A New Approach on Many Objective Diversity Measurement

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Abstract. In this paper, we introduce two measurements for computing the diversity and spread of non-dominated solutions in the objective space. These measurements compute the angular positions of solutions in the objective space and are able to find a percentage which indicates the distribution of solutions in the space. Also, because we are able to compute the positions of the solutions, the spread of solutions along the non-dominated front can also be measured. This is more important when we evaluate solutions of a problem with a large number of objectives, the objective space of which cannot be illustrated graphically. These measurements are being examined to measure distribution of several sets of non-dominated solutions in the objective space.

1 Introduction

Most of the Multi-objective Optimization (MO) methods approximate the Pareto-optimal front by a set of non-dominated solutions. The approximated solutions (non-dominated solutions) must have obtained a) good convergence to the Pareto-optimal front, and b) good diversity and spread along the front. The quality of an approximated set can be obtained by comparing it with the set of Pareto-optimal solutions and this can only be achieved for problems, the Pareto-optimal front of which is available. In the 90's, the diversity and convergence of solutions to the Pareto-optimal front (in the objective space) are being evaluated by by visual inspection. This kind of evaluations is possible for 2- and 3-objective test functions. However, in general the quality of the approximated sets must be measured by a quantitative metric. Different metrics have been studied in [10, 7, 5, 5, 15]. Here, we categorize these methods into two groups. The first group consists of methods which compare two approximated sets. Indeed, they deliver no information about the diversity, spread and convergence of one approximated set. But in fact two approximated sets are compared. Many scientist use these methods to observe the improvements of the new approaches. Most of the existing measurements belong to this group.

The second group consists of methods which deliver information about an approximated set. These methods are valuable when there are no information about the Pareto-optimal front or when there is no possibility to observe the

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approximated set graphically, for example for m-objective problems ($m \geq 4$). This measurements can also be used as the stopping criteria in the iterative methods like Multi-Objective Evolutionary Algorithms (MOEAs).

In this paper, we study the latter methods and introduce two quality measurements, which evaluate the diversity and spread of a set of non-dominated solutions. These methods compute the percentage of the space occupied by the solutions and also calculate the spread of solutions. They are applied to the obtained results of several test problems and the qualities of solution sets are studied.

This paper is organized as follows: In the following, we study the background and the existing methods. Section 2 is dedicated to the diversity and spread measurements. Several experiments and the results are explained in Section 3 and Section 4 concludes the paper with a summary and future work.

1.1 Background

It is stated by Deb [1] that the quality of a set of non-dominated solutions can be measured in terms of *convergence* and *diversity* of the solutions. It has also been studied by Zitzler et al. [18] that for comparing two approximated sets, several quality measurements are required.

Here, we consider those methods, which evaluate the spread and distribution of solutions. Figure 1 illustrates examples of spread and diversity of solutions. In this figure, the Pareto-optimal front is illustrated by a solid line and we consider that the MO method has approximated the front with 5 non-dominated solutions. In Figure 1 (a), the non-dominated solutions are located with equal distances to each other. Also, they have a good spread on the Pareto-optimal front, since two of the approximated solutions, namely A and B are lying on the so-defined *extreme* solutions. Extreme solutions show the upper bounds of the objective functions over the Pareto-optimal front. One component of these solutions must be the maximal value of each objective [2]. The solutions in Figure 1 (b) have a good spread but not a good distribution and in contrast to these, the solutions in Figure 1 (c) have good distribution and not a good spread.

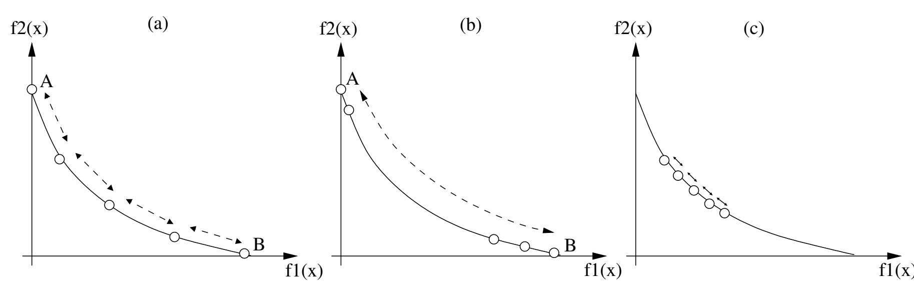

Fig. 1. Examples of solutions with different diversities and spreads along a Pareto-optimal front. The Pareto-optimal front is shown by solid line.

