Using the Chu Construction for generalizing formal concept analysis

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Abstract. The goal of this paper is to show a connection between FCA generalisations and the Chu construction on the category ChuCors, the category of formal contexts and Chu correspondences. All needed categorical properties like categorical product, tensor product and its bifunctor properties are presented and proved. Finally, the second order generalisation of FCA is represented by a category built up in terms of the Chu construction.

Keywords: formal concept analysis, category theory, Chu construction

1 Introduction

The importance of category theory as a foundational tool was discovered soon after its very introduction by Eilenberg and MacLane about seventy years ago. On the other hand, Formal Concept Analysis (FCA) has largely shown both its practical applications and its capability to be generalized to more abstract frameworks, and this is why it has become a very active research topic in the recent years; for instance, a framework for FCA has been recently introduced in [16] in which the sets of objects and attributes are no longer unstructured but have a hypergraph structure by means of certain ideas from mathematical morphology. On the other hand, for an application of the FCA formalism to other areas, in [8] the authors introduce a representation of algebraic domains in terms of FCA.

The Chu construction [5] is a theoretical method that, from a symmetric monoidal closed (autonomous) category and a dualizing object, generates a *-autonomous category. This construction, or the closely related notion of Chu space, has been applied to represent quantum physical systems and their symmetries [1, 2].

This paper continues with the study of the categorical foundations of formal concept analysis. Some authors noticed the property of being a cartesian closed category of certain concept structures that can be approximated [7, 17]; others have provided a categorical construction of certain extensions of FCA [9]; morphisms have received a categorical treatment in [14] as a means for the modelling of communication.

There already exist some approaches [6] which consider the Chu construction in terms of FCA. In the current paper, we continue the previous study by the authors on the categorical foundation of FCA [10,12,13]. Specifically, the goal of this paper is to highlight the importance of the Chu construction in the research area of categorical description of the theory of FCA and its generalisations. The Chu construction plays here the role of some recipe for constructing a suitable category that covers the second order generalisation of FCA.

The structure of this paper is the following: in Section 2 we recall the preliminary notions required both from category theory and formal concept analysis. Then, the various categorical properties of the input category which are required (like the existence of categorical and tensor product) are developed in detail in Sections 3 and 4. An application of the Chu construction is presented in Section 5 where it is also showed how to construct formal contexts of second order from the category of classical formal contexts and Chu correspondences (ChuCors).

2 Preliminaries

In order to make the manuscript self-contained, the fundamental notions and its required properties are recalled in this section.

Definition 1. *A* formal context *is any triple* $\mathcal{C} = \langle \mathcal{B}, \mathcal{A}, \mathcal{R} \rangle$ *where* B *and* A *are finite sets and* $R \subseteq B \times A$ *is a binary relation. It is customary to say that* B *is a set of* objects*,* A *is a set of* attributes *and* R *represents a relation between objects and attributes.*

On a given formal context (B, A, R)*, the derivation (or concept-forming) operators are a pair of mappings* $\uparrow\colon 2^B \to 2^A$ *and* $\downarrow\colon 2^A \to 2^B$ *such that if* $X \subseteq B$*, then* ${\uparrow}X$ *is the set of all attributes which are related to every object in* X *and, similarly, if* $Y \subseteq A$*, then* ${\downarrow}Y$ *is the set of all objects which are related to every attribute in* Y*.*

In order to simplify the description of subsequent computations, it is convenient to describe the concept forming operators in terms of characteristic functions, namely, considering the subsets as functions on the set of Boolean values. Specifically, given $X \subseteq B$ and $Y \subseteq A$, we can consider mappings ${\uparrow}X\colon A \to \{0,1\}$ and ${\downarrow}Y\colon B \to \{0,1\}$

1. ${\uparrow}X(a) = \bigwedge\limits_{b \in B} \big((b \in X) \Rightarrow ((b,a) \in R)\big)$ for any $a \in A$
2. ${\downarrow}Y(b) = \bigwedge\limits_{a \in A} \big((a \in Y) \Rightarrow ((b,a) \in R)\big)$ for any $b \in B$

where the infimum is considered in the set of Boolean values and $\Rightarrow$ is the truth-function of the implication of classical logic.

Definition 2. *A* formal concept *is a pair of sets* $\langle X, Y \rangle \in 2^B \times 2^A$ *which is a fixpoint of the pair of concept-forming operators, namely,* ${\uparrow}X = Y$ *and* ${\downarrow}Y = X$*. The object part* X *is called the* extent *and the attribute part* Y *is called the* intent*.*

There are two main constructions relating two formal contexts: the bonds and the Chu correspondences. Their formal definitions are recalled below:

Definition 3. *Consider* $\mathcal{C}_1 = \langle B_1, A_1, R_1 \rangle$ *and* $\mathcal{C}_2 = \langle B_2, A_2, R_2 \rangle$ *two formal contexts. A* bond *between* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *is any relation* $\beta \in 2^{B_1 \times A_2}$ *such that its columns are extents of* $\mathcal{C}_1$ *and its rows are intents of* $\mathcal{C}_2$*. All bonds between such contexts will be denoted by* $\text{Bonds}(\mathcal{C}_1, \mathcal{C}_2)$*.*

