A generalized framework to consider positive and negative attributes in formal concept analysis.

J. M. Rodriguez-Jimenez, P. Cordero, M. Enciso and A. Mora

Universidad de Málaga, Andalucía Tech, Spain.
{pcordero,enciso}@uma.es
{amora,jmrodriguez}@ctima.uma.es

Abstract. In Formal Concept Analysis the classical formal context is analized taking into account only the positive information, i.e. the presence of a property in an object. Nevertheless, the no presence of a property in an object also provides a significant knowledge which can only be partially considered with the classical approach. In this work we have modified the concept forming operators to allow the treatment of both, positive and negative attributes which come from respectively, the presence and absence of the properties. In this work we define the new operators and we prove that they are Galois connections. Finally, we have also studied the correspondence between the formal context in the new framework and the extended concept lattice, providing new interesting properties.

1 Introduction

Data analysis of information is a well established discipline with tools and techniques well developed to challenge the identification of hide patterns in the data. Data mining, and general Knowledge Discovering, helps in the decision making process using pattern recognition, clustering, association and classification methods. One of the popular approaches used to extract knowledge is mining the patterns of the data expressed as implications (functional dependencies in database community) or association rules.

Traditionally, implications and similar notions have been built using the *positive* information, i.e. information induced by the presence of attributes in objects. In Manilla et al. [6] an extended framework for enriched rules was introduced, considering negation, conjunction and disjunction. Rules with negated attributes were also considered in [1]: "if we buy caviar, then we do not buy canned tuna".

In the framework of formal concept analysis, some authors have proposed the mining of implications with positive and negative attributes from the apposition of the context and its negation $(\mathbb{K}|\overline{\mathbb{K}})$ [2, 4]. Working with $(\mathbb{K}|\overline{\mathbb{K}})$ conduits to a huge exponential problem and also as R. Missaoui et.al. shown in [9] real applications use to have sparse data in the context $\mathbb{K}$ whereas dense data in $\overline{\mathbb{K}}$ (or viceversa), and therefore *"generate a huge set of candidate itemsets and a tremendous set of uninteresting rules"*.

R. Missaoui et al. [7, 8] propose the mining from a formal context $\mathbb{K}$ of a subset of all *mixed* implications, i.e. implication with positive and negative attributes, representing the presence and absence of properties. As far as we know, the approach of these authors use, for first time in this problem, a set of inference rules to manage negative attributes.

In [11] we followed the line proposed by Missaoui and presented an algorithm, based on the NextClosure algorithm, that allows to obtain mixed implications. The proposed algorithm returns a feasible and complete basis of mixed implications by performing a reduced number of requests to the formal context. Beyond the benefits provided by the inclusion of negative attributes in terms of expressiveness, Revenko and Kuznetsov [10] use negative attributes to tackle the problem of nding some types of errors in new object intents is introduced. Their approach is based on nding implications from an implication basis of the context that are not respected by a new object. Their work illustrates the great benefit that a general framework for negative and positive attributes would provide.

In this work we propose a deeper study of the algebraic framework for Formal Concept Analysis taking into account positive and negative information. The first step is to consider an extension of the classical forming-concept operators, proving to be Galois connection. As in the classical framework, this fact will allows to built the two usual dual concept lattices, but in this case, as we shall see, the correspondence among concept lattices and formal contexts reveal several characteristics which induce interesting properties. The main aim of this work is to establish a formal full framework which allows to develop in the future new methods and techniques dealing with positive and negative information.

In Section 2 we present the background of this work: the notions related with format concept analysis and negative attributes. Section 3 introduces the main results which constitutes the contribution of this paper. Finally we end with a conclusion section.