By considering this example, one possible way to evaluate the distribution of solutions is to calculate the distances between the non-dominated solutions. Deb et al. [2] proposed a method to measure the distribution of non-dominated solutions as follows: They compute the Euclidian distance d_i between the solutions in the non-dominated set A and then calculate the average of them as $\bar{d}$. Then, the distribution metric $\Delta_s^{'}$ is as follows:

$$\Delta_s^{'} = \sum_{i=1}^{|A|-1} \frac{|d_i - \bar{d}|}{|A| - 1} \tag{1}$$

This method is able to compute the distribution of solutions in 2-objective spaces. However, it cannot be used for problems with more than two objectives, as consecutive sorting is involved. Another variation of this method is proposed by Deb et al. [1, 2], which also considers the spread of solutions as follows:

$$\Delta = \frac{d_f + d_l + \sum_{i=1}^{|A|-1} |d_i - \bar{d}|}{d_f + d_l + (|A| - 1)\bar{d}} \tag{2}$$

where d_f and d_l are the Euclidian distances between the extreme solutions and the boundary solutions of A. Also this method works only for 2-objective problems. Another method called *Spacing* is introduced by Schott in [12]. This method is based on computing the shortest distances between the non-dominated solutions along each axis. In the cases that solutions are gathered in small groups along the non-dominated front, the distances between the groups are not considered, because only the shortest distances are computed and therefore, it may be misleading.

However, there are also other techniques for computing the spread and distribution of solutions like *Maximum Spread* [16], *Chi-Square-Like Deviation* [1] and *Uniform Distribution* [13]. Maximum Spread method [16] uses a similar technique in S metric [15]. In S metric, a hyper-volume between the solutions and a reference point is calculated, where in Maximum Spread method, a hyper-volume between the minimum and maximum values among each axis and a reference point is calculated. Other methods like *Entropy approach* [5] and *Sparsity measure* [3] are utilized to compare the diversity of solutions of two non-dominated sets, the outcome of which is a value of having no information when considering only one approximated set.

In fact, what we are going to emphasize here is that the existing methods do not evaluate one approximated set, but they compare two approximated sets. Here, there are still some open questions, e.g., what percentage of the objective space is occupied by the non-dominated solutions? Where are the solutions concentrated? The answers to these questions are valuable particularly for high dimensional fronts (e.g., for 4- and higher-objective spaces) or when there is no information about the true set of optimal solutions.

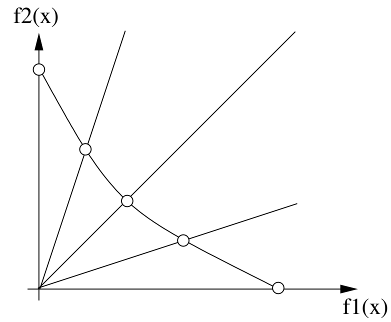

Fig. 2. 2-objective Sigma diversity metric. Black points illustrate a set of non-dominated solutions and the lines are the so-called reference lines.

2 Sigma Diversity Metric

Sigma diversity metric is a method which considers the positions of solutions by a vector called Sigma vector (value). This method is introduced in [9]. Indeed, it is inspired from the polar and spherical coordinate axis for 2- and 3-objective spaces. For higher dimensional objective spaces, we cannot define a coordinate axis which presents a simple distribution like in polar or spherical coordinates. Therefore, the Sigma diversity metric is suggested to calculate the positions of the solutions in the objective space.

Figure 2 shows the idea of using the Sigma diversity metric for a 2-objective space. In this figure, $|A|$ number of solutions are illustrated. Also, $|A|$ lines are drawn from the origin, these lines are called *reference lines*. The angle between each of the two neighboring reference lines is equal to $\frac{\pi}{2(|A|-1)}$. A possible good diversity of solutions is to have one solution on each line or enclosed between two lines.

This metric uses the idea of the Sigma method introduced in [9]. In the following, we explain how we can compute the diversity of solutions. Consider Figure 3 (left). To each line $f_2 = af_1$ a value σ is assigned as follows:

$$\sigma = \frac{f_1^2 - f_2^2}{f_1^2 + f_2^2} \tag{3}$$

In fact, all the points on the line $f_2 = af_1$ have the same σ values: $\sigma_i = (1 - a^2)/(1 + a^2)$.

In the general case, σ is defined as a vector of $\binom{m}{2}$ elements, where m is the dimension of the objective space. In this case, each element of $\boldsymbol{\sigma}$ is the combination of two coordinates in terms of the Equation 3. For example for three coordinates of f_1, f_2 and f_3, it is defined as follows:

$$\boldsymbol{\sigma} = \begin{pmatrix} f_1^2 - f_2^2 \\ f_2^2 - f_3^2 \\ f_3^2 - f_1^2 \end{pmatrix} / (f_1^2 + f_2^2 + f_3^2) \tag{4}$$

