04351 Summary – Spatial Representation: Discrete vs. Continuous Computational Models

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Abstract. Topological notions and methods are used in various areas of the physical sciences and engineering, and therefore computer processing of topological data is important. Separate from this, but closely related, are computer science uses of topology: applications to programming language semantics and computing with exact real numbers are important examples. The seminar concentrated on an important approach, which is basic to all these applications, i.e. spatial representation.

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1 Introduction to the problem area

Topological notions and methods are used in various areas of the physical sciences and engineering, and therefore computer processing of topological data is important. Separate from this, but closely related, are computer science uses of topology: applications to programming language semantics and computing with exact real numbers are important examples. The seminar concentrated on an important approach, which is basic to all these applications, i.e. spatial representation.

Due to the digital nature of most applications, the structures used in computer science are different from the mathematical structures that are classically used in engineering and that are based on the continuum. Typical features of these digital structures are asymmetry and partiality. Whereas classical spaces contain only the ideal elements that are the result of a computation (approximation) process, spaces that also allow reasoning on such processes in a formal way must as well contain the partial (and finite) objects appearing during a computation. Only they can be observed in finite time.

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2 Contents of the seminar

The seminar was devoted to the study of several topological structures. The leading example of such is the domain (in Scott's sense), and it is closely related to locales. Here, the finitely observable properties of a process are the primary objects of study. The ideal entities, which are the first class citizens of classical mathematical structures, are obtained as derived objects. These have given rise to a constructive treatment of topological spaces, Formal Topology.

Constructive theories have the important property that algorithms can be derived from their proofs. One of their typical features is that besides equality a further basic relation is used, apartness, which is stronger than inequality. The intuition is that two objects are apart if they can be separated by disjoint open sets.

A continuous model of spatial representation to represent a classical space, is usually found by representing it as the space of maximal points of a suitable domain, possibly with additional structure such as a partial metric or a measurement. This gives a handle on the computational aspects of the spaces. Several researchers have started to apply this scheme, or variations of it, in the foundations of physics.

Viewing space (or space-time) by the use of domains is a clear case of the application of ideas derived from computer science in the direction of physics. There are several other reasons for wanting to consider discrete models of space and time. For the purposes of digital computation, this scarcely needs elaboration. It is striking that, motivated in particular by problems in quantum gravity, several theoretical physicists have proposed discrete spatio-temporal models which are strikingly reminiscent of those seen in computer science (though developed independently). For example we can compare the causal sets of physics with event structures and other process models in computer science. It is hardly surprising that posets and graphs figure extensively in "discrete" models, whatever the discipline. Going beyond these basic structures, we have (abstract) simplicial complexes, extensively used for spatial modelling in computer science, but also proposed in physics. The question may be asked: What is the relation between these basic discrete structures and the more conventional continuous models? The latter should be obtainable in some sense in the limit from the former, and the inverse limit construction within an appropriate category, is often involved in the construction. Early such topological constructions due to Alexandroff and Freudenthal (in the 1920's and 1930's) seemed irrelevant and were largely ignored for many years. Later, similar constructions were discovered in topology (Kopperman/Wilson), computer science (Smyth), logic (Martin-Löf), shape theory (Porter), and physics (Sorkin, Raptis), among others.

Another such basic discrete structure is the matroid; these are most often seen in computer science in connection with combinatorial algorithms. They are important in this seminar due to their use as framework for geometry. This program has been carried out extensively by Faure and Frölicher, in the development of projective geometry in a form suitable for theoretical physics. Independently, several researchers have worked with (mainly, oriented) matroids to develop an axiomatic discrete geometry suitable for image processing and other applications.

It might be objected that matroids are by definition finite, and thus too restricted for a foundations of geometry, but in fact it is easy to extend the definition to allow infinite matroids, by considering them as certain closure spaces. This is what Faure and Frölicher do, and by taking as their morphisms (partial) continuous maps between closure spaces, they are able for the first time to exhibit satisfactory categories of matroids and of projective spaces. As shown in the seminar, by using a slightly more general notion of morphism, one obtains a Cartesian closed category of closure spaces.

3 Scientific highlights

It was the aim of this and earlier Dagstuhl Seminars on Topology in Computer Science to bring together people working in different fields and to foster interaction between them. This time we say 43 participants from Canada, Czech Republic, Denmark, Germany, Italy, Ireland, Japan, New Zealand, Poland, Russia, Slovenia, South Africa, United Kingdom and USA.