2 Preliminares

2.1 Formal Concept Analysis

In this section, the basic notions related with Formal Concept Analysis (FCA) [12] and attribute implications are briefly presented. See [3] for a more detailed explanation. A *formal context* is a triple $\mathbb{K} = \langle G, M, I\rangle$ where G and M are finite non-empty sets and $I \subseteq G \times M$ is a binary relation. The elements in G are named objects, the elements in M attributes and $\langle g, m\rangle \in I$ means that the object g has the attribute m. From this triple, two mappings $\uparrow: 2^G \to 2^M$ and $\downarrow: 2^M \to 2^G$, named concept-forming operators, are defined as follows: for any $X \subseteq G$ and $Y \subseteq M$,

$$X^{\uparrow} = \{m \in M \mid \langle g, m\rangle \in I \text{ for all } g \in X\} \tag{1}$$

$$Y^{\downarrow} = \{g \in G \mid \langle g, m\rangle \in I \text{ for all } m \in Y\} \tag{2}$$

$X^{\uparrow}$ is the subset of all attributes shared by all the objects in X and $Y^{\downarrow}$ is the subset of all objects that have the attributes in Y. The pair $(\uparrow, \downarrow)$ constitutes a Galois connection between 2^G and 2^M and, therefore, both compositions are closure operators.

A pair of subsets $\langle X, Y\rangle$ with $X \subseteq G$ and $Y \subseteq M$ such $X^{\uparrow} = Y$ and $Y^{\downarrow} = X$ is named a *formal concept*. X is named the *extent* and Y the *intent* of the concept. These extents and intents coincide with closed sets wrt the closure operators because $X^{\uparrow\downarrow} = X$ and $Y^{\downarrow\uparrow} = Y$. Thus, the set of all the formal concepts is a lattice, named *concept lattice*, with the relation

$$\langle X_1, Y_1\rangle \leq \langle X_2, Y_2\rangle \text{ if and only if } X_1 \subseteq X_2 \text{ (or equivalently, } Y_2 \subseteq Y_1) \qquad (3)$$

This concept lattice will be denoted by $\mathfrak{B}(G, M, I)$.

The concept lattice can be characterized in terms of *attribute implications* being expressions $A \to B$ where $A, B \subseteq M$. An implication $A \to B$ holds in a context $\mathbb{K}$ if $A^{\downarrow} \subseteq B^{\downarrow}$. That is, any object that has all the attributes in A has also all the attributes in B. It is well known that the sets of attribute implications that are valid in a context satisfies the Armstrong's Axioms:

`[Ref]` Reflexivity: If $B \subseteq A$ then $\vdash A \to B$.
`[Augm]` Augmentation: $A \to B \vdash A \cup C \to B \cup C$.
`[Trans]` Transitivity: $A \to B, B \to C \vdash A \to C$.

A set of implications Σ is considered an *implicational system for* $\mathbb{K}$ if: an implication holds in $\mathbb{K}$ if and only if it can be inferred, by using Armstrong's Axioms, from Σ.

Armstrong's axioms allow us to define the closure of attribute sets wrt an implicational system (the closure of a set A is usually denoted as A^+) and it is well-known that closed sets coincide with concepts. On the other hand, several kind of implicational systems has been defined in the literature being the most used the so-called Duquenne-Guigues (or stem) basis [5]. This basis satisfies that its cardinality is minimum among all the implicational systems and can be obtained from a context by using the renowned NextClosure Algorithm [3].

2.2 Negatives attributes

As we have mentioned in the introduction, classical FCA only discover knowledge limited to positive attributes in the context, but it does not consider information relative to the absence of properties (attributes). Thus, the Duquenne-Guigues basis obtained from Table 1 is $\{e \to bc, d \to c, bc \to e, a \to b\}$. Moreover, the implications $b \to c$ either $b \to d$ do not hold in Table 1 and therefore they can not be derived from the basis by using the inference system. Nevertheless, both implications correspond with different situations. In the first case, some objects have attributes b and c (e.g. objects o_1 and o_3) whereas another objects (e.g. o_2) have the attribute b and do not have c. By the other side, in the second case, any object that has the attribute b does not have the attribute d.

I	a	b	c	d	e
o_1		×	×		×
o_2	×	×			
o_3		×	×		×
o_4			×	×	

Table 1. A formal context

A more general framework is necessary to deal with this kind of information. In [11], we have tackled this issue focusing on the problem of mining implication with positive and negative attributes from formal contexts. As a conclusion of that work we emphasized the necessity of a full development of an algebraic framework.

