On the notion of fuzzy adjunctions between fuzzy orders

I. P. Cabrera, P. Cordero, F. García-Pardo, and M. Ojeda-Aciego

1. Introduction

Adjunctions (also called isotone Galois connections) between two mathematical structures provide a means of linking both theories allowing for mutual cooperative advantages. A number of results can be found in the literature concerning sufficient or necessary conditions for a Galois connection between ordered structures to exist.

In previous works [8], the authors studied the existence and construction of the right adjoint to a given mapping f, but *in a more general framework*: the initial setting is to consider a mapping $f\colon A \to B$ from a (fuzzy or pre-) ordered set A into an unstructured set B, and then characterize those situations in which B can be (fuzzily or pre) ordered and an isotone mapping $g\colon B \to A$ can be built such that the pair (f, g) is an adjunction.

A *fuzzy order* is understood as a fuzzy relation satisfying reflexivity, antisymmetry and $\otimes$-transitivity. It is worth to recall that, in a fuzzy setting, reflexivity and antisymmetry are conflicting properties [3] and some authors [5] opted for dropping reflexivity from the ordered structures used. Our choice in [7, 6] was to introduce the notion of fuzzy Galois connection in a straightforward way for a fuzzy order, together with the following very specific version of antisymmetry:

Antisymmetry: Condition $\rho_U(a, b) = \rho_U(b, a) = 1$ implies $a = b$, for all $a, b \in U$.

The definition given in [7] was the expected extension of that in the crisp case. Namely,

Definition 1. Let $\mathbb{A} = (A, \rho_A)$, $\mathbb{B} = (B, \rho_B)$ be fuzzy orders, and two mappings $f\colon A \to B$ and $g\colon B \to A$. The pair (f, g) forms a **fuzzy adjunction** between A and B, denoted $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$ if, for all $a \in A$ and $b \in B$, the equality $\rho_A(a, g(b)) = \rho_B(f(a), b)$ holds.

In principle, the term *fuzzy adjunction* is not fully justified, since mappings f and g are both crisp. In this paper, we explain the way in which the given definition is related to fuzzy mappings, hence, explaining the suitability of this notion to work in a completely fuzzy environment.

2. Preliminary definitions

On underlying fuzzy framework is that of the $\mathbb{L}$-fuzzy sets, where $\mathbb{L} = (L, \vee, \wedge, \top, \bot, \otimes, \to)$ is a *residuated lattice*. A $\mathbb{L}$*-fuzzy set* X is a mapping from the universe set, say A, to the lattice L, i.e. $X\colon A \to L$, where $X(u)$ means the degree in which u belongs to X.

A fuzzy binary relation on A is a fuzzy subset of $A \times A$, that is $R\colon A \times A \to L$, and it is said to be:

- *Reflexive* if $R(a, a) = \top$ for all $a \in A$.
- $\otimes$-*Transitive* if $R(a, b) \otimes R(b, c) \leq R(a, c)$ for all $a, b, c \in A$.
- *Symmetric* if $R(a, b) = R(b, a)$ for all $a, b \in A$.

From now on, when no confusion arises, we will omit the prefix "$\mathbb{L}$-".

Definition 2. A fuzzy relation R on A is said to be a:

- *Fuzzy equivalence* if R is a reflexive, $\otimes$-transitive and symmetric fuzzy relation on A.
- *Fuzzy equality* if R is a fuzzy equivalence relation satisfying that $R(a, b) = \top$ implies $a = b$, for all $a, b \in A$.

We will use the infix notation for fuzzy equivalence relation, that is: for $\approx\colon A \times A \to L$ a fuzzy equivalence relation, we denote $a_1 \approx a_2$ to refer to $\approx(a_1, a_2)$.

Our approach to fuzzy ordered structures is based on the following definitions, see [4]:

Definition 3. Let $\approx_A$ be a fuzzy equivalence relation on A. A fuzzy binary relation $\rho_A\colon A \times A \to L$ is said to be

- $\approx_A$-*reflexive* if $(a_1 \approx_A a_2) \leq \rho_A(a_1, a_2)$ for all $a_1, a_2 \in A$.
- $\otimes$-$\approx_A$-*antisymmetric* if $\rho_A(a_1, a_2) \otimes \rho_A(a_2, a_1) \leq (a_1 \approx_A a_2)$ for all $a_1, a_2 \in A$.