We heard talks on domain theory, geometry, logic and constructive approaches, topology as well as in the newly emerging area that interrelates theoretical computer science with the foundations of physics.

Among the talks, one by Victor Selivanov developed a large part of descriptive set theory for algebraic domains, in parallel with the well known descriptive set theory for Polish spaces. The Borel hierarchy and the Hausdorff difference hierarchy as well as Wadge degrees were introduced. Some natural examples were given in which the Wadge reducibility behaves much better than in classical spaces as the reals.

Homeira Pajoohesh showed how every domain can be equipped with a partial metric (perhaps valued in a quantale other than the extended reals, such as a power of the unit interval). This partial metric then gives rise to the order and approximating relation of the domain. A related talk by Steve Matthews, who originated partial metrics, showed that they can be represented by classical metrics with base points.

Another pair of talks by Rafael Sorkin and Graham Brightwell concerned the structure of spacetime in quantum gravity. Sorkin's talk was a review of the causal set program in quantum gravity where spacetime is viewed as a discrete poset: the order structure represents causality. Sorkin described "stochastic growth models" of spacetime as a "toy" version of quantum gravity. In these models points are added by a stochastic process. Brightwell - an expert on random graphs - described the combinatorics of this model and showed that these descrete models could be Lorentz invariant despite being discrete precisely because they were randomized.

Prakash Panangaden described joint work with Keye Martin which was originally inspired by Sorkin's programme. In this work the causal structure of spacetime is modelled as a poset and this poset is studied from the point of view of domain theory. One basic question that can be asked - and was answered - is whether the spacetime causal structure is enough to recover the topology. The notions of Scott and Lawson topology turned out to be crucial ideas in this. It was also shown that spacetime can be reconstructed from a countable dense subset. Finally it was shown that globally hyperbolic spacetime exactly correspond to a class of domains called interval domains.

Timothy Porter and Jonathan Gratis brought new aspects ("fractafolds" and differential geometry), connections (e.g. Chu spaces), and applications (feature extraction; economics/social choice) of our main theme, discrete vs. continuous geometry. Gratis proposed an extension of the inverse limit approach to "MRZ" (Mallios/Raptis/Zapatrin) calculus.

Region geometry (a species of point-free geometry) is another of our ongoing themes, and John Stell developed an extension from spatial to space-time regions. Mike Smyth reviewed the idea, previously developed with Rueiher Tsaur, that Helly graphs are the discrete version of hyperconvex spaces (these are the injectives in the category of metric spaces), and showed that it could be extended to non-reflexive graphs.

4 Open questions

Many important and interesting new results were presented during the seminar. Shortly after the seminar Martin Kovár found a solution to an old problem of D.E. Cameron (1977). He had already presented a partial solution during the seminar and got the main new ideas during his train trip from Dagstuhl to Brno.

The approximation of topological spaces by finite spaces was discussed during this and previous related seminars. In this area the approximation of maps is important, and contains many open problems now being resolved, such as how best to represent such maps and how to use the theory of such representations to redevelop and reinterpret the classical theory of continuous functions.

The physics related talks spawned a number of interesting questions on the general theme of recovering or approximating familiar geometric notions in a discrete setting. The best example of this is the derivative operator. Sorkin described how it seems possible to describe a derivative operator on a discrete poset. Panangaden raised the question of recovering the metric structure on spacetime from an appropriate domain theoretic source: perhaps from a notion like measurement. A related topic is analysing other causality conditions (other than global hyperbolicity) from a domain theoretic viewpoint.

Oriented Projective geometry (OPG), as developed by Stolfi, is a (non-axiomatic) foundation for CAD that combines projective geometry (the use of homogeneous coordinates) and convexity. Stolfi works entirely with continuous models, and one problem is how OPG can be done in a discrete setting. A somewhat radical approach is required, given that finite projective spaces are precisely those matroids in which the rank function is modular (Frölicher and Faure give an extensive exposition), and it is known that no such matroid is orientable (and so cannot have convexity).

Orthogonality (of subspaces of Hilbert space) is the central concept of quantum logic, and the search for an abstract theory of orthogonality has been one of the main areas of research in the subject. Combinatorial versions, in terms of lattices, grahs and other structures are widely studied. Connections with orthogonality in oriented matroid theory are yet to be explored.