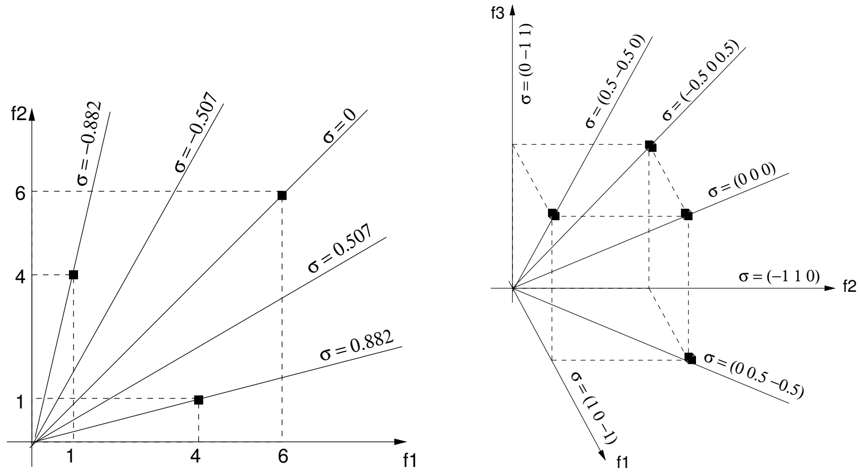

Fig. 3. Sigma values (vectors) in 2- and 3-objective spaces.

Different values of $\boldsymbol{\sigma}$ for different values of f_1, f_2 and f_3 are shown in Figure 3 (right). In the general case, when a point has the same position in each dimension (e.g., $f_1 = f_2 = f_3$ in 3 dimensional space), $\boldsymbol{\sigma} = \mathbf{0}$.

It means that each point in the objective space can be described by a Sigma vector. All the points along a line have the same Sigma vectors and the solutions lying on the lines which are very close two each other have similar Sigma vectors. This is the idea which is used to construct the Sigma diversity metric.

Diversity Metric Before computing the diversity of an approximated set, a set of reference lines must be calculated. The number of reference lines must be equal to the number of approximated solutions [8]. It must be emphasized that the reference lines must be calculated once for each number of objectives and can be stored in a table. Then the Sigma diversity metric can be computed as follows:

- Calculate reference lines.
- Compute the Sigma vector of each reference line (*reference Sigma vector*).
- Keep a binary *Flag* of initial zero value beside each reference Sigma vector. The Flag of each reference Sigma vector can only turn to 1, when at least one solution has a Sigma vector equal to it or within a distance (Euclidian distance) less than d. The value of d depends on the test function, however it should be decreased for high number of reference lines.
- A counter C counts the reference lines with Flags equal to 1 and the diversity metric D becomes:

$$D = \frac{C}{\text{number of reference lines}} \tag{5}$$

The Sigma diversity measurement expresses at what percentage the non-dominated solutions are distributed in the objective space.

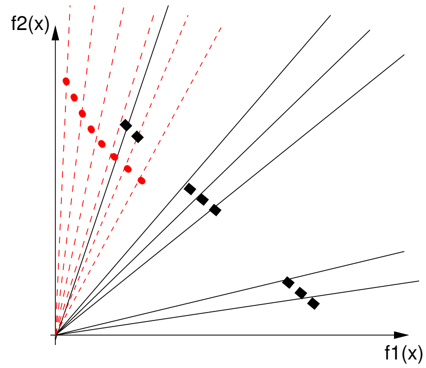

Fig. 4. Different spreads of solutions. Both of these sets of solutions occupy the same percentage of the space, but with different spreads.

In the case that the extreme solutions lie on the coordinate axis, we are able to obtain a high value for D, i.e., 100%. For discrete or disconnected sets of solutions, the value of D can never reach the highest value.

Discussion We know from D in Equation (5), that the solutions are distributed along the non-dominated front with D percent. If the value of D is high, it means that the solutions are well distributed. But when D is small, it means the solutions are

- either concentrated in one part of the space, or
- distributed in small groups along the front.

Indeed, there is a difference between these two kinds of solutions. Figure 4 shows the difference between two sets of solutions with the same D values. These two sets of solutions have different spreads and the Sigma diversity metric cannot distinguish between them. In the next subsection, we investigate how to calculate the spread of solutions. Also, this metric can only be used for non-dominated solutions located in the positive part of the objective space. If the solutions are not in the positive part of the space, they must be transformed into the positive part.

2.1 Median Sigma Method ($\tilde{\boldsymbol{\sigma}}$)

Let's consider a 2-objective space. As it is shown in Figure 3 (left), the Sigma value is changing from 1 to 0 and to -1, by changing the angle between the line $f_2 = a.f_1$ and coordinate axis f_1. We can use this property and find the *median* of the Sigma values of the solutions. If our solutions have good diversity and spread, then the median of their Sigma values is zero. Negative and positive values of the median mean that the solutions are concentrated in the left and right hand side of the front (i.e., left and right hand side of the line with $\boldsymbol{\sigma} = \mathbf{0}$), respectively. However, the zero value of median, is also valid when the solutions are all concentrated in the middle of the non-dominated front.