A *fuzzy order* with respect to $\otimes$ and $\approx$, shortly $\otimes$-$\approx_A$ *fuzzy order*, is a fuzzy binary relation that is $\approx_A$-reflexive, $\otimes$-$\approx_A$-antisymmetric, and $\otimes$-transitive.

The triplet $\mathcal{A} = (A, \approx_A, \rho_A)$ will be called $\otimes$-$\approx_A$- *fuzzy ordered set* or simply *fuzzy ordered set*, when no confusion can arise.

3. Fuzzy functions and fuzzy adjunction

A number of different approaches to the notion of fuzzy function can be found in the literature. The main problem with the definition resides in that the fuzziness of the function would imply that the function itself should be a fuzzy set (in some sense).

This difficulty can be overcome with the use of suitable fuzzy equivalences in the domain and the codomain of the function. Thus, one arrives to the following definition [1, 9]:

Definition 4. Let $\approx_A$ and $\approx_B$ be fuzzy equivalence relations on the sets A and B, respectively. A *partial fuzzy function* from A to B is a mapping $\mu\colon A \times B \to L$ satisfying the following conditions:

(Ext1) $\mu(a_1, b) \otimes (a_1 \approx_A a_2) \leq \mu(a_2, b)$ for all $a_1, a_2 \in A$ and $b \in B$.

(Ext2) $\mu(a, b_1) \otimes (b_1 \approx_B b_2) \leq \mu(a, b_2)$ for all $a \in A$ and $b_1, b_2 \in B$.

(Part) $\mu(a, b_1) \otimes \mu(a, b_2) \leq (b_1 \approx_B b_2)$ for all $a \in A$ and $b_1, b_2 \in B$

Moreover, μ is said to be a *perfect fuzzy function* whenever the following condition holds:

(Tot) For all $a \in A$ there exists $b \in B$ such that $\mu(a, b) = \top$.

An alternative approach was introduced in [2], which used the notion of *compatibility.*

Definition 5. Let $\approx_A$ and $\approx_B$ be fuzzy equivalence relations on the sets A and B, respectively. A mapping $\mu\colon A \times B \to L$ is said to be *compatible wrt* $\approx_A$ and $\approx_B$ if the following condition holds:

(Comp) $(a_1 \approx_A a_2) \otimes (b_1 \approx_B b_2) \otimes \mu(a_1, b_1) \leq \mu(a_2, b_2)$ for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

It is not difficult to show that compatibility is an equivalent version of the two types of extensionality properties in Definition 4. Specifically, we have the following

Lemma 6. *Let $\approx_A$ and $\approx_B$ be fuzzy equivalence relations on the sets A and B, respectively and $\mu\colon A\times B\to L$ a fuzzy relation. Then μ satisfies **(Ext1)** and **(Ext2)** if and only if μ satisfies **(Comp)**.*

PROOF. Suppose that μ verifies **(Ext1)** and **(Ext2)**. Then,

$$(a_1 \approx_A a_2) \otimes (b_1 \approx_B b_2) \otimes \mu(a_1, b_1) =$$

$$(b_1 \approx_B b_2) \otimes (\mu(a_1, b_1) \otimes (a_1 \approx_A a_2)) \leq (b_1 \approx_B b_2) \otimes \mu(a_2, b_1) \leq \mu(a_2, b_2)$$

Conversely, assume now that μ verifies **(Comp)**. Then,

$$\mu(a_1, b) \otimes (a_1 \approx_A a_2) = \mu(a_1, b) \otimes (a_1 \approx_A a_2) \otimes (b \approx_B b) \leq \mu(a_2, b) \quad \text{and}$$

$$\mu(a, b_1) \otimes (b_1 \approx_B b_2) = \mu(a, b_1) \otimes (b_1 \approx_B b_2) \otimes (a \approx_A a) \leq \mu(a, b_2)$$

The interesting part of the previous approaches is that, given a fuzzy function, there always exists a crisp function which, somehow, represents it. Formally:

Definition 7. Let $\mu\colon A \times B \to L$ be a fuzzy function. A *crisp description* of μ is a partial mapping $f\colon A \to B$ such that $\mathrm{dom}(f) = \{a \in A \mid \mu(a, b) = \top \text{ for some } b \in B\}$ and $\mu(a, f(a)) = \top$ for all $a \in \mathrm{dom}(f)$.