The Chu correspondence between contexts can be seen as an alternative inter-contextual structure which, instead, links intents of $\mathcal{C}_1$ and extents of $\mathcal{C}_2$. Namely,

Definition 4. *Consider* $\mathcal{C}_1 = \langle B_1, A_1, R_1 \rangle$ *and* $\mathcal{C}_2 = \langle B_2, A_2, R_2 \rangle$ *two formal contexts. A* Chu correspondence *between* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *is any pair of multimappings* $\varphi = \langle \varphi_L, \varphi_R \rangle$ *such that*

- $\varphi_L \colon B_1 \to \text{Ext}(\mathcal{C}_2)$
- $\varphi_R \colon A_2 \to \text{Int}(\mathcal{C}_1)$
- $\uparrow_2(\varphi_L(b_1))(a_2) = \downarrow_1(\varphi_R(a_2))(b_1)$ *for any* $(b_1, a_2) \in B_1 \times A_2$

All Chu correspondences between such contexts will be denoted by $\text{Chu}(\mathcal{C}_1, \mathcal{C}_2)$*.*

The notions of bond and Chu correspondence are interchangeable; specifically, we will use the bond β_φ associated to a Chu correspondence φ from $\mathcal{C}_1$ to $\mathcal{C}_2$ defined for $b_1 \in B_1, a_2 \in A_2$ as follows:

$$\beta_\varphi(b_1, a_2) = \uparrow_2 (\varphi_L(b_1))(a_2) = \downarrow_1 (\varphi_R(a_2))(b_1)$$

The set of all bonds (resp. Chu correspondences) between any two formal contexts endowed with set inclusion as ordering have a complete lattice structure. Moreover, both complete lattices are dually isomorphic.

In order to formally define the composition of two Chu correspondences, we need to introduce the extension principle below:

Definition 5. *Given a mapping* $\varphi \colon X \to 2^Y$ *we define its extended mapping* $\varphi_+ \colon 2^X \to 2^Y$ *defined by* $\varphi_+(M) = \bigcup_{x \in M} \varphi(x)$*, for all* $M \in 2^X$*.*

The set of formal contexts together with Chu correspondences as morphisms forms a category denoted by ChuCors. Specifically:

- *objects* formal contexts
- *arrows* Chu correspondences
- *identity arrow* $\iota \colon \mathcal{C} \to \mathcal{C}$ of context $\mathcal{C} = \langle B, A, R \rangle$
  - $\iota_L(o) = \downarrow\uparrow(\{b\})$, for all $b \in B$
  - $\iota_R(a) = \uparrow\downarrow(\{a\})$, for all $a \in A$
- *composition* $\varphi_2 \circ \varphi_1 \colon \mathcal{C}_1 \to \mathcal{C}_3$ *of arrows* $\varphi_1 \colon \mathcal{C}_1 \to \mathcal{C}_2$, $\varphi_2 \colon \mathcal{C}_2 \to \mathcal{C}_3$ (where $\mathcal{C}_i = \langle B_i, A_i, R_i \rangle$, $i \in \{1, 2, 3\}$)
  - $(\varphi_2 \circ \varphi_1)_L : B_1 \to 2^{B_3}$ and $(\varphi_2 \circ \varphi_1)_R : A_3 \to 2^{A_1}$
  - $(\varphi_2 \circ \varphi_1)_L(b_1) = \downarrow_3\uparrow_3 (\varphi_{2L+}(\varphi_{1L}(b_1)))$
  - $(\varphi_2 \circ \varphi_1)_R(a_3) = \uparrow_1\downarrow_1 (\varphi_{1R+}(\varphi_{2R}(a_3)))$

The category ChuCors is *-autonomous and equivalent to category of complete lattices and isotone Galois connection, more results on this category and its L-fuzzy extensions can be found in [10,12,13,15].

3 Categorical product on ChuCors

In this section, the category ChuCors is proved to contain all finite categorical products, that is, it is a Cartesian category. To begin with, it is convenient to recall the notion of categorical product.

Definition 6. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two objects in a category. By a* product *of $\mathcal{C}_1$ and $\mathcal{C}_2$ we mean an object $\mathcal{P}$ with arrows $\pi_i\colon \mathcal{P} \to \mathcal{C}_i$ for $i \in \{1,2\}$ satisfying the following condition: For any object $\mathcal{D}$ and arrows $\delta_i\colon \mathcal{D} \to \mathcal{C}_i$ for $i \in \{1,2\}$, there exists a unique arrow $\gamma\colon \mathcal{D} \to \mathcal{P}$ such that $\gamma \circ \pi_i = \delta_i$ for all $i \in \{1,2\}$.*

The construction will use the notion of disjoint union of two sets $S_1 \uplus S_2$ which can be formally described as $(\{1\} \times S_1) \cup (\{2\} \times S_2)$ and, therefore, their elements will be denoted as ordered pairs (i, s) where $i \in \{1,2\}$ and $s \in S_i$. Now, we can proceed with the construction:

Definition 7. *Consider $\mathcal{C}_1 = \langle B_1, A_1, R_1\rangle$ and $\mathcal{C}_2 = \langle B_2, A_2, R_2\rangle$ two formal contexts. The* product *of such contexts is a new formal context*

$$\mathcal{C}_1 \times \mathcal{C}_2 = \langle B_1 \uplus B_2, A_1 \uplus A_2, R_{1\times 2}\rangle$$

where the relation $R_{1\times 2}$ is given by

$$((i,b),(j,a)) \in R_{1\times 2} \text{ if and only if } \big((i = j) \Rightarrow (b,a) \in R_i\big)$$

for any $(b,a) \in B_i \times A_j$ and $(i,j) \in \{1,2\} \times \{1,2\}$.