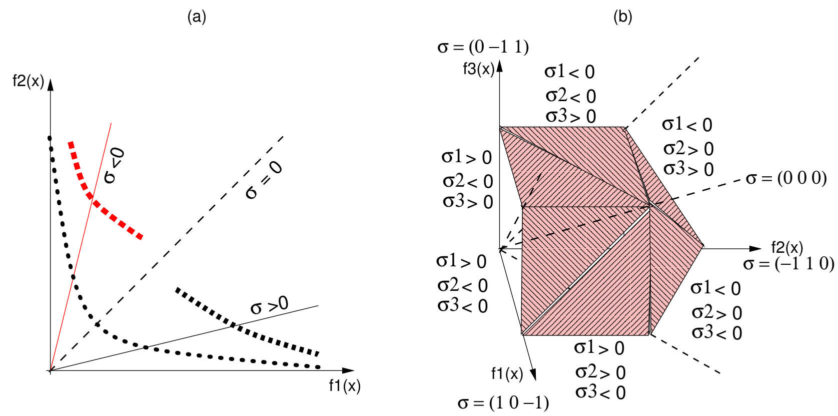

Fig. 5. Properties of the median Sigma method in (a) 2- and (b) 3-objective spaces

The same idea applies to higher number of objectives. The line in the middle has the Sigma vector equal to $\mathbf{0}$ (see Figure 3 (right)). This time the median vector should be considered. Let us consider $\sigma = \{\boldsymbol{\sigma}_1, \cdots, \boldsymbol{\sigma}_{|A|}\}$ as the set of the Sigma vectors of the solutions in the the set $|A|$. Then, the jth element of the median vector $\tilde{\boldsymbol{\sigma}}$ is defined as follows:

$$\tilde{\sigma}_j = \begin{cases} \sigma_{l,j}, & \text{if } |A| = 2l + 1 \\ \frac{1}{2}(\sigma_{(l+1),j} + \sigma_{l,j}) & \text{if } |A| = 2l \end{cases} \tag{6}$$

Figure 5 shows different parts of the space for different median values for 2- and 3-objective spaces. In this figure, σ_i is the ith element of the Sigma vector. Calculating the median value is useful when the diversity measure D has a small value. Indeed, this measurement completes the information on evaluating the diversity and spread of the solutions along a non-dominated set.

Discussion The Sigma diversity metric and the median Sigma method evaluate the diversity and spread of obtained solutions along the approximated Pareto-optimal front. The advantages of these measurements in comparison to other diversity and spread metrics are as follows:

- computing the metric is possible for any desired number of objectives.
- they evaluate *one* approximated front, where most of the other methods compare two approximated fronts. Indeed, the output of the Sigma diversity metric is a percentage of the objective space; the other methods presents a scalar value representing a hyper volume or a measure which are used to compare two sets.
- the median Sigma method computes the positions of the solutions in the objective space. This can be used to describe the positions of solutions with high number of objectives, where the graphical illustration is impossible.

The Sigma diversity metric requires information about d, the neighborhood defined around each reference line. The value of d depends highly on the shape of

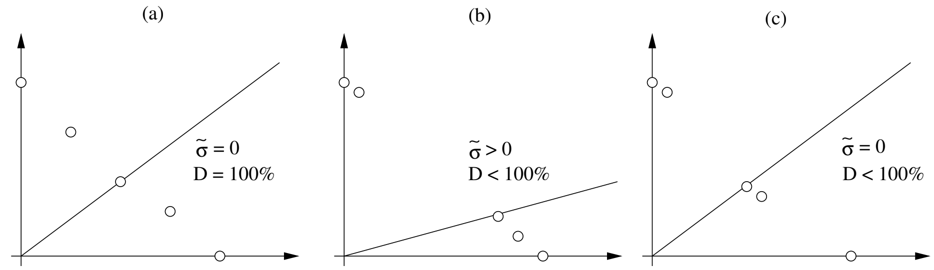

Fig. 6. An example of different non-dominated sets with different diversities of solutions. (a) Solutions are well-distributed: $D = 100\%$ and $\tilde{\sigma} = 0$. (b),(c) Solutions are not well-distributed: $D \neq 100\%$. Median Sigma value, $\tilde{\sigma}$, indicates the spread of solutions.

the Pareto-optimal front. Computing the reference lines should be done once for each number of objectives and the corresponding values can be stored in a table. Also, this measure highly depens on the position of the origin and focuses very much on the middle region of the Pareto-optimal front. If the Pareto-optimal front is a line perpendicular the $\boldsymbol{\sigma} = \mathbf{0}$ ray, the rays cut the front at more or less equal distances. But, if the Pareto-optimal front has a strong bend (knee), then the different segments of the front have very different lengths.