It is worth to take into account the following facts concerning a fuzzy function, say $\mu\colon A \times B \to L$, and its crisp description:

- There is always a partial mapping $f\colon A \to B$ such that f is a crisp description of μ.
- Obviously, μ is a perfect fuzzy function, i.e. it satisfies **(Tot)**, if and only if any crisp description of μ is a total map.
- In general, a fuzzy function need not have a unique crisp description. However, if the fuzzy relation $\approx_B$ on B is a fuzzy equality, then there exists just one crisp description for μ. Indeed, given f_1 and f_2 two crisp descriptions for μ, observe that $\top = \mu(a, f_1(a)) = \mu(a, f_2(a))$, then by **(Part)**,

  $$\top = \mu(a, f_1(a)) \otimes \mu(a, f_2(a)) \leq (f_1(a) \approx_B f_2(a))$$

  which implies that $(f_1(a) \approx_B f_2(a)) = \top$, thus $f_1(a) = f_2(a)$.

The previous considerations lead us to discuss the potential one-to-one correspondence of fuzzy functions and their crisp descriptions together with possibly extra conditions.

Definition 8. Let $\approx_A$ and $\approx_B$ be fuzzy equivalence relations on the sets A and B, respectively. A mapping $f\colon A \to B$ is said to be *compatible* with $\approx_A$ and $\approx_B$ if $(a_1 \approx_A a_2) \leq (f(a_1) \approx_B f(a_2))$ for all $a_1, a_2 \in A$.

The following two technical lemmas are the key to the proof of the correspondence between fuzzy functions and crisp descriptions.

Lemma 9. *Let $\approx_A$ and $\approx_B$ be fuzzy equivalence relations on the sets A and B, respectively and let $f\colon A \to B$ be a mapping which is compatible with $\approx_A$ and $\approx_B$. Then, there exists a perfect fuzzy function $\mu\colon A \times B \to L$ defined by $\mu(a,b) = (f(a) \approx_B b)$ for all $a \in A$ and $b \in B$ such that f is a crisp description of μ.*

Proof. Trivially, f is a crisp description of μ because $\approx_B$ is reflexive:

$\mu(a, f(a)) = (f(a) \approx_A f(a)) = \top$ for all $a \in A$.

Furthermore, this equality states that property **(Tot)** holds.

Let us see now that μ is a fuzzy function:

(Ext1) Since f is a map which is compatible with $\approx_A$ and $\approx_B$, for all $a_1, a_2 \in A$ and $b \in B$,

$$\mu(a_1, b) \otimes (a_1 \approx_A a_2) = (f(a_1) \approx_B b) \otimes (a_1 \approx_A a_2) \leq$$

$$(f(a_1) \approx_B b) \otimes (f(a_1) \approx_B f(a_2)).$$

Applying symmetry and $\otimes$-transitivity of $\approx_B$, we can rewrite

$$(f(a_1) \approx_B b) \otimes (f(a_1) \approx_B f(a_2)) = (f(a_2) \approx_B f(a_1)) \otimes (f(a_1) \approx_B b) \leq$$

$$(f(a_2) \approx_B b) = \mu(a_2, b).$$

(Ext2) By definition of μ and the $\otimes$-transitive property of $\approx_B$

$$\mu(a, b_1) \otimes (b_1 \approx_B b_2) = (f(a) \approx_B b_1) \otimes (b_1 \approx_B b_2) \leq (f(a) \approx_B b_2) = \mu(a, b_2)$$

(Part) By definition of μ and the symmetric and $\otimes$-transitive properties of $\approx_B$,

$$\mu(a, b_1) \otimes \mu(a, b_2) = (f(a) \approx_B b_1) \otimes (f(a) \approx_B b_2) =$$

$$(b_1 \approx_B f(a)) \otimes (f(a) \approx_B b_2) \leq (b_1 \approx_B b_2).$$

Lemma 10. *Let $\approx_A$ and $\approx_B$ be fuzzy equivalence relations on the sets A and B, respectively and let $\mu\colon A \times B \to L$ be a perfect fuzzy function. Then, every crisp description $f\colon A \to B$ of μ is compatible with both $\approx_A$ and $\approx_B$.*