First, we begin with the introduction of an extended notation that allows us to consider the negation of attributes. From now on, the set of attributes is denoted by M, and its elements by the letter m, possibly with subindexes. That is, the lowercase character m is reserved for positive attributes. We use $\overline{m}$ to denote the negation of the attribute m and $\overline{M}$ to denote the set $\{\overline{m} \mid m \in M\}$ whose elements will be named negative attributes.

Arbitrary elements in $M \cup \overline{M}$ are going to be denoted by the first letters in the alphabet: a, b, c, etc. and $\overline{a}$ denotes the opposite of a. That is, the symbol a could represent a positive or a negative attribute and, if $a = m \in M$ then $\overline{a} = \overline{m}$ and if $a = \overline{m} \in \overline{M}$ then $\overline{a} = m$.

Capital letters A, B, C ... denote subsets of $M \cup \overline{M}$. If $A \subseteq M \cup \overline{M}$, then $\overline{A}$ denotes the set of the opposite of attributes $\{\overline{a} \mid a \in A\}$ and the following sets are defined:

- $\text{Pos}(A) = \{m \in M \mid m \in A\}$
- $\text{Neg}(A) = \{m \in M \mid \overline{m} \in A\}$
- $\text{Tot}(A) = \text{Pos}(A) \cup \text{Neg}(A)$

Note that $\text{Pos}(A), \text{Neg}(A), \text{Tot}(A) \subseteq M$.

Once we have introduced the notation, we are going to summarize some results concerning the mining of knowledge from contexts in terms of implications with negative and positive attributes [11]. A trivial approach could be obtained by adding new columns to the context with the opposite of the attributes [4]. That is, given a context $\mathbb{K} = \langle G, M, I \rangle$, a new context $(\mathbb{K}|\overline{\mathbb{K}}) = \langle G, M \cup \overline{M}, I \cup \overline{I} \rangle$ is considered, where $\overline{I} = \{\langle g, \overline{m} \rangle \mid g \in G, m \in M, \langle g, m \rangle \notin I\}$. For example, if $\mathbb{K}$ is the context depicted in Table 1, the context $(\mathbb{K}|\overline{\mathbb{K}})$ is those presented in Table 2. Obviously, the classical framework and its corresponding machinery can be used to manage the new context and, in this (direct) way, negative attributes are considered. However, this rough approach induces a non trivial growth of the formal context and, consequently, algorithms have a worse performance.

In our opinion, a deeper study was done by R. Missaoui et al. in [7] where an evolved approach has been provided. For first time –as far as we know– inference rules for the management of positive and negative attributes are introduced [8].

$I \cup \overline{I}$	a	b	c	d	e	$\overline{a}$	$\overline{b}$	$\overline{c}$	$\overline{d}$	$\overline{e}$
o_1		×	×		×	×			×	
o_2	×	×						×	×	×
o_3		×	×		×	×			×	
o_4			×	×		×	×			×

Table 2. The formal context $(\mathbb{K}|\overline{\mathbb{K}})$

The authors also developed new methods to mine mixed attribute implications by means of the key notion [9].

In [11], we have developed a method to mine mixed implications whose main goal has been to avoid the management of the large $(\mathbb{K}|\overline{\mathbb{K}})$ contexts, so that the performance of the corresponding method has a controlled cost.

First, we extend the definitions of concept-forming operators, formal concept and attribute implication.

