omega \models x \text{ for all } \sqsubseteq\text{-minimal } \omega \in \Omega\}$$

An inconsistent belief state is represented by an empty relation $\sqsubseteq_\bot$: for every pair $\omega, \omega' \in \Omega, \omega \not\sqsubseteq_\bot \omega'$.

A modified Grove-Construction [**?**] is used to construct the revision operation from a given plausibility relation:

Definition 2 ($\sqsubseteq$ to *)

$$x \in K_e^{*_\sqsubseteq} \textit{ iff } \begin{cases} [e] \subseteq [x] & \textit{if } \sqsubseteq = \sqsubseteq_\bot \\ \omega \models x \textit{ for every } \omega \sqsubseteq\textit{-minimal in } [e] & \textit{otherwise.} \end{cases}$$

The plausibility ordering is stipulated to evolve as follows. TWO SPECIAL CASES:

1. If $[e] = \emptyset$ then, and only then, $\sqsubseteq_e^\circ = \sqsubseteq_\bot$
2. Else, if $\sqsubseteq = \sqsubseteq_\bot$, then $\omega_1 \sqsubseteq_e^\circ \omega_2$ iff either $\omega_1 \models e$ or $\omega_2 \models \neg e$.

GENERAL CASE:

When the prior preorder is nonempty ($\sqsubseteq \neq \sqsubseteq_\bot$) and the evidence is satisfiable ($[e] \neq \emptyset$),

1. If $\omega_1 \models e$ and $\omega_2 \models e$ then $\omega_1 \sqsubseteq_e^\circ \omega_2$ iff $\omega_1 \sqsubseteq \omega_2$
2. If $\omega_1 \models \neg e$ and $\omega_2 \models \neg e$ then $\omega_1 \sqsubseteq_e^\circ \omega_2$ iff $\omega_1 \sqsubseteq \omega_2$
3. If $\omega_1 \models e$ and $\omega_2 \models \neg e$ then $\omega_1 \sqsubset_e^\circ \omega_2$

2.1 Need for a Dynamic Contraction Operator

Just as there is a need for itareted belief revision, there is a *prima facia* case for iterated belief expansion and iterated belief contraction. The former is trivial: $(K_x^+)_y^+ = Cn(Cn(K \cup \{x\}) \cup \{y\}) = Cn(K \cup \{x, y\})$. Iterated Belief Contraction, however, does not soccumb to such easy solution. Just like revision, contraction involves choice; hence iterated belief contraction would presuppose an account of contracting from a choice mechanism. It is little curprise that the rationality postulates of belief contraction offered by the AGM does notprovide a cogent account of iterated belief contraction:

Closure:	K_x^- is a theory
Inclusion:	$K_x^- \subseteq K$ (inclusion)
Vacuity:	If $x \notin Cn(K)$ then $K_x^- = K$
Success:	If $\emptyset \not\vdash x$ then $x \notin Cn(K_x^-)$
Preservation:	If $Cn(x) = Cn(y)$ then $K_x^- = K_y^-$
Recovery:	$K \subseteq Cn((K_x^-) \cup \{x\})$
Intersection:	$(K_x^-) \cap (K_y^-) \subseteq K_{x\wedge y}^-$
Conjunction:	If $x \notin K_{x\wedge y}^-$ then $K_{(x\wedge y)}^- \subseteq K_x^-$

Interestingly, there is very little discussion in the literature regarding iterated belief contraction – an exception being [5]. Presumably the reason behind such reluctance is the fact that, in some sense or other, belief revision and belief expansion are "natural" operations where as belief contraction is a "theoretical construct". Despite the persuasion of literary critics to view the *willing suspension of disbelief* as a constituting ingredient of poetic faith[1], in the belief change literature, belief contraction remains a second class citizen. However, even if

[1]Samuel Taylor Coleridge in *Biographia Literaria*.

belief contraction is not as natural as other forms of belief change, iterated belief contraction deserves the researchers' attention – if not for anything else, for the sake of completeness. As we see below, there are more compelling reasons to study iterated belief contraction.