Proof. Given $a_1, a_2 \in A$, since f is a crisp description of μ, we have that $\mu(a_i, f(a_i)) = \top$ for $i \in \{1, 2\}$. Then,

$$(a_1 \approx_A a_2) = \mu(a_1, f(a_1)) \otimes \mu(a_2, f(a_2)) \otimes (a_1 \approx_A a_2).$$

Now, by the condition (Ext1), we have that $\mu(a_1, f(a_1)) \otimes (a_1 \approx_A a_2) \leq \mu(a_2, f(a_1))$. Thus, we obtain that

$$\mu(a_1, f(a_1)) \otimes \mu(a_2, f(a_2)) \otimes (a_1 \approx_A a_2) \leq \mu(a_2, f(a_1)) \otimes \mu(a_2, f(a_2)).$$

And, by the condition (Part),

$$\mu(a_2, f(a_1)) \otimes \mu(a_2, f(a_2)) \leq (f(a_1) \approx_B f(a_2)).$$

Therefore, $(a_1 \approx_A a_2) \leq (f(a_1) \approx_B f(a_2))$, for all $a_1, a_2 \in A$.

We are now in situation to state and prove the promised result about equivalence between fuzzy mappings and their crisp descriptions.

Theorem 11. *Let $\approx_A$ be a fuzzy equivalence on A and let $\approx_B$ be a fuzzy equality on B. There exists a bijection between the perfect fuzzy functions defined from A to B and the crisp mappings from A to B which are compatible with $\approx_A$ and $\approx_B$.*

Proof. Given $\mu\colon A \times B \to L$ a perfect fuzzy function, the crisp description $f\colon A \to B$ of μ is compatible with $\approx_A$ and $\approx_B$, by Lemma 10.

Conversely, given a mapping $f\colon A \to B$ which is compatible with $\approx_A$ and $\approx_B$, by Lemma 9, the fuzzy relation $\mu \in L^{A\times B}$ defined by $\mu(a,b) = (f(a) \approx_B b)$, for all $a \in A$ and $b \in B$, is a fuzzy function, whose crisp description is precisely f, since we have that $\mu(a, f(a)) = (f(a) \approx_B f(a)) = \top$.

Moreover, the crisp description $f\colon A \to B$ of a perfect fuzzy function μ satisfies that $\mu(a,b) = (f(a) \approx_B b)$. In effect, $\mu(a,b) = \mu(a,b)\otimes\mu(a,f(a)) \leq (b \approx_B f(a))$; on the other hand, $(b \approx_B f(a)) = (b \approx_B f(a)) \otimes \mu(a, f(a)) \leq \mu(a,b)$.

It is worth to remark that, under the hypotheses of the theorem, for every perfect fuzzy function $\mu\colon A \times B \to L$ there exists a unique mapping $f\colon A \to B$ such that $\mu(a, f(a)) = \top$ and $\mu(a,b) = (f(a) \approx_B b)$, for all $a \in A$ and $b \in B$. As a result, we can safely work with crisp mappings which are compatible wrt the fuzzy equivalences.

Now, a reasonable approach to the fuzzified notion of adjunction would be the following:

Definition 12. Let $\mathcal{A} = (A, \approx_A, \rho_A)$ and $\mathcal{B} = (B, \approx_B, \rho_B)$ be two fuzzy ordered sets. Let $f\colon A \to B$ and $g\colon B \to A$ be two mappings which are compatible with $\approx_A$ and $\approx_B$. The pair (f, g) is said to be a *fuzzy adjunction* between $\mathcal{A}$ and $\mathcal{B}$ if the following conditions hold

(G1) $(a_1 \approx_A a_2) \otimes \rho_A(a_2, g(b)) \leq \rho_B(f(a_1), b)$

(G2) $(b_1 \approx_B b_2) \otimes \rho_B(f(a), b_1) \leq \rho_A(a, g(b_2))$

for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$.

It turns out that the previous definition is equivalent to the naï ve and straightforward definition given in [7].

Theorem 13. *Let $\mathcal{A} = (A, \approx_A, \rho_A)$ and $\mathcal{B} = (B, \approx_B, \rho_B)$ be two fuzzy ordered sets. Let $f\colon A \to B$ and $g\colon B \to A$ be two mappings which are compatible with $\approx_A$ and $\approx_B$, respectively.*

Then, the pair (f, g) is a fuzzy adjunction between $\mathcal{A}$ and $\mathcal{B}$ if and only if $\rho_A(a, g(b)) = \rho_B(f(a), b)$ for all $a \in A$ and $b \in B$.