Lemma 1. *The above defined contextual product fulfills the property of the categorical product on the category* ChuCors*.*

Proof. We define the projection arrows $\langle \pi_{iL}, \pi_{iR}\rangle \in \mathrm{Chu}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{C}_i)$ for $i \in \{1,2\}$ as follows

- $\pi_{iL}\colon B_1 \uplus B_2 \to \mathrm{Ext}(\mathcal{C}_i) \subseteq 2^{B_i}$
- $\pi_{iR}\colon A_i \to \mathrm{Int}(\mathcal{C}_1 \times \mathcal{C}_2) \subseteq 2^{A_1 \cup A_2}$
- such that for any $(k,x) \in B_1 \uplus B_2$ and $a_i \in A_i$ the following equality holds

$$\uparrow_i (\pi_{iL}(k,x))(a_i) = \downarrow_{1\times 2} (\pi_{iR}(a_i))(k,x)$$

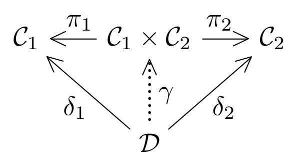
The definition of the projections is given below

$$\pi_{iL}(k,x)(b_i) = \begin{cases} \downarrow_i\uparrow_i (\chi_x)(b_i) & \text{for } k = i \\ \downarrow_i\uparrow_i \left(\overline{0}\right)(b_i) & \text{for } k \neq i \end{cases} \quad \text{for any } (k,x) \in B_1 \uplus B_2 \text{ and } b_i \in B_i$$

$$\pi_{iR}(a_i)(k,y) = \begin{cases} \uparrow_i\downarrow_i (\chi_{a_i})(y) & \text{for } k = i \\ \uparrow_k\downarrow_k \left(\overline{0}\right)(y) & \text{for } k \neq i \end{cases} \quad \text{for any } (k,y) \in A_1 \uplus A_2 \text{ and } a_i \in A_i.$$

The proof that the definitions above actually provide a Chu correspondence is just a long, although straightforward, computation and it is omitted.

Now, one has to show that to any formal context $\mathcal{D} = \langle E, F, G\rangle$, where $G \subseteq E \times F$ and any pair of arrows (δ_1, δ_2) with $\delta_i\colon \mathcal{D} \to \mathcal{C}_i$ for all $i \in \{1,2\}$, there exists a unique morphism $\gamma\colon \mathcal{D} \to \mathcal{C}_1 \times \mathcal{C}_2$ such that the following diagram commutes:



We give just the definition of γ as a pair of mappings $\gamma_L\colon E \to 2^{B_1 \uplus B_2}$ and $\gamma_R\colon A_1 \uplus A_2 \to 2^F$

- $\gamma_L(e)(k,x) = \delta_{kL}(e)(x)$ for any $e \in E$ and $(k,x) \in B_1 \uplus B_2$.
- $\gamma_R(k,y)(f) = \delta_{kR}(y)(f)$ for any $f \in F$ and $(k,y) \in A_1 \uplus A_2$.

Checking the condition of categorical product is again straightforward but long and tedious and, hence, it is omitted. □

We have just proved that binary products exist, but a cartesian category requires the existence of *all finite products*. If we recall the well-known categorical theorem which states that if a category has a terminal object and binary product, then it has all finite products, we have just to prove the existence of a terminal object (namely, the nullary product) in order to prove ChuCors to be cartesian.

Any formal context of the form $\langle B, A, B \times A\rangle$ where the incidence relation is the full cartesian product of the sets of objects and attributes is (isomorphic to) the terminal object of ChuCors. Such formal context has just one formal concept $\langle B, A\rangle$; hence, from any other formal context there is just one Chu correspondence to $\langle B, A, B \times A\rangle$.

4 Tensor product and its bifunctor property

Apart from the categorical product, another product-like construction can be given in the category ChuCors, for which the notion of transposed context $\mathcal{C}^*$ is needed.

Given a formal context $\mathcal{C} = \langle B, A, R\rangle$, its transposed context is $\mathcal{C}^* = \langle A, B, R^t\rangle$, where $R^t(a,b)$ holds iff $R(b,a)$ holds. Now, if $\varphi \in \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2)$, one can consider $\varphi^* \in \mathrm{Chu}(C_2^*, C_1^*)$ defined by $\varphi_L^* = \varphi_R$ and $\varphi_R^* = \varphi_L$.