2.2 Argument from the Levi Identity

As mentioned before, Levi Identity $K^*_x = (K^-_{\neg x})^+_x$ has generally been used to define the revision operation via the expansion and contraction operation, the latter two being viewed to be more basic. This fact has been however largely ignored in the belief revision literature. In particular, The computation of $(K^*_x)^*_y$ has been carried out directly by providing an account of how a belief state evolves in light of newly accepted information, instead of decomposing the expression $(K^*_x)^*_y$ into $(((K^-_{\neg x})^+_x)^-_{\neg y})^+_y$. The primacy of $+$ and $-$ over $*$ however manadtes that we provide an account of how the belief set $(((K^-_{\neg x})^+_x)^-_{\neg y})^+_y$ should be computed. It follows then that we should in general be asking outselves how to compute sets such as

1. $(K^+_x)^+_y$
2. $(K^-_x)^+_y$
3. $(K^+_x)^-_y$, and
4. $(K^-_x)^-_y$.

Out of these, the first two constructs, the second operation in them being an expansion operation, are relatively innocuous. The last two constructs, however, deserve more careful attention: since the second operation is a contraction operation, it will be dependent on the intermediate epistemic state that would have evolved from the initial epistemic state associated with K. Notice that the last construct is exactly the problem of iterated belief contraction.

We need a reasoned account of iterated belief change (expansion and contraction) in which the second operation is a contraction operation. Such an account can be provided by providing an account of how an epistemic state gets modified when subjected to expansion or contraction by a sentence. The aim of the next section is to explore this issue.

3 Approach

Given a contraction function, we can construct a revision function by first contracting everything in the belief set that would cause the addition of x to lead to inconsistency and then expanding the belief set by x. That is the intuition behind the Levi Identity. It is the Levi Identity that embodies the idea that revision is reducible to contraction and expansion – the idea that forces us to examine different combinations of contraction and expansion, different forms of *belief change*. However, the Levi Identity, as traditionally conceived, involves modification of a *belief set*, where as iterated belief revision (and dynamic belief revision) involves revision of the preorder over possible worlds: $\sqsubseteq_x^\circ$ is taken to be the resultant preorder, when the given preorder $\sqsubseteq$ is revised in light of an accepted input sentence x. It is therefore desirable to obtain an analogue of the Levi Identity:

> NEW LEVI IDENTOTY $\sqsubseteq_x^\circ = (\sqsubseteq_{\neg x}^\ominus)_x^\oplus$
> where $\ominus$ is a preorder contraction operation, and $\oplus$ is a preorder expansion operator.

Therefore, our aim now is to define these two new operators $\sqsubseteq_x^\ominus$ and $\sqsubseteq_x^\oplus$ similar to $\sqsubseteq_x^\circ$ in such a way that this analogy is preserved. Once this aim is achieved, it will be sufficient to characterise any belief change by using only the contraction and expansion preorders.

3.1 Preorder Expansion – semantics

Belief expansion is the simplest form of belief change. In the AGM account, belief expansion is captured by: $K_x^+ = Cn(K \cup \{x\})$. Semantically speaking, $[K_x^+] = [K] \cap [x]$: the result of accepting information x results in a state that entertains exactly those worlds that satisfy all the old beliefs as well as the accepted piece of information. It follows that expansion does not handle inconsistency very well – if the new piece of information conflicts with the current beliefs, the agent ends up believing anything and everything.

As mentioned in section 2, expansion can be defined in terms of belief revision (the "Third Identity"). This motivates the way we define the expansion preorder as follows:

$$\sqsubseteq_x^\oplus = \begin{cases} \sqsubseteq_x^\circ & \text{if } \neg x \notin K \\ \sqsubseteq_\perp & \text{otherwise} \end{cases}$$

In other words, if there exists a world $\omega \in [x]$ such that ω is $\sqsubseteq$-minimal, then the belief would be the same whether we expand it by x or revise it by x. However, if the current belief does not allow any world ω which is consistent to $[x]$, then its expansion to include x will result in an inconsistent state.

With this definition of the dynamic expansion operator, we are now in the position to construct the dynamic contraction operator.