Figure 6 illustrates an example of three different non-dominated sets with different spreads and diversities of solutions. In Figure 6 (a), the solutions are well-distributed. Therefore, both of the Sigma diversity measure and the median Sigma value are satisfactory, i.e., $D = 100\%$ and $\tilde{\sigma} = 0$. In Figure 6 (b), there is a large gap between the solutions and therefore, the Sigma diversity measure is less than the desired value. The positive median value indicates that most of the solutions are concentrated on the right hand side of the non-dominated front. In Figure 6 (c), solutions are not well-distributed. Therefore, the Sigma diversity measure is less than 100%. These solutions have better spread than those in Figure 6 (b). This is also indicated by the median Sigma value.

3 Experiments

The diversity measurements are applied to the results of different test problems with 2, 3 and 4 objectives. These tests problems[3] are chosen from [4, 15] (shown in Table 1). By studying the shape of the Pareto-optimal fronts, it can be concluded that for ZDT3, ZDT6, CP3, and DLZT we cannot obtain 100% as the diversity value. Because the solutions are not continuously distributed in the space. On the other hand, the other test functions should have a diversity of about 100%. The solutions are obtained by running two different methods (A and B)[4], which deliver solutions with different diversities. The optimal sets have

[3] The ZDT6 test problem is constructed like ZDT6 in [1]; here the second objective is modified.

[4] Method A is based on a Multi-Objective Particle Swarm Optimization method [9] and method B is the SPEA2 algorithm [17].

Table 1. Test Functions

test	Function	
ZDT1	$g(x_2, \cdots, x_n) = 1 + 9(\sum_{i=2}^{n} x_i)/(n-1)$ $h(f_1, g) = 1 - \sqrt{f_1/g}$ $f_1(x_1) = x_1$ $f_2(\boldsymbol{x}) = g(x_2, \cdots, x_n) \cdot h(f_1, g)$	$x_i \in [0,\ 1]$ $n = 30$ $i = 1, 2, \ldots, n$
ZDT3	$g(x_2, \cdots, x_n) = 1 + 9(\sum_{i=2}^{n} x_i)/(n-1)$ $h(f_1, g) = 1 - \sqrt{f_1/g} - (f_1/g)\sin(10\pi f_1)$ $f_1(x_1) = x_1$ $f_2(\boldsymbol{x}) = g(x_2, \cdots, x_n) \cdot h(f_1, g) + 1$	$x_i \in [0,\ 1]$ $n = 30$ $i = 1, 2, \ldots, n$
ZDT4	$g(x_2, \cdots, x_n) = 1 + 10(n-1) + (\sum_{i=2}^{n}(x_i^2 - 10cos(4\pi x_i))$ $h(f_1, g) = 1 - \sqrt{f_1/g}$ $f_1(x_1) = x_1$ $f_2(\boldsymbol{x}) = g(x_2, \cdots, x_n) \cdot h(f_1, g)$	$x_1 \in [0,\ 1]$ $x_i \in [-5,\ 5]$ $n = 10$ $i = 2, \ldots, n$
ZDT6	$g(x_2, \cdots, x_n) = 1 + 9[(\sum_{i=2}^{n} x_i)/9]^{0.25}$ $h(f_1, g) = 1 - \sqrt{x_1/g}$ $f_1(x_1) = 1 - exp(-4x_1)sin^6(6\pi x_1)$ $f_2(\boldsymbol{x}) = g(x_2, \cdots, x_n) \cdot h(f_1, g)$	$x_i \in [0,\ 1]$ $n = 10$ $i = 1, 2, \ldots, n$
CP3	$f_1(\boldsymbol{x}) = (1 + x_3)(x_1^3 x_2^2 - 10x_1 - 4x_2)$ $f_2(\boldsymbol{x}) = (1 + x_3)(x_1^3 x_2^2 - 10x_1 + 4x_2)$ $f_3(\boldsymbol{x}) = 3(1 + x_3)x_1^2$	$1 \leq x_1 \leq 3.5$ $-2 \leq x_2 \leq 2$ $0 \leq x_3 \leq 1$
DLZT	$f_1(\boldsymbol{x}) = x_1$ $f_2(\boldsymbol{x}) = x_2$ $f_3(\boldsymbol{x}) = 3.5 - \sum_{i=1}^{n} 2x_i sin(n\pi x_i)$	$x_i \in [0,\ 1]$ $n = 3$ $i = 1, 2, \ldots, n$
GSPm	$f_1(\boldsymbol{x}) = (1 + x_m^2)\cos(x_1\pi/2)\cdots\cos(x_{m-1}\pi/2)$ $f_2(\boldsymbol{x}) = (1 + x_m^2)\cos(x_1\pi/2)\cdots\sin(x_{m-1}\pi/2)$ $\vdots$ $f_{m-1}(\boldsymbol{x}) = (1 + x_m^2)\cos(x_1\pi/2)\sin(x_2\pi/2)$ $f_m(\boldsymbol{x}) = (1 + x_m^2)\sin(x_1\pi/2)$	$x_i \in [0,\ 1]$ $n = m$ $i = 1, \ldots, m-1$ $x_m \in [-1,\ 1]$

the same cardinality (50 for 2-, 100 for 3- and 4-objective test functions). It must be mentioned that the solutions are not compared in terms of their convergence to the Pareto-optimal set.

