Upper error bounds for approximations of stochastic differential equations with Markovian switching

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We consider stochastic differential equations with Markovian switching (SDEwMS). An SDEwMS is an ordinary stochastic differential equation with drift and diffusion coefficients depending not only on the current state of the solution but also on the current state of a right-continuous Markov chain taking values in a finite state space.

Let W be a one-dimensional Brownian motion on the unit interval and let r be a right-continuous Markov chain with state space $S := \{1, 2, \ldots, N\}$ and transition probabilities

$$P\{r(t+\delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\,\delta^n + o(\delta^n), & \text{if } i \neq j, \\ 1 + \gamma_{ij}\,\delta^n + o(\delta^n), & \text{if } i = j, \end{cases} \tag{1}$$

where $\delta > 0$ and $n \geq 1$ is a free parameter. In (1) we use γ_{ij} to denote the transition rate from i to j satisfying $\gamma_{ij} > 0$ for $i \neq j$ and

$$\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}.$$

According to (1) the probability for switching from state i to another state j during a small time interval of length δ is proportional to δ^n. The general form of an SDEwMS is given by

$$dy(t) = f(y(t), r(t))\,dt + g(y(t), r(t))\,dW(t), \quad 0 \leq t \leq 1 \tag{2}$$

with initial value $y(0)$ and initial state $r(0)$ of the Markov chain.

We analyze numerical methods for pathwise approximation of equations (2). The starting point for these investigations was a paper by Yuan and Mao, in which the continuous Euler approximation is studied. For an equidistant discretization $t_k = k/m$, $k = 0, \ldots, m$ of the unit interval the continuous Euler approximation is defined by $X_0 = y(0)$ and

$$X(t) = X_k + f(X_k, r_k^{\Delta}) \cdot (t - t_k) + g(X_k, r_k^{\Delta}) \cdot (W(t) - W(t_k)) \tag{3}$$

for $t \in [t_k, t_{k+1})$, where $\{r_k^{\Delta}, k = 0, 1, \ldots, m-1\}$ is a discrete Markov chain and $X_k := X(t_k)$. Yuan and Mao have shown that under natural assumptions

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the mean-square error $e(X) := \sup_{t\in[0,1]} \left(E(y(t) - X(t))^2\right)^{1/2}$ satisfies the uniform upper bound

$$e(X) \leq c \cdot m^{-1/2}$$

with an unspecified constant c. We consider the equidistant continuous Milstein approximation which is obtained by adding the term

$$1/2 \cdot \left(g \cdot g^{(1,0)}\right) (X_k, r_k^\Delta) \cdot \left((W(t) - W(t_k))^2 - (t - t_k)\right)$$

to the right hand side of (3) and present an upper bound for $e(X)$. It turns out that $e(X)$ can be estimated from above by

$$c \cdot m^{-\min\{n/2,1\}}.$$

This result means that there is a strong connection between the power of the step-size appearing in the upper bound and the intensity of the switching. Consequently, the Milstein approximation yields a better upper bound only if the probability of switching in another state is reduced by choosing a switching parameter $n \geq 2$.