Definition 8. *The tensor product of formal contexts $\mathcal{C}_i = \langle B_i, A_i, R_i\rangle$ for $i \in \{1,2\}$ is defined as the formal context $\mathcal{C}_1 \boxtimes \mathcal{C}_2 = \langle B_1 \times B_2, \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2^*), R_\boxtimes\rangle$ where*

$$R_\boxtimes((b_1,b_2),\varphi) = \downarrow_2(\varphi_L(b_1))(b_2).$$

Mori studied in [15] the properties of the tensor product above, and proved that ChuCors with $\boxtimes$ is a symmetric and monoidal category. Those results were later extended to the L-fuzzy case in [10]. In both papers, the structure of the formal concepts of a product context was established as an ordered pair formed by a bond and a set of Chu correspondences.

Lemma 2. *Let $\mathcal{C}_i = \langle B_i, A_i, R_i\rangle$ for $i \in \{1,2\}$ be two formal contexts, and let $\langle \beta, X\rangle \in \mathrm{Bonds}(\mathcal{C}_1, \mathcal{C}_2^*) \times 2^{\mathrm{Chu}(\mathcal{C}_1,\mathcal{C}_2^*)}$ be an arbitrary formal concept of $\mathcal{C}_1 \boxtimes \mathcal{C}_2$. Then $\beta = \bigwedge_{\psi\in X} \beta_\psi$ and $X = \{\psi \in \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2^*) \mid \beta \leq \beta_\psi\}$.*

Proof. Let X be an arbitrary subset of $\mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2^*)$. Then, for all $(b_1, b_2) \in B_1 \times B_2$, we have

$$\begin{aligned}\downarrow_{\mathcal{C}_1\boxtimes\mathcal{C}_2}(X)(b_1, b_2) &= \bigwedge_{\psi\in\mathrm{Chu}(\mathcal{C}_1,\mathcal{C}_2^*)} \big((\psi \in X) \Rightarrow \downarrow_2(\psi_L(b_1))(b_2)\big)\\ &= \bigwedge_{\psi\in X} \downarrow_2(\psi_L(b_1))(b_2) = \bigwedge_{\psi\in X} \beta_\psi(b_1, b_2)\end{aligned}$$

Let β be an arbitrary subset of $B_1 \times B_2$. Then, for all $\psi \in \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2^*)$

$$\begin{aligned}\uparrow_{\mathcal{C}_1\boxtimes\mathcal{C}_2}(\beta)(\psi) &= \bigwedge_{(b_1,b_2)\in B_1\times B_2} \big(\beta(b_1, b_2) \Rightarrow \downarrow_2(\psi_L(b_1))(b_2)\big)\\ &= \bigwedge_{(b_1,b_2)\in B_1\times B_2} \big(\beta(b_1, b_2) \Rightarrow \beta_\psi(b_1, b_2)\big)\end{aligned}$$

Hence $\uparrow_{\mathcal{C}_1\boxtimes\mathcal{C}_2}(\beta) = \{\psi \in \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2^*) \mid \beta \leq \beta_\psi\}$ □

We now introduce the notion of product of one context with a Chu correspondence.

Definition 9. *Let $\mathcal{C}_i = \langle B_i, A_i, R_i\rangle$ for $i \in \{0,1,2\}$ be formal contexts, and consider $\varphi \in \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2)$. Then, the pair of mappings*

$$(\mathcal{C}_0 \boxtimes \varphi)_L \colon B_0 \times B_1 \to 2^{B_0\times B_2} \qquad (\mathcal{C}_0 \boxtimes \varphi)_R \colon \mathrm{Chu}(\mathcal{C}_0, \mathcal{C}_2) \to 2^{\mathrm{Chu}(\mathcal{C}_0,\mathcal{C}_1)}$$

is defined as follows:

- $(\mathcal{C}_0 \boxtimes \varphi)_L(b, b_1)(o, b_2) = \downarrow_{\mathcal{C}_0\boxtimes\mathcal{C}_2}\uparrow_{\mathcal{C}_0\boxtimes\mathcal{C}_2}(\gamma_\varphi^{b,b_1})(o, b_2)$ *where* $\gamma_\varphi^{b,b_1}(o, b_2) = \big((b = o) \wedge \varphi_L(b_1)(b_2)\big)$ *for any* $b, o \in B_0$, $b_i \in B_i$ *with* $i \in \{1,2\}$
- $(\mathcal{C}_0 \boxtimes \varphi)_R(\psi_2)(\psi_1) = \big(\psi_1 \leq (\psi_2 \circ \varphi^*)\big)$ *for any* $\psi_i \in \mathrm{Chu}(\mathcal{C}_0, \mathcal{C}_i)$

As one could expect, the result is a Chu correspondence between the products of the contexts. Specifically,

Lemma 3. *Let $\mathcal{C}_i = \langle B_i, A_i, R_i\rangle$ be formal contexts for $i \in \{0,1,2\}$, and consider $\varphi \in \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2)$. Then $\mathcal{C}_0 \boxtimes \varphi \in \mathrm{Chu}(\mathcal{C}_0 \boxtimes \mathcal{C}_1, \mathcal{C}_0 \boxtimes \mathcal{C}_2)$.*

Proof. $(\mathcal{C}_0 \boxtimes \varphi)_L(b, b_1) \in \mathrm{Ext}(\mathcal{C}_0 \boxtimes \mathcal{C}_2)$ for any $(b, b_1) \in B_0 \times B_1$ follows directly from its definition. $(\mathcal{C}_0 \boxtimes \varphi)_R(\psi) \in \mathrm{Int}(\mathcal{C}_0 \boxtimes \mathcal{C}_1)$ for any $\psi \in \mathrm{Chu}(\mathcal{C}_0, \mathcal{C}_1)$ follows from Lemma 2.