3.1 Results

Here, the results of different test functions are analyzed separately. Results of 2- and 3-objective test functions are studied for continuous and disconnected-continuous Pareto-optimal fronts. Then the results of 4-objective test functions are analyzed.

2-objective continuous Pareto-optimal front

Test functions ZDT1 and ZDT4 have convex and continuous Pareto-optimal fronts. We estimate that the Sigma diversity metric finds a diversity of 100% and the median values must be very close to zero. Figure 7 shows the results of methods A and B for ZDT1 (top row) and ZDT4 (bottom row) test functions.

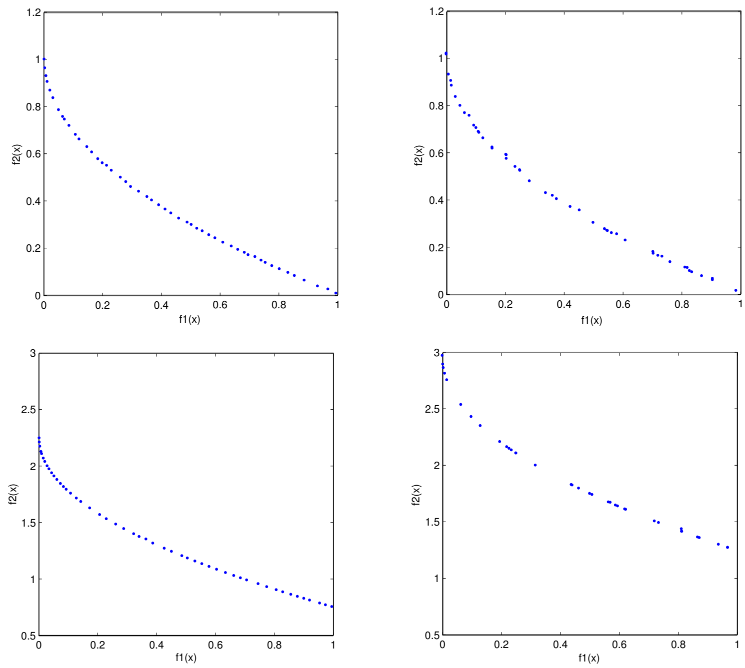

Fig. 7. Continuous Pareto-optimal fronts. Solutions of ZDT1 (first row) and ZDT4 (second row) in the objective space. The left and right columns illustrate the results of the methods A and B, respectively.

We observe that the results of A (in Figures 7 (top-left)) have a better diversity and spread in the objective space than the solutions of B (in Figures 7 (top-right)). The results of the Sigma diversity metric and the median Sigma method are recorded in Table 2. In this table, the diversity of solutions is shown by using the Sigma diversity metric D and the spread of solutions by the median Sigma values $\tilde{\sigma}$. Large values of D indicate a better diversity, and low absolute values of $\tilde{\sigma}$ show that the solutions are symmetrically distributed along the front. The solutions of method A occupy 84% whereas the solutions of method B occupy 68% of the space for the test function ZDT1. These solutions have a median value of 0.07 and -0.13 for methods A and B, respectively. It means that the solutions of A have a better spread than B. The solutions of ZDT4 test function have lower values of Sigma diversity metric. One reason can be that these solutions are not converged to the Pareto-optimal front.

2-objective continuous-disconnected Pareto-optimal front

Test functions ZDT3 and ZDT6 have non-convex and continuous and disconnected Pareto-optimal fronts. Therefore, we estimate that the Sigma diversity metric must have a value of less than 100%, but the median values must still be very close to zero. Figure 8 shows the results of methods A and B. We observe that the results of A (Figures 8 (top-left)) have a better diversity and spread in the objective space than the solutions of B (Figures 8 (top-right)). As it can be