PROOF. Assume that for all $a \in A$ and $b \in B$ the equality $\rho_A(a, g(b)) = \rho_B(f(a), b)$ holds.

Let $a_1, a_2 \in A$ and $b \in B$. Since f is a map which is compatible with $\approx_A$ and $\approx_B$, then

$$(a_1 \approx_A a_2) \otimes \rho_A(a_2, g(b)) \leq (f(a_1) \approx_B f(a_2)) \otimes \rho_A(a_2, g(b)).$$

By the hypothesis, we obtain that

$$(f(a_1) \approx_B f(a_2)) \otimes \rho_A(a_2, g(b)) \leq (f(a_1) \approx_B f(a_2)) \otimes \rho_B(f(a_2), b).$$

As ρ_B is $\approx_B$-reflexive and transitive, we have that

$$(f(a_1) \approx_B f(a_2)) \otimes \rho_B(f(a_2), b) \leq \rho_B(f(a_1), f(a_2)) \otimes \rho_B(f(a_2), b) \leq \rho_B(f(a_1), b).$$

Therefore, $(a_1 \approx_A a_2) \otimes \rho_A(a_2, g(b)) \leq \rho_B(f(a_1), b)$ for all $a_1, a_2 \in A$ and $b \in B$. Analogously, the condition $(G2)$ holds.

Conversely, assume now that conditions $(G1)$ and $(G2)$ hold. Applying condition $(G1)$, for $a \in A$ and $b \in B$, we have that $(a \approx_A a) \otimes \rho_A(a, g(b)) \leq \rho_B(f(a), b)$. Being $\approx_A$ reflexive, it is deduced that $\rho_A(a, g(b)) \leq \rho_B(f(a), b)$ for all $a \in A$ and $b \in B$. Analogously, $\rho_B(f(a), b) \leq \rho_A(a, g(b))$ for all $a \in A$ and $b \in B$. Therefore, $\rho_A(a, g(b)) = \rho_B(f(a), b)$ for all $a \in A$ and $b \in B$.

4. Conclusions and future work

Theorem 13 above states that the straightforward approach to the fuzzy notion of adjunction (or isotone Galois connection) makes perfect sense and, moreover, opens up two different ways to the generalization, depending on whether one would consider underlying fuzzy equalities/equivalences within the fuzzy order or not. A thorough study of both possibilities will be developed as future work.

Acknowledgements

This work has been partially supported by the Spanish Science Ministry projects TIN12-39353-C04-01 and TIN11-28084.

[1] M. Demirci. Fuzzy functions and their applications. *J. Math. Anal. Appl*, 252:495–517, 2000.

[2] R. Belohlavek Fuzzy Relational Systems: Foundations and Principles. Kluwer Academic Publishers, Norwell, MA, USA. 2002

[3] U. Bodenhofer. A similarity-based generalization of fuzzy orderings preserving the classical axioms. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 8(5):593–610, 2000.

[4] U. Bodenhofer, B. De Baets and J. Fodor. A compendium of fuzzy weak orders: Representations and constructions. *Fuzzy Sets and Systems* 158(8):811–829, 2007.

[5] J. Fodor and M. Roubens. *Fuzzy Preference Modelling and Multicriteria Decision Support*. Kluwer Academic Publishers, Dordrecht, 1994

[6] F. García-Pardo, I.P. Cabrera, P. Cordero, and M. Ojeda-Aciego. On adjunctions between fuzzy preordered sets: necessary conditions. *Lecture Notes in Artificial Intelligence* 8536:211–221, 2014.

[7] F. García-Pardo, I.P. Cabrera, P. Cordero, and M. Ojeda-Aciego. On the construction of fuzzy Galois connections. *Proc. of XVII Spanish Conference on Fuzzy Logic and Technology*, pages 99-102, 2014.

[8] F. García-Pardo, I.P. Cabrera, P. Cordero, M. Ojeda-Aciego, and F.J. Rodríguez. On the definition of suitable orderings to generate adjunctions over an unstructured codomain. *Information Sciences* 286: 173–187, 2014.

[9] S. Gottwald. Fuzzy Sets and Fuzzy Logic. Foundations of Application—from a Mathematical Point of View. Vieweg, Braunschweig, Wiesbaden 1993.