3.2 Contraction Operations Satisfying the New Levi Identity

The use of a system of nested spheres of worlds to visually represent the preorder $\sqsubseteq$ is wellknown [3]: a world more central in the system represents a more plausible world than one relatively less central. A sphere is a set of possible worlds for a given belief set and a system of spheres is a set of nested spheres which can be considered as an ordering of plausibility over the worlds; the more plausible worlds lying closer to the centre of the system of spheres. The smallest sphere at the centre of the system represents the current beliefs in the sense that it consists of exactly the worlds that satisfy the current beliefs. Two boundary cases of such representation of a belief state are:

1. FULL PREORDER. If $\omega \sqsubseteq \omega'$ for all worlds ω and ω', the system of sphere is conflated to a single sphere. It represents the state of complete epistemic innocence – the agent in question holds no contingent beliefs whatsoever. It is the state of null information $\sqsubseteq_\top$: the associated belief set is $Cn(\emptyset)$.

2. EMPTY PREORDER. If $\omega \sqsubseteq \omega'$ for *no* two worlds ω and ω', the state in question represents the "epistemic hell", a state in which the agent believes every conceivable state of affairs. This is the state of full information $\sqsubseteq_\perp$: the associated belief set is $K_\perp = Cn(\perp)$.

In order to contract from the state of null information, nothing needs to be done. So,

$$(\sqsubseteq_\top)^\ominus_x = \sqsubseteq_\top \text{ for all } x.$$

However, contracting from the state of full information is not so obvious. To contract $K_\perp$ by x, we need to allow some world which is consistent to $\neg x$ to be included in the resultant belief state. So it is reasonable to suggest that $(K_\perp)^-_x = Cn(\{\neg x\})$. However, in this case, the agent will end up believing $\neg x$ but this should not be allowed as there is no evidence to support either x or $\neg x$. Therefore, it is more appropriate that the agent will lose all the information and start his/her epistemic life again when he/she reaches this state $(K_\perp)^-_x = Cn(\emptyset)$. Accordingly we postulate that

$$(\sqsubseteq_\perp)^\ominus_x = \sqsubseteq_\top \text{ for all } x.$$

Now let us look at how state contraction should function in the principal case. Conside the initial state $\sqsubseteq$ be represented by a sustem of spheres $[K_0] \subseteq [K_1] \subseteq [K_2] \subseteq [K_3]$ where, as illustrated in Figure 1, $[K_0] = 1$, $[K_1] = [K_0] \cup 2 \cup 5$, $[K_2] = [K_1] \cup 3 \cup 6$ and $[K_3] = [K_2] \cup 4 \cup 7$. We are interested in contracting this state by the belief x, where $[\neg x] = 5 \cup 6 \cup 7$. In order to satisfy the original Levi Identity, it will suffice if the state resulting from this contraction centers on $[K_0] \cup 5$, since that would ensure that if K_0 was to be revised by $\neg x$, the resulting theory will hold of exactly those worlds that are minimal in $[\neg x]$, that is, 5. This effectively is the relevant faithfulness condition for belief contraction.