Consider an arbitrary $b \in B_0$, $b_1 \in B_1$ and $\psi_2 \in \mathrm{Chu}(\mathcal{C}_0, \mathcal{C}_2^*)$

$$
\begin{aligned}
&{\uparrow_{\mathcal{C}_0 \boxtimes \mathcal{C}_2}} \big((\mathcal{C}_0 \boxtimes \varphi)_L(b, b_1) \big)(\psi_2) \\
&\qquad = {\uparrow_{\mathcal{C}_0 \boxtimes \mathcal{C}_2}} {\downarrow_{\mathcal{C}_0 \boxtimes \mathcal{C}_2}} {\uparrow_{\mathcal{C}_0 \boxtimes \mathcal{C}_2}} \, (\gamma_\varphi^{b,b_1})(\psi_2) \\
&\qquad = {\uparrow_{\mathcal{C}_0 \boxtimes \mathcal{C}_2}} \, (\gamma_\varphi^{b,b_1})(\psi_2) \\
&\qquad = \bigwedge_{(o,b_2) \in B_0 \times B_2} \big(\gamma_\varphi^{b,b_1}(o, b_2) \Rightarrow {\downarrow}(\psi_{2R}(b_2))(o) \big) \\
&\qquad = \bigwedge_{(o,b_2) \in B_0 \times B_2} \Big(\big((o = b) \wedge \varphi_L(b_1)(b_2) \big) \Rightarrow {\downarrow}(\psi_{2R}(b_2))(o) \Big) \\
&\qquad = \bigwedge_{o \in B_0} \bigwedge_{b_2 \in B_2} \Big((o = b) \Rightarrow \big(\varphi_L(b_1)(b_2) \Rightarrow {\downarrow}(\psi_{2R}(b_2))(o) \big) \Big) \\
&\qquad = \bigwedge_{o \in B_0} \Big((o = b) \Rightarrow \bigwedge_{b_2 \in B_2} (\varphi_L(b_1)(b_2) \Rightarrow {\downarrow}(\psi_{2R}(b_2))(o)) \Big) \\
&\qquad = \bigwedge_{b_2 \in B_2} \big(\varphi_L(b_1)(b_2) \Rightarrow {\downarrow}(\psi_{2R}(b_2))(b) \big) \\
&\qquad = \bigwedge_{b_2 \in B_2} \Big(\varphi_L(b_1)(b_2) \Rightarrow \bigwedge_{a \in A} (\psi_{2R}(b_2)(a) \Rightarrow R(b, a)) \Big) \\
&\qquad = \bigwedge_{a \in A} \Big(\bigvee_{b_2 \in B_2} (\varphi_L(b_1)(b_2) \wedge \psi_{2R}(b_2)(a)) \Rightarrow R(b, a) \Big) \\
&\qquad = \bigwedge_{a \in A} \Big(\big(\psi_{2R+}(\varphi_L(b_1))(a) \big) \Rightarrow R(b, a) \Big) \\
&\qquad = {\downarrow}(\psi_{2R+}(\varphi_L(b_1))(b) = {\downarrow}{\uparrow}{\downarrow}(\psi_{2R+}(\varphi_L(b_1))(b) = {\downarrow}((\varphi \circ \psi_2)_R(b_1))(b)
\end{aligned}
$$

Note the use above of the extended mapping as given in Definition 5 in relation to the composition of Chu correspondences.

On the other hand, we have

$$
\begin{aligned}
&{\downarrow_{\mathcal{C}_0 \boxtimes \mathcal{C}_1}} ((\mathcal{C}_0 \boxtimes \varphi)_R(\psi_2))(b, b_1) \\
&\qquad = \bigwedge_{\psi_1 \in \mathrm{Chu}(\mathcal{C}_0, \mathcal{C}_1)} ((\mathcal{C}_0 \boxtimes \varphi)_R(\psi_2)(\psi_1) \Rightarrow {\downarrow}(\psi_{1R}(b_1))(b)) \\
&\qquad = \bigwedge_{\psi_1 \in \mathrm{Chu}(\mathcal{C}_0, \mathcal{C}_1)} ((\psi_1 \geq \varphi \circ \psi_2) \Rightarrow {\downarrow}(\psi_{1R}(b_1))(b)) \\
&\qquad = \bigwedge_{\substack{\psi_1 \in \mathrm{Chu}(\mathcal{C}_0, \mathcal{C}_1) \\ \psi_1 \geq \varphi \circ \psi_2}} {\downarrow} (\psi_{1R}(b_1))(b) \\
&\qquad = {\downarrow}((\varphi \circ \psi_2)_R(b_1))(b)
\end{aligned}
$$