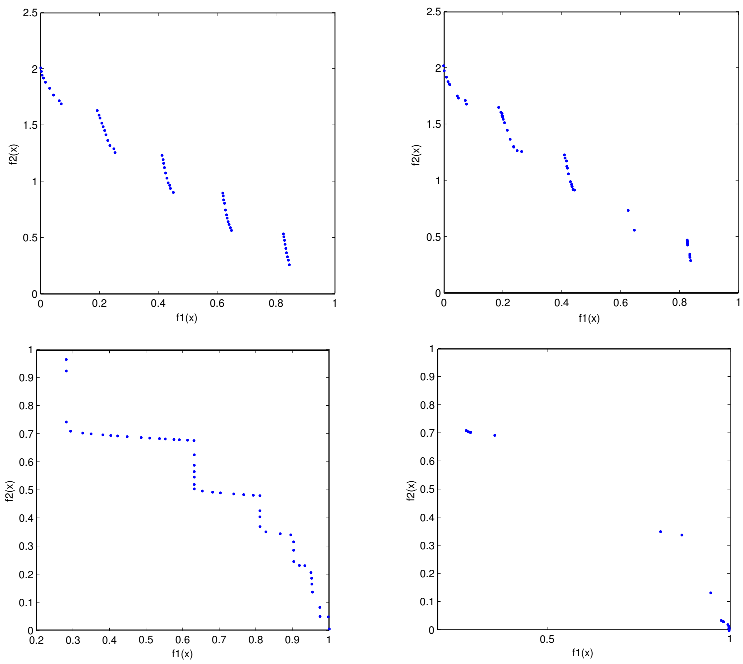

Fig. 8. Continuous and disconnected Pareto-optimal fronts. Solutions of ZDT3 (first row) and ZDT6 (second row) in the objective space. The left and right columns illustrate the results of the methods A and B, respectively.

observed in Table 2, all of the solutions have a Sigma diversity value less than 100%. The solutions of method B for the test function ZDT6 occupy only 20% of the objective space and are all concentrated in the right hand side of the front, close to the f_1 axes. This can be concluded from the median value of 0.99.

3-objective continuous Pareto-optimal front

GSP3 and CP3 test functions have continuous Pareto-optimal fronts. The solutions of methods A and B for these test functions are shown in Figure 9. The solutions of the test function GSP3 must cover the whole objective space and therefore the Sigma diversity metric must be very close to 100%. But the solutions of the test function DLZT cover only a narrow surface of the objective space and the Sigma diversity metric cannot reach the value of 100%. The corresponding values of Sigma diversity metric and the median Sigma vectors are shown in Table 3. It can be concluded that method A can obtain solutions

Table 2. Diversity measures of 2-objective test functions

Test	D_A	$\tilde{\sigma}_A$	D_B	$\tilde{\sigma}_B$
ZDT1	84%	+0.07	68%	-0.13
ZDT3	66%	-0.20	50%	-0.37
ZDT4	74%	-0.58	44%	-0.54
ZDT6	80%	+0.29	20%	+0.99

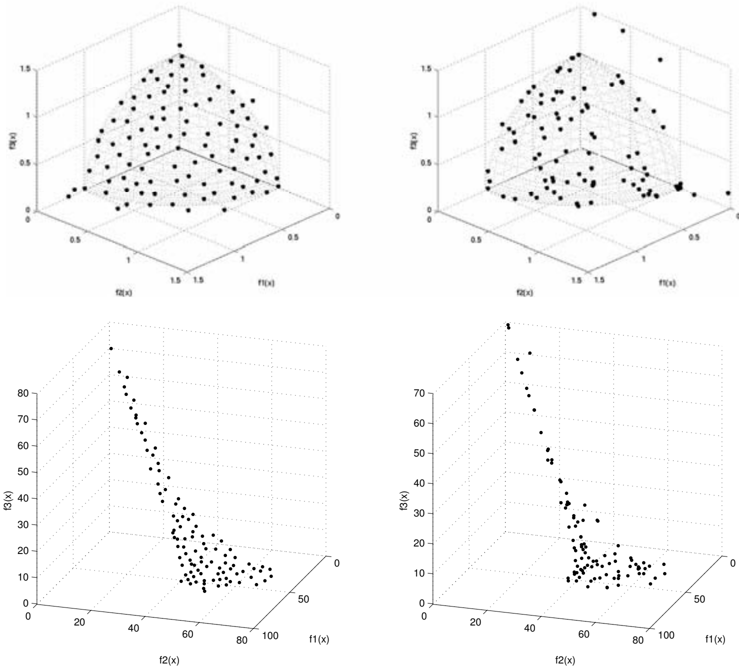

Fig. 9. Continuous Pareto-optimal fronts. Solutions of GSP3 (first row) and CP3 (second row) in the objective space. The left and right columns illustrate the results of the methods A and B, respectively.

with better diversity than method B for the test function GSP3. However, both of these methods find solutions with good spread all over the Pareto-optimal front. Furthermore, we compute the location of the solutions of the CP3 test function in the objective space. Let us consider Figure 5 (right) and the median Sigma vectors in Table 3. Method A delivers solutions with $\sigma_1 \leq 0$,$\sigma_2 = 0.38$, and $\sigma_3 = -0.37$. In Figure 5 (right), these values indicate the region between the vectors (0 0 0), $(-1\ 1\ 0)$, and the 45 degree line passing on the f1-f2 (axis) plane. It means that the median value of the solutions lies in this region. In other