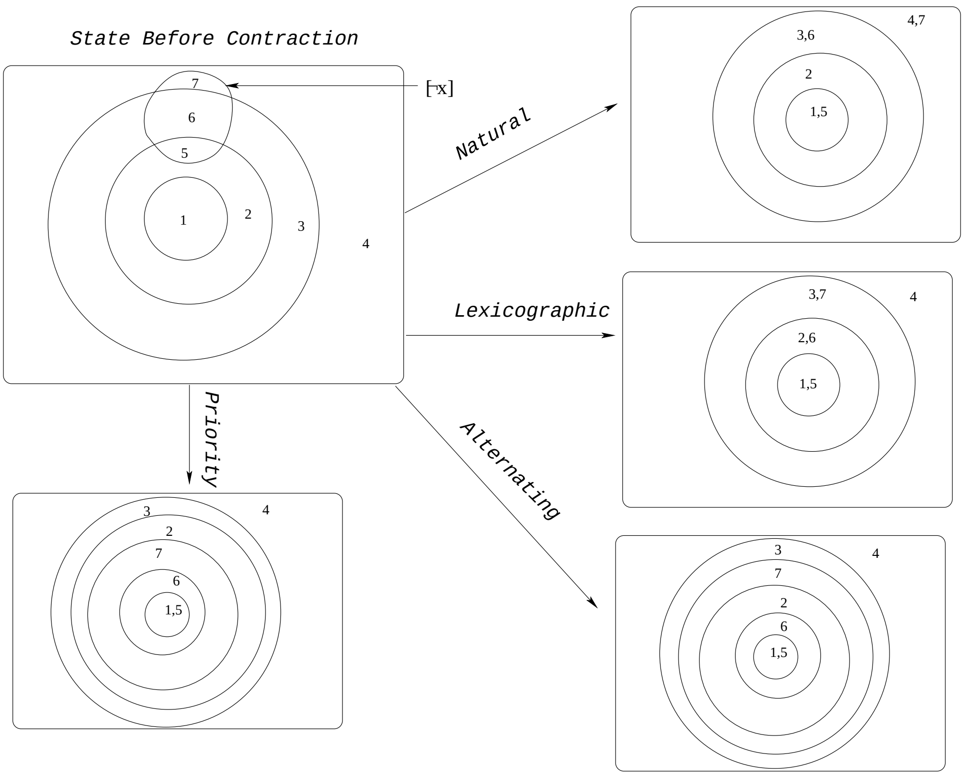

Figure 1: States before and after the contraction

The new Levi Identity imposes one more constraint on this. It effectively says that, the resultant state after the contraction, apart from having $[K_0]$ as the center, must ensure that the prior ordering of worlds inside $[x]$ (respectively outside $[x]$) should not be disturbed. That is, for any two worlds ω and ω' that are both inside $[x]$ (or both inside $[\neg x]$), $\omega \sqsubseteq_x^{\ominus} \omega'$ iff $\omega \sqsubseteq \omega'$. This condition is quite appealing, and is reminiscent of conditions well known in the context of iterated belief revision (Spohn's OCF's, Darwiche & Pearl's account, as well as Nayak's Lexicographic Revision respect this condition). It turns out that in the current context, the new Livi Identity is liberal enough to allow many different construction of the state contraction operation $\ominus$. In Figure 1 we illustrate four such constructions, each of which looks reasonable, and satisfies the new Levi

Identity.

1. NATURAL CONTRACTION: The only modification in the starting state effected is due to the faithfulness requirement. All other worlds are left as before.

2. LEXICOGRAPHIC CONTRACTION: Faithfulness puts $[K_0]$ and the worlds in 5 at the same footing. All other worlds are "shifted" accordingly, thus for instance, worlds in 2 and in 6 are viewed to be at par with each other.

3. ALTERNATING CONTRACTION: Faithfulness is respected. Then, repeatedly, the next best worlds in $[x]$ and $[\neg x]$ are alternated, with $[\neg x]$ being given priority.

4. PRIORITY CONTRACTION: All worlds in $[\neg x]$ are given more priority than all worlds in $[x]$, subject to the satisfaction of Faithfulness. Faithfulness is respected.

It is easily noticed that all these four constructions of a state contraction operation will satisfy the New Levi Identity. Hence, if we must identify a unique state contraction operation, further reasonable principles must be identified and adhered to. We find such a principle in the generalisation of the Harper Identity, as discussed below.

3.3 New Harper Identity

In the context of classical belief change, the Levi Identity is equivalent to the Harper Identity; $K_x^- = K \cap K_{\neg x}^*$. Semantically the Harper Identity says that the $\sqsubseteq$-minimal worlds in Ω and the $\sqsubseteq$-minimal worlds in $[\neg x]$ are to be given equivalent status in the state resulting from the contraction of $\sqsubseteq$ by x. This Identity is easily generalised as follows:

> Let $B_i, 0 \leq i \leq n-1$ be the n bands ($\sqsubseteq$-equivalence classes) of worlds generated by the the pre-contraction state $\sqsubseteq$, where B_0 consists of the $\sqsubseteq$-minimal worlds in Ω and each $\omega \in B_i \sqsubset$ each $\omega' \in B_j$ for every $i < j$. Let k be the smallest index such that $B_k \cap [\neg x] \neq \emptyset$. Define $C_i = B_{k+i} \cap [\neg x]$ for $0 \leq i \leq n-k$, and $C_i = \emptyset$ for $n-k+1 \leq i \leq n$. The bands in $\sqsubseteq_x^{\ominus}$ are given by $D_i = B_i \cup C_i$, for $0 \leq i \leq n-1$.