Hence ${\uparrow_{\mathcal{C}_0 \boxtimes \mathcal{C}_2}} \, ((\mathcal{C}_0 \boxtimes \varphi)_L(b, b_1))(\psi_2) = {\downarrow_{\mathcal{C}_0 \boxtimes \mathcal{C}_1}} \, ((\mathcal{C}_0 \boxtimes \varphi)_R(\psi_2))(b, b_1)$. So if $\varphi \in \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2)$ then $\mathcal{C}_0 \boxtimes \varphi \in \mathrm{Chu}(\mathcal{C}_0 \boxtimes \mathcal{C}_1, \mathcal{C}_0 \boxtimes \mathcal{C}_2)$. □

Given a fixed formal context $\mathcal{C}$, the tensor product $\mathcal{C} \boxtimes (-)$ forms a mapping between objects of ChuCors assigning to any formal context $\mathcal{D}$ the formal context $\mathcal{C} \boxtimes \mathcal{D}$. Moreover to any arrow $\varphi \in \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2)$ it assigns an arrow $\mathcal{C} \boxtimes \varphi \in \mathrm{Chu}(\mathcal{C} \boxtimes \mathcal{C}_1, \mathcal{C} \boxtimes \mathcal{C}_2)$. We will show that this mapping preservers the unit arrows and the composition of Chu correspondences. Hence the mapping forms an endofunctor on ChuCors, that is, a covariant functor from the category ChuCors to itself.

To begin with, let us recall the definition of functor between two categories:

Definition 10 (See [4]). *A covariant functor* $F\colon \mathrm{C} \to \mathrm{D}$ *between categories* C *and* D *is a mapping of objects to objects and arrows to arrows, in such a way that:*

- *For any morphism* $f\colon A \to B$*, one has* $F(f)\colon F(A) \to F(B)$
- $F(g \circ f) = F(g) \circ F(f)$
- $F(1_A) = 1_{F(A)}$.

Lemma 4. *Let* $\mathcal{C} = \langle B, A, R\rangle$ *be a formal context.* $\mathcal{C} \boxtimes (-)$ *is an endofunctor on* ChuCors*.*

Proof. Consider the unit morphism $\iota_{\mathcal{C}_1}$ of a formal context $\mathcal{C}_1 = \langle B_1, A_1, R_1\rangle$, and let us show that $(\mathcal{C} \boxtimes \iota_{\mathcal{C}_1}) = \iota_{\mathcal{C} \boxtimes \mathcal{C}_1}$. In other words, $\mathcal{C} \boxtimes (-)$ respects unit arrows in ChuCors.

$$
\begin{aligned}
&\uparrow_{\mathcal{C}\boxtimes\mathcal{C}_1}\big((\mathcal{C} \boxtimes \iota_{\mathcal{C}_1})(b, b_1)\big)(\psi) \\
&\quad = \bigwedge_{(o,o_1)\in B\times B_1} \Big(\big((o = b) \wedge \iota_{\mathcal{C}_1 L}(b_1)(o_1)\big) \Rightarrow \downarrow_1 (\psi_L(o))(o_1)\Big) \\
&\quad = \bigwedge_{o_1\in B_1} \big(\downarrow_1\uparrow_1 (\chi_{b_1})(o_1) \Rightarrow \downarrow_1 (\psi_L(b))(o_1)\big) \\
&\quad = \bigwedge_{o_1\in B_1} \Big(\downarrow_1\uparrow_1 (\chi_{b_1})(o_1) \Rightarrow \bigwedge_{a_1\in A_1} \big(\psi_L(b)(a_1) \Rightarrow R(o_1, a_1)\big)\Big) \\
&\quad = \bigwedge_{o_1\in B_1} \bigwedge_{a_1\in A_1} \Big(\downarrow_1\uparrow_1 (\chi_{b_1})(o_1) \Rightarrow \big(\psi_L(b)(a_1) \Rightarrow R(o_1, a_1)\big)\Big) \\
&\quad = \bigwedge_{o_1\in B_1} \bigwedge_{a_1\in A_1} \Big(\psi_L(b)(a_1) \Rightarrow \big(\downarrow_1\uparrow_1 (\chi_{b_1})(o_1) \Rightarrow R(o_1, a_1)\big)\Big) \\
&\quad = \bigwedge_{a_1\in A_1} \Big(\psi_L(b)(a_1) \Rightarrow \bigwedge_{o_1\in B_1} \big(\downarrow_1\uparrow_1 (\chi_{b_1})(o_1) \Rightarrow R(o_1, a_1)\big)\Big) \\
&\quad = \bigwedge_{a_1\in A_1} \big(\psi_L(b)(a_1) \Rightarrow \uparrow_1\downarrow_1\uparrow_1 (\chi_{b_1})(a_1)\big) \\
&\quad = \bigwedge_{a_1\in A_1} \big(\psi_L(b)(a_1) \Rightarrow R_1(b_1, a_1)\big) \\
&\quad = \downarrow_1 (\psi_L(b))(b_1)
\end{aligned}
$$