Definition 1. *Let $\mathbb{K} = \langle G, M, I\rangle$ be a formal context. We define the operators $\Uparrow: 2^G \to 2^{M\cup\overline{M}}$ and $\Downarrow: 2^{M\cup\overline{M}} \to 2^G$ as follows: for $X \subseteq G$ and $Y \subseteq M \cup \overline{M}$,*

$$X^{\Uparrow} = \{m \in M \mid \langle g, m\rangle \in I \textit{ for all } g \in X\} \\ \cup \{\overline{m} \in \overline{M} \mid \langle g, m\rangle \notin I \textit{ for all } g \in X\} \tag{4}$$

$$Y^{\Downarrow} = \{g \in G \mid \langle g, m\rangle \in I \textit{ for all } m \in Y\} \\ \cap \{g \in G \mid \langle g, m\rangle \notin I \textit{ for all } \overline{m} \in Y\} \tag{5}$$

Definition 2. *Let $\mathbb{K} = \langle G, M, I\rangle$ be a formal context. A* mixed formal concept *in $\mathbb{K}$ is a pair of subsets $\langle X, Y\rangle$ with $X \subseteq G$ and $Y \subseteq M \cup \overline{M}$ such $X^{\Uparrow} = Y$ and $Y^{\Downarrow} = X$.*

Definition 3. *Let $\mathbb{K} = \langle G, M, I\rangle$ be a formal context and let $A, B \subseteq M \cup \overline{M}$, the context $\mathbb{K}$ satisfies a* mixed attribute implication *$A \to B$, denoted by $\mathbb{K} \models A \to B$, if $A^{\Downarrow} \subseteq B^{\Downarrow}$.*

For example, in Table 1, as we previously mentioned, two different situations were presented. Thus, in this new framework we have that $\mathbb{K} \not\models b \to d$ and $\mathbb{K} \models b \to \overline{d}$ whereas $\mathbb{K} \not\models b \to c$ either $\mathbb{K} \not\models b \to \overline{c}$.

Now, we are going to introduce the mining method for mixed attribute implications. The method is strongly based on the set of inference rules built by supplementing Armstrong's axioms with the following ones, introduced in [8]: let $a, b \in M \cup \overline{M}$ and $A \subseteq M \cup \overline{M}$,

`[Cont]` Contradiction: $\vdash a\overline{a} \to M\overline{M}$.
`[Rft]` Reflection: $Aa \to b \vdash A\overline{b} \to \overline{a}$.

The closure of an attribute set A wrt a set of mixed attribute implications Σ, denoted as $A^{\#}$, is defined as the biggest set such that $A \to A^{\#}$ can be inferred from Σ by using Armstrong's Axioms plus `[Cont]` and `[Rft]`. Therefore, a mixed implication $A \to B$ can be inferred from Σ if and only if B is a subset of the closure of A, i.e. $B \subseteq A^{\Uparrow\!\Downarrow}$.

The proposed mining method, depicted in Algorithm 1, uses the inference rules in such a way that it is not centered around the notion of key, but it extends, in a proper manner, the classical NextClosure algorithm [3].

Algorithm 1: Mixed Implications Mining

Data: $\mathbb{K} = \langle G, M, I\rangle$
Result: Σ set of implications

begin
  $\Sigma := \varnothing$;
  $Y := \varnothing$;
  while $Y < M$ **do**
    foreach $X \subseteq Y$ **do**
      $A := (Y \smallsetminus X) \cup \overline{X}$;
      if `Closed`(A, Σ) **then**
        $C := A^{\Downarrow\Uparrow}$;
        if $A \neq C$ **then** $\Sigma := \Sigma \cup \{A \to C \smallsetminus A\}$;
    $Y :=$ `Next`(Y) `// i.e. successor of` Y `in the lectic order`
  return Σ

The algorithm to calculate the mixed implicational system doesn't need to exhaustive traverse all the subsets of mixed attributes, but only those ones that are closed w.r.t. the set of implications previously computed. The **Closed** function is defined having linear cost and is used to discern when a set of attributes is not closed and thus, the context is not visited in this case.