It is easily verified that the new Harper Identity subsumes the new Levi Identity. Furthermore, Lexicographic contraction is the only state contraction operation described above that satisfies the new Harper Identity.

3.4 Test Case

We have noticed that the new Levi Identity and the new Harper Identity argue in favour of adopting Lexicographic Contraction as the correct state contraction operation. In this section we examine a test case to see how this operation fares *vis a vis* our intuitive judgment about iterated contraction. We consider a variant of a well known example due to Darwiche and Pearl [4]:

We initially believe on independent grounds that x is `smart` and that x is `rich`. That is, removing `smart` leaves `rich` undisturbed, and similarly, removing `rich` leaves `smart` undisturbed. The question is, what should we believe if we were to first remove `smart` followed by removal of `rich`. That is, what should be $K^{\ominus}_{smart})^{\ominus}_{rich}$? Intuitively, the resultant belief set should have nothing interesting to say about `smart` and `rich`.

Figure 2 below illustrates this scenario.

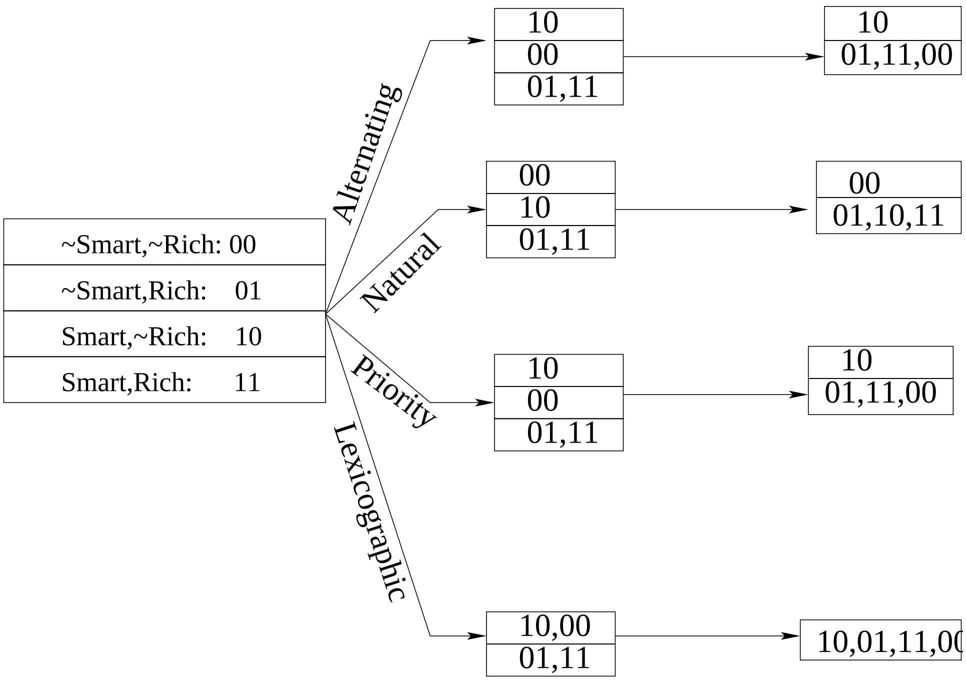

Figure 2: A variant of Smart-Rich problem

It is easily verified that the result of using Lexicigraphic Contraction in this case of iterated belief contraction concurs with our intuitive expectation. In contrast, the other operations leave residual beliefs – the Natural Contraction allows the agent to retain the belief $\texttt{smart} \vee \texttt{rich}$, while both Priority and Alternating retain $\texttt{smart} \rightarrow \texttt{rich}$.

Thus, this test case adds further credence to Lexicographic Contraction.