and, on the other hand, we have

$$\begin{aligned}
&\uparrow_{\mathcal{C}\boxtimes\mathcal{C}_1}(\iota_{\mathcal{C}\boxtimes\mathcal{C}_1}(b,b_1))(\psi)\\
&\qquad = \uparrow_{\mathcal{C}\boxtimes\mathcal{C}_1}(\chi_{(b,b_1)})(\psi)\\
&\qquad = \bigwedge_{(o,o_1)\in B\times B_1}\big(\chi_{(b,b_1)}(o,o_1) \Rightarrow \downarrow_1(\psi_L(o))(o_1)\big)\\
&\qquad = \downarrow_1(\psi_L(b))(b_1)
\end{aligned}$$

As a result, we have obtained $\uparrow_{\mathcal{C}\boxtimes\mathcal{C}_1}((\mathcal{C}\boxtimes\iota_{\mathcal{C}_1})(b,b_1))(\psi) = \uparrow_{\mathcal{C}\boxtimes\mathcal{C}_1}(\iota_{\mathcal{C}\boxtimes\mathcal{C}_1}(b,b_1))(\psi)$ for any $(b,b_1)\in B\times B_1$ and any $\psi\in \mathrm{Chu}(\mathcal{C},\mathcal{C}_1)$; hence, $\iota_{\mathcal{C}\boxtimes\mathcal{C}_1} = (\mathcal{C}\boxtimes\iota_{\mathcal{C}_1})$.

We will show now that $\mathcal{C}\boxtimes(-)$ preserves the composition of arrows. Specifically, this means that for any two arrows $\varphi_i \in \mathrm{Chu}(\mathcal{C}_i,\mathcal{C}_{i+1})$ for $i\in\{1,2\}$ it holds that $\mathcal{C}\boxtimes(\varphi_1\circ\varphi_2) = (\mathcal{C}\boxtimes\varphi_1)\circ(\mathcal{C}\boxtimes\varphi_2)$.

$$\begin{aligned}
&\uparrow_{\mathcal{C}\boxtimes\mathcal{C}_3}\big((\mathcal{C}\boxtimes(\varphi_1\circ\varphi_2))_L(b,b_1)\big)(\psi_3)\\
&\qquad = \bigwedge_{(o,b_3)\in B\times B_3}\Big(\big((o=b)\wedge(\varphi_1\circ\varphi_2)_L(b_1)(b_3)\big) \Rightarrow \downarrow(\psi_{3R}(b_3))(o)\Big)\\
&\qquad = \bigwedge_{b_3\in B_3}\big((\varphi_1\circ\varphi_2)_L(b_1)(b_3) \Rightarrow \downarrow(\psi_{3R}(b_3))(b)\big)\\
&\qquad\quad \text{(by similar operations to those in the first part of the proof)}\\
&\qquad = \downarrow\big(((\varphi_1\circ\varphi_2)\circ\psi_3)_L(b_1)\big)(b)
\end{aligned}$$

On the other hand, and writing F for $\mathcal{C}\boxtimes -$ in order to simplify the resulting expressions, we have

$$\begin{aligned}
&\uparrow_{F\mathcal{C}_3}((F\varphi_1\circ F\varphi_2)_L(b,b_1))(\psi_3)\\
&\quad = \uparrow_{F\mathcal{C}_3}\downarrow_{F\mathcal{C}_3}\uparrow_{F\mathcal{C}_3}\big((F\varphi_2)_{L+}\big((F\varphi_1)_L(b,b_1)\big)\big)(\psi_3)\\
&\quad = \bigwedge_{(o,b_3)\in B\times B_3}\Bigg(\\
&\qquad \bigvee_{(j,b_2)\in B\times B_2}\Big(\big((F\varphi_1)_L(b,b_1)(j,b_2)\wedge(F\varphi_2)_L(j,b_2)(o,b_3)\big) \Rightarrow \downarrow(\psi_{3R}(b_3))(o)\Big)\\
&\quad = \bigwedge_{b_3\in B_3}\bigwedge_{b_2\in B_2}\Big(\big(\varphi_{1L}(b_1)(b_2)\wedge\varphi_{2L}(b_2)(b_3)\big) \Rightarrow \downarrow(\psi_{3R}(b_3))(b)\Big)\\
&\quad = \bigwedge_{b_3\in B_3}\Big(\bigvee_{b_2\in B_2}\big(\varphi_{1L}(b_1)(b_2)\wedge\varphi_{2L}(b_2)(b_3)\big) \Rightarrow \downarrow(\psi_{3R}(b_3))(b)\Big)\\
&\quad = \bigwedge_{b_3\in B_3}\Big(\varphi_{2L+}(\varphi_{1L}(b_1))(b_3) \Rightarrow \downarrow(\psi_{3R}(b_3))(b)\Big)\\
&\quad = \bigwedge_{b_3\in B_3}\Big((\varphi_1\circ\varphi_2)_L(b_1)(b_3) \Rightarrow \downarrow(\psi_{3R}(b_3))(b)\Big)
\end{aligned}$$

From the previous equalities we see that $\mathcal{C} \boxtimes (\varphi_1 \circ \varphi_2) = (\mathcal{C} \boxtimes \varphi_1) \circ (\mathcal{C} \boxtimes \varphi_2)$. Hence, composition is preserved.

