Optimal Approximation of Elliptic Problems II: Wavelet Methods

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1. Introduction. We study the optimal approximation of the solution of an operator equation

$$\mathcal{A}(u) = f, \tag{1}$$

where $\mathcal{A}$ is a boundedly invertible linear operator

$$\mathcal{A} : H \to G \tag{2}$$

from a Hilbert space H into another Hilbert space G. We have in mind the more specific situation of an elliptic operator equation, i.e.,

$$\mathcal{A} \ : H_0^s(\Omega) \to H^{-s}(\Omega), \quad s > 0, \tag{3}$$

where $\Omega \subset \mathbf{R}^d$ is a bounded Lipschitz domain. A typical example we shall primary be concerned with is the Poisson equation

$$\begin{aligned} -\triangle u &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{4}$$

Our aim is to answer the following fundamental questions:

- In which sense can we say that an approximation scheme is optimal?
- What happens in the special case of elliptic partial differential equations?
- Do there exist optimal bases and methods, respectively?

2. Basic Concepts. We use linear and nonlinear mappings S_n for the approximation of the solution u to (1). Let us consider the worst case error

$$e(S_n, F, H) = \sup_{\|f\|_F \le 1} \|\mathcal{A}^{-1}(f) - S_n(f)\|_H,$$

where F is a normed (or quasi-normed) space, $F \subset G$. For a given basis $\mathcal{B}$ of H we consider the class $\mathcal{N}_n(\mathcal{B})$ of all (linear or nonlinear) mappings of the form

$$S_n(f) = \sum_{k=1}^{n} c_k \, h_{i_k},$$

where the c_k and the i_k depend in an arbitrary way on f. Then the nonlinear widths $e_{n,C}^{\text{non}}(S, F, H)$ are given by

$$e_{n,C}^{\text{non}}(S, F, H) = \inf_{\mathcal{B} \in \mathcal{B}_C} \inf_{S_n \in \mathcal{N}_n(\mathcal{B})} e(S_n, F, H).$$

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Here $\mathcal{B}_C$ denotes a set of Riesz bases for H where C indicates the stability of the basis. We compare nonlinear approximations with linear approximations. Here we consider the class $\mathcal{L}_n$ of all continuous linear mappings $S_n : F \to H$,

$$S_n(f) = \sum_{i=1}^{n} L_i(f) \cdot \tilde{h}_i$$

with arbitrary $\tilde{h}_i \in H$. The worst case error of optimal linear mappings is given by

$$e_n^{\mathrm{lin}}(S, F, H) = \inf_{S_n \in \mathcal{L}_n} e(S_n, F, H).$$

The third class of approximation methods that we study is the class of continuous mappings $\mathcal{C}_n$, given by arbitrary continuous mappings $N_n : F \to \mathbf{R}^n$ and $\phi_n : \mathbf{R}^n \to H$. Again we define the worst case error of optimal continuous mappings by

$$e_n^{\mathrm{cont}}(S, F, H) = \inf_{S_n \in \mathcal{C}_n} e(S_n, F, H),$$

where $S_n = \phi_n \circ N_n$. These numbers, or slightly different numbers, were, e.g., studied by [4, 5, 6]. The three different widths are related as follows:

Theorem 1. *Assume that $S : F \to H$ with Hilbert spaces F and H. Then, under some additional technical conditions*

$$e_n^{\mathrm{lin}}(S, F, H) = e_n^{\mathrm{cont}}(S, F, H) \asymp e_{n,C}^{\mathrm{non}}(S, F, H). \tag{5}$$

Therefore we conclude that optimal linear mappings have the same order as the best n–term approximation.

3. Elliptic Problems. The next step is to apply this general concept to the special case of elliptic operator equations. It turns out that nonlinear approximation methods do not yield a better rate of convergence compared with linear schemes. The order of convergence only depends on the smoothness of the right–hand side.