Function Closed(A,Σ): boolean

Data: $A \subseteq M \cup \overline{M}$ with Pos(A)$\cap$Neg(A) $= \varnothing$ and Σ being a set of mixed implications.
Result: '`true`' if A is closed wrt Σ or '`false`' otherwise.

begin
  foreach $B \to C \in \Sigma$ **do**
    if $B \subseteq A$ *and* $C \not\subseteq A$ **then** exit and **return** `false`;
    if $B \smallsetminus A = \{a\}$, $A \cap \overline{C} \neq \varnothing$, *and* $\overline{a} \notin A$ **then** exit and **return** `false`;
  return `true`

---

3 Mixed concept lattices

As we have mentioned, the goal of this paper is to develop a deep study of the generalized algebraic framework. In this section we are going to introduce the main results of this paper providing the properties of the generalized concept lattice. The main pillar of our new framework are the two concept-forming operators introduced in Equations 4 and 5. The following theorem ensures that the pair of these operators is a Galois connection:

Theorem 1. *Let $\mathbb{K} = \langle G, M, I\rangle$ be a formal context. The pair of concept-forming operators* $(^{\Uparrow}, ^{\Downarrow})$ *introduced in Definition 1 is a Galois Connection.*

Proof. We need to prove that, for all subsets $X \subseteq G$ and $Y \subseteq M \cup \overline{M}$,

$$X \subseteq Y^{\Downarrow} \text{ if and only if } Y \subseteq X^{\Uparrow}$$

First, assume $X \subseteq Y^{\Downarrow}$. For all $a \in Y$, we distinguish two cases:

1. If $a \in \mathrm{Pos}(Y)$, exists $m \in M$ with $a = m$ and, for all $g \in X$, since $X \subseteq Y^{\Downarrow}$, $\langle g, m\rangle \in I$ and therefore $a = m \in X^{\Uparrow}$.
2. If $a \in \mathrm{Neg}(Y)$, exits $m \in M$ with $a = \overline{m}$ and, for all $g \in X$, since $X \subseteq Y^{\Downarrow}$, $\langle g, m\rangle \notin I$ and therefore $a = \overline{m} \in X^{\Uparrow}$.

Conversely, assume $Y \subseteq X^{\Uparrow}$ and $g \in X$. To ensure that $g \in Y^{\Downarrow}$, we need to prove that $\langle g, a\rangle \in I$ for all $a \in \mathrm{Pos}(Y)$ and $\langle g, \overline{a}\rangle \notin I$ for all $a \in \mathrm{Neg}(Y)$, which is straightforward from $Y \subseteq X^{\Uparrow}$. □

Therefore, above theorem ensures that $^{\Uparrow\circ\Downarrow}$ and $^{\Downarrow\circ\Uparrow}$ are closure operators. Furthermore, as in the classical case, both closure operators provide two dually isomorphic lattices. We denote by $\mathfrak{B}^{\sharp}(G, M, I)$ to the lattice of mixed concepts with the relation

$$\langle X_1, Y_1\rangle \leq \langle X_2, Y_2\rangle \text{ iff } X_1 \subseteq X_2 \text{ (or equivalently, iff } Y_1 \supseteq Y_2)$$

Moreover, as in the classical FCA, mixed implications and mixed concept lattice make up the two sides of the same coin, i.e. the information mined from the mixed formal context may be dually represented by means of a set of mixed attribute implications or a mixed concept lattice.

As we shall see later in this section, unlike the classical FCA, mixed concept lattices are restricted to an specific lattice subclass. There exist specific properties that lattices may observe to be considered a valid lattice structure which corresponds to a mixed formal context. In fact, this is one of the main goal of this paper, the characterization of the lattices in the mixed formal concept analysis.

In Table 3 six different lattices are depicted. In the classical framework, all of them may be associated with formal contexts, i.e. in the classical framework any lattice corresponds with a collection of formal context. Nevertheless, in the mixed attribute framework this property does not hold anymore. Thus, in Table 3, as we shall prove later in this paper, lattices 3 and 5 cannot be associated with a mixed formal context.

The following two definitions characterizes two kind of significant sets of attributes that will be used later:

Definition 4. *Let $\mathbb{K} = \langle G, M, I\rangle$ be a formal context. A set $A \subseteq M \cup \overline{M}$ is named consistent set if* $\mathrm{Pos}(A) \cap \mathrm{Neg}(A) = \varnothing$.