As a result, the mapping $\mathcal{C} \boxtimes (-)$ forms a functor from ChuCors to itself. □

All the previous computations can be applied to the first argument without any problems, hence we can directly state the following proposition.

Proposition 1. *The tensor product forms a bifunctor* $- \boxtimes -$ *from* ChuCors × ChuCors *to* ChuCors.

5 The Chu construction on ChuCors and second order formal concept analysis

The Chu construction [5] is a theoretical process that, from a symmetric monoidal closed (autonomous) category and a dualizing object, generates a *-autonomous category.

In the following, the construction will be applied on ChuCors and the dualizing object $\bot = \langle \{\diamond\}, \{\diamond\}, \neq \rangle$ as inputs. In this section it is shown how second order FCA is connected to the output of such construction.

The category generated by the Chu construction and ChuCors and $\bot$ will be denoted by CHU(ChuCors, $\bot$):

- Its objects are triplets of the form $\langle \mathcal{C}, \mathcal{D}, \rho \rangle$ where
  - $\mathcal{C}$ and $\mathcal{D}$ are objects of the input category ChuCors (i.e. formal contexts)
  - ρ is an arrow in $\mathrm{Chu}(\mathcal{C} \boxtimes \mathcal{D}, \bot)$
- Its morphisms are pairs of the form $\langle \varphi, \psi \rangle \colon \langle \mathcal{C}_1, \mathcal{C}_2, \rho_1 \rangle \to \langle \mathcal{D}_1, \mathcal{D}_2, \rho_2 \rangle$ where $\mathcal{C}_i$ and $\mathcal{D}_i$ are formal contexts for $i \in \{1, 2\}$ and
  - φ and ψ are elements from $\mathrm{Chu}(\mathcal{C}_1, \mathcal{D}_1)$ and $\mathrm{Chu}(\mathcal{D}_2, \mathcal{C}_2)$, respectively, such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{C}_1 \boxtimes \mathcal{D}_2 & \xrightarrow{\mathcal{C}_1 \boxtimes \psi} & \mathcal{C}_1 \boxtimes \mathcal{C}_2 \\ {\scriptstyle \varphi \boxtimes \mathcal{D}_2}\downarrow & & \downarrow{\scriptstyle \rho_1} \\ \mathcal{D}_1 \boxtimes \mathcal{D}_2 & \xrightarrow[\rho_2]{} & \bot \end{array}$$

or, equivalently, the following equality holds

$$(\mathcal{C}_1 \boxtimes \psi) \circ \rho_1 = (\varphi \boxtimes \mathcal{D}_2) \circ \rho_2$$

There are some interesting facts in the previous construction with respect to the second order FCA recently introduced in [11]:

1. To begin with, every object $\langle \mathcal{C}_1, \mathcal{C}_2, \rho \rangle$ in CHU(ChuCors$_{\mathcal{L}}$, $\bot$), and recall that $\rho \in \mathrm{Chu}(\mathcal{C}_1 \boxtimes \mathcal{C}_2, \bot)$, can be represented as a formal context of second order. Simply take into account that, from basic properties of the tensor product, we can obtain $\mathrm{Chu}(\mathcal{C}_1 \boxtimes \mathcal{C}_2, \bot) \cong \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2^*)$.

Specifically, as ChuCors is a closed monoidal category, we have that for every three formal contexts $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ the following isomorphism holds

$$\text{ChuCors}(\mathcal{C}_1 \boxtimes \mathcal{C}_2, \mathcal{C}_3) \cong \text{ChuCors}(\mathcal{C}_1, \mathcal{C}_2 \multimap \mathcal{C}_3)$$

and recall that $\mathcal{C}_2 \multimap \bot \cong \mathcal{C}_2^*$ because ChuCors is *-autonomous.

2. Similarly, any formal context of second order is representable by an object of $\text{CHU}(\text{ChuCors}_{\mathcal{L}}, \bot)$.

6 Conclusions and future work

After introducing the basic definitions needed from category theory and formal concept analysis, in this paper we have studied two different product constructions in the category ChuCors, namely the categorical product and the tensor product. The existence of products allows to represent tables and, hence, binary relations; the tensor product is proved to fulfill the required properties of a bifunctor, which enables us to consider the Chu construction on the category ChuCors. As a first application, we have sketched the representation of second order formal concept analysis [11] in terms of the Chu construction on the category ChuCors.

The use of different subcategories of ChuCors as input to the Chu construction seems to be an interesting way of obtaining different existing generalizations of FCA. For future work, we are planning to provide representations based on the Chu construction for one-sided FCA, heterogeneous FCA, multi-adjoint FCA, etcetera.

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