4 Partial Characterisation

In this section we provide some technical results that partially characterise the Lexicographic contraction operation. Before the results are pesented, we provide a rigorous definition of this operation. The preorder operation $\sqsubseteq_e^\ominus$ based on the total preorder $\sqsubseteq$ and the sentence x to be removed with the following constraints:

Condition 1:	If $\omega, \omega' \in [x]$ then $\omega \sqsubseteq_x^\ominus \omega'$ if and only if $\omega \sqsubseteq \omega'$
Condition 2:	If $\omega, \omega' \in [\neg x]$ then $\omega \sqsubseteq_x^\ominus \omega'$ if and only if $\omega \sqsubseteq \omega'$
Condition 3:	If $\omega \in [x], \omega' \in [\neg x]$ then $\omega \sqsubseteq_x^\ominus \omega'$ if and only if for every chain $\omega'_0 \sqsubset \omega'_1 \sqsubset \cdots \sqsubset \omega'$ in $[\neg x]$ of length n, there exists a chain $\omega_0 \sqsubset \omega_1 \sqsubset \cdots \sqsubset \omega$ in $[x]$ of length n
Condition 4:	If $\omega \in [\neg x], \omega' \in [x]$ then $\omega \sqsubseteq_x^\ominus \omega'$ if and only if for every chain $\omega'_0 \sqsubset \omega'_1 \sqsubset \cdots \sqsubset \omega'$ in $[x]$ of length n there exists a chain $\omega_0 \sqsubset \omega_1 \sqsubset \cdots \sqsubset \omega$ in $[\neg x]$ of length n

Here we simply list some of the important properties of $\ominus$. The complete characterisation and proof of the properties will be provided in the complete paper.

1. $(K_x^-)_y^- = K_y^-$ only if $K_y^- \subseteq K_x^-$.
   This property shows that in the iterated contraction $(K_x^-)_y^-$, the effect of the first contraction is washed away by the second contraction only if $K_y^- \subseteq K_x^-$. This is understandable since removal of y removes more information then removal of x.

2. $(K_x^-)_y^- = K_x^-$ if and only if either $\models y$ or $y \notin K_x^-$
   This property shows that in the iterated contraction $(K_x^-)_y^-$, the second contraction will have no effect if and only if either y is a tautology or y does not exist in the belief set after the first contraction.

3. If $y \vdash x$ then $(K_x^-)_y^- = K_{x \vee y}^-$

4. If $x \vdash y$ and $y \notin K_x^-$ then $(K_x^-)_y^- = K_{x \wedge y}^-$

5. If $x \in K_y^-$ and $y \in K_x^-$ then $(K_x^-)_y^- = K_x^- \cap K_y^-$

6. If $x \vdash y$ and $y \in K_x^-$ then $(K_x^-)_y^- = K_x^- \cap K_y^-$.
   Properties 5 and 6 concern scenarios in which $(K_x^-)_y^- = K_x^- \cap K_y^-$. In these cases, the order of the contractions is not important as the resultant belief set after the two contractions is always the intersection of the two belief sets after each individual contraction. The set of worlds in the

resultant belief sets is the union of the current worlds K and the minimal world of $[\neg x]$ and the minimal world of $[\neg y]$.

7. If $x \notin K_y^-$, $y \in K_x^-$ and $x \not\vdash y$ then $(K_x^-)_y^- = K_x^- \cap K_{x \vee y}^-$.

5 Conclusions

In this paper we started with the idea of using the expansion and contraction operation instead of the revision operation, in the context of iterated belief change. This was motivated by the Levi Identity. We noticed that it naturally led to the problem of iterated contraction.

In the process we defined state expansion and state contraction operations. For this purpose we used an analogue of the Levi Identity appropriate for state transformations. It turned out that this analogue is not strong enough to determine a unique state contraction operation. We then argued that an analogue of the Harper Identity leads to a reasoned account of state contraction, which naturally corresponds to the idea behind lexicographic Revision. An examination of a test case lends further support to such a state contraction operation.

We have provided a partial characterisation of this operation. Complete characterisation of this operation, as well as its generalisation to complement belief merging will be taken up in our future works.

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