Theorem 2. *Assume that $S : H^{-s}(\Omega) \to H_0^s(\Omega)$ is an isomorphism. Here $\Omega \subset \mathbf{R}^d$ is a bounded Lipschitz domain. Then for all $C \geq 1$*

$$e_n^{\mathrm{lin}}(S, H^{-s+t}(\Omega), H^s(\Omega)) \asymp e_{n,C}^{\mathrm{non}}(S, H^{-s+t}(\Omega), H^s(\Omega)) \asymp n^{-t/d}. \tag{6}$$

However, there is an important difference between regular and nonregular elliptic problems. In the regular case, a Galerkin scheme based on a sequence of uniformly refined spaces is sufficient to obtain the optimal order of convergence, whereas for the nonregular case the optimal linear method requires the precomputation of a suitable basis which is usually a prohibitive task. This leads us to the following problem: Can we find a basis for which best n–term approximation produces the optimal order of convergence, without any precomputation? We especially focus on a *wavelet basis* $\mathbf{\Psi} = \{\psi_\lambda, \lambda \in \mathcal{J}\}$. The indices $\lambda \in \mathcal{J}$ typically encode several types of information, namely the *scale* often denoted $|\lambda|$, the spatial location and also the type of the wavelet. Ψ is assumed to fulfill the following requirements:

- the wavelets are *local* in the sense that

$$\mathrm{diam}(\mathrm{supp}\psi_\lambda) \asymp 2^{-|\lambda|}, \quad \lambda \in \mathcal{J};$$

- the wavelets satisfy the *cancellation property*

$$|\langle v, \psi_\lambda \rangle| \ \lesssim \ 2^{-|\lambda|\widetilde{m}} \|v\|_{H^{\widetilde{m}}(\mathrm{supp}\, \psi_\lambda)},$$

  where $\widetilde{m}$ denotes some suitable parameter;

- the wavelet basis induces characterizations of Besov spaces of the form

$$\|f\|_{B^s_q(L_p(\Omega))} \asymp \left(\sum_{|\lambda|=j_0}^{\infty} 2^{j(s+d(\frac{1}{2}-\frac{1}{p}))q} \left(\sum_{\lambda \in \mathcal{J}, |\lambda|=j} |\langle f, \tilde{\psi}_\lambda \rangle|^p \right)^{q/p} \right)^{1/q}, \ s > d\left(\frac{1}{p} - 1\right)_+$$

  where $\tilde{\Psi} = \{\tilde{\psi}_\lambda : \lambda \in \mathcal{J}\}$ denotes the *dual basis*, $\langle \psi_\lambda, \tilde{\psi}_\nu \rangle = \delta_{\lambda,\nu}, \quad \lambda, \nu \in \mathcal{J}$.

It turns out that for the special case of the Poisson equation (4) best n–term wavelet approximation is still suboptimal, but nevetheless superior when compared with uniform schemes. Moreover, for more specific domains, i.e., for polygon domains, wavelet methods are indeed optimal.

Theorem 3. *For the problem* (4)*, best n-term wavelet approximation produces the worst case error estimate:*

$$e(S_n, H^{t-1}(\Omega), H^1(\Omega)) \leq C\, n^{-(\frac{(t+1)}{3}-\varrho)/d} \quad \text{for all} \quad \varrho > 0, \tag{7}$$

provided that $\frac{1}{2} < t \leq \frac{3d}{2(d-1)} - 1$.

Theorem 4. *For problem* (4) *in a polygonal domain in* $\mathbf{R}^2$*, best n-term wavelet approximation is almost optimal in the sense that*

$$e(S_n, H^{t-1}(\Omega), H^1(\Omega)) \leq C n^{-(t-\varrho)/2}, \qquad \text{for all} \quad \varrho > 0. \tag{8}$$

The proofs of these results are based on regularity estimates of the exact solution of (4) in specific scales of Besov spaces as developed in [1, 2].

Details of the analysis outlined above can be found in [3].

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