Table 3. Diversity measures of the 3-objective test functions

Test	D_A	$\tilde{\sigma}_A$	D_B	$\tilde{\sigma}_B$
GP3	88%	$\begin{pmatrix} +0.01 \\ -0.00 \\ -0.01 \end{pmatrix}$	71%	$\begin{pmatrix} +0.00 \\ +0.02 \\ -0.00 \end{pmatrix}$
DLZT	19%	$\begin{pmatrix} -0.00 \\ -0.81 \\ +0.80 \end{pmatrix}$	16%	$\begin{pmatrix} +0.00 \\ -0.83 \\ +0.84 \end{pmatrix}$
CP3	22%	$\begin{pmatrix} -0.00 \\ +0.38 \\ -0.37 \end{pmatrix}$	28%	$\begin{pmatrix} +0.03 \\ +0.38 \\ -0.42 \end{pmatrix}$

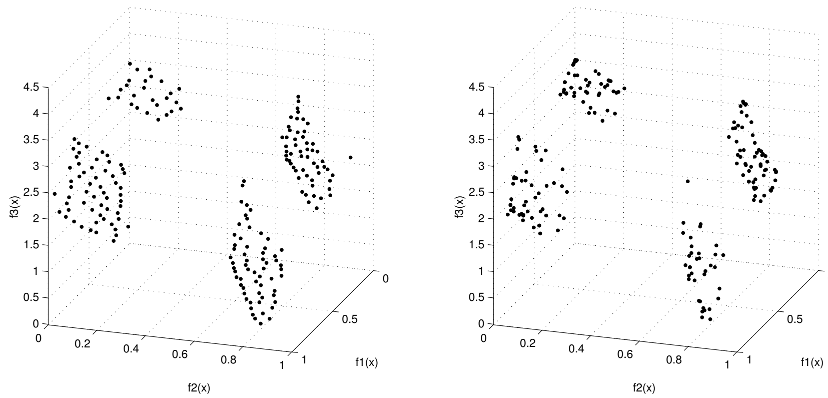

Fig. 10. Continuous and disconnected Pareto-optimal front. Non-dominated sets obtained by method A (left) and B (right)

words, the zero value of σ_1 means that solutions have a symmetric spread on the f1-f2 planes with various f3 values, and also tend more toward the f1-f2 plane. This can also be observed in Figure 9 (second row). The Pareto-optimal front gets narrower for higher values of f3.

3-objective disconnected-continuous Pareto-optimal front

Test function DLZT has a continuous and disconnected Pareto-optimal front. Therefore, the the Sigma diversity metric cannot reach the value of 100%. Figures 10 (left)-(right) show the results of method A and B, respectively. The results of Sigma diversity metric indicate that the results of method A have better diversity than method B in Table 3. This can also be observed in Figure 10. The median Sigma vectors have the same values, it means the solutions have the same spread all over the objective space.

4-objective continuous Pareto-optimal front

Table 4 shows the diversity values of the GSP4 test function. By these results, we conclude that method A has obtained solutions with better diversity than method B. The solutions of method A have a good spread, because they also have a median Sigma vector very close to zero vector. The solutions of method B do not have a good spread and diversity. These solutions are distributed symmetrically on the f1-f2 axis plane ($\sigma_1 = 0.00$). Because $\sigma_2 = -0.14$ and $\sigma_3 = +0.12$, we conclude the solutions are close to f_3 axes[5]. The values of $\sigma_4 = -0.14$ and $\sigma_5 = +0.12$ are close to zero, it means the solutions are symmetrically distributed on the f4-f1 and f1-f3 planes.

[5] $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_6)^T$ (where T is the transpose operator). Here, σ_1, σ_2, σ_3, σ_4, σ_5, σ_6 are related to f1-f2, f2-f3, f3-f4, f4-f1, f1-f3, f2-f4 axis, respectively.

Table 4. Diversity measures on the 4-objective test function GSP4 (T indicates the transpose operator)

method	D	$\tilde{\sigma}^T$
A	79%	$(+0.00, -0.00, +0.01, -0.02, -0.00, -0.00)$
B	62%	$(-0.00, -0.14, +0.12, +0.06, +0.03, +0.18)$

4 Summary

Obtaining a well-distributed set of solutions is one of the goals of multi-objective optimization. In this paper, we have studied and introduced Sigma diversity metric and the median Sigma method for measuring the diversity and spread of a set of non-dominated solutions. These measurements are able to provide information like the percentage of the objective space being occupied by solutions, or their positions in terms of their angular coordinates in the objective space. These methods are tested on the results of several test problems. In the 2- and 3-objective spaces the evaluations confirm the observations. The measurements are also applied to the results in a 4-objective space. These measurements can evaluate solutions with several number of objectives and also give us a view of the positions of the solutions in the objective space. This is more important when we deal with problems of a large number of objectives, where a graphical illustration is not possible. The Sigma diversity metric depends on several parameters, like the Euclidian distance around each reference line (d) and the number of reference lines. Also, this method assumes positive objective values. Therefore, solutions with negative objectives must be transformed into the positive part of the axis.

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