The set of consistent sets are going to be denoted by $\mathbb{C}$tts, i.e.

$$\mathbb{C}\mathrm{tts} = \{A \subseteq M \cup \overline{M} \mid \mathrm{Pos}(A) \cap \mathrm{Neg}(A) = \varnothing\}$$

If $A \in \mathbb{C}$tts then $|A| \leq |M|$ and, in the particular case where $|A| = |M|$, we have $\mathrm{Tot}(A) = M$. This situation induces the notion of *full* set:

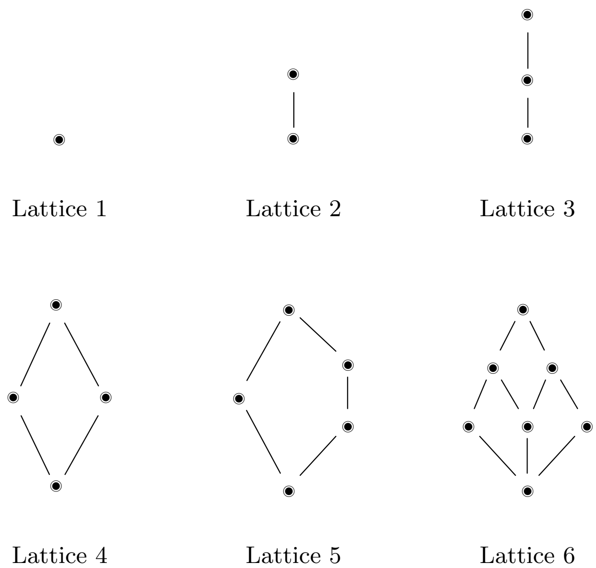


Table 3. Scheletons of some lattices

Definition 5. *Let $\mathbb{K} = \langle G, M, I\rangle$ be a formal context. A set $A \subseteq M \cup \overline{M}$ is said to be full consistent set if $A \in \mathbb{C}\mathrm{tts}$ and $\mathrm{Tot}(A) = M$.*

The following lemma, which characterize the boundary cases, is straightforward from Definition 1.

Lemma 1. *Let $\mathbb{K} = \langle G, M, I\rangle$ be a formal context. Then $\varnothing^{\Uparrow} = M \cup \overline{M}$, $\varnothing^{\Downarrow} = G$ and $(M \cup \overline{M})^{\Downarrow} = \varnothing$.*

In the classical framework, the concept lattice $\mathcal{B}(G, M, I)$ is bounded by $\langle M^{\downarrow}, M\rangle$ and $\langle G, G^{\uparrow}\rangle$. However, in this generalized framework, as a direct consequence from above lemma, the lower and upper bounds of $\mathcal{B}^{\sharp}(G, M, I)$ are $\langle \varnothing, M \cup \overline{M}\rangle$ and $\langle G, G^{\Uparrow}\rangle$ respectively.

Lemma 2. *Let $\mathbb{K} = \langle G, M, I\rangle$ be a formal context. The following properties hold:*

1. *For all $g \in G$, $\{g\}^{\Uparrow}$ is a full consistent set.*
2. *For all $g_1, g_2 \in G$, if $g_1 \in \{g_2\}^{\Uparrow\Downarrow}$ then $\{g_1\}^{\Uparrow} = \{g_2\}^{\Uparrow}$.* [1]
3. *For all $X \subseteq G$, $X^{\Uparrow} = \bigcap_{g\in X}\{g\}^{\Uparrow}$.*

Proof. 1. It is obvious because, for all $m \in M$, $\langle g, m\rangle \in I$ or $\langle g, m\rangle \notin I$ and $\{g\}^{\Uparrow} = \{m \in M \mid \langle g, m\rangle \in I\} \cup \{\overline{m} \in \overline{M} \mid \langle g, m\rangle \notin I\}$ being a disjoint union. Thus, $\mathrm{Tot}(\{g\}^{\Uparrow}) = M$ and $\mathrm{Pos}(\{g\}^{\Uparrow}) \cap \mathrm{Neg}(\{g\}^{\Uparrow}) = \varnothing$.

[1] That is, g_1 and g_2 have exactly the same attributes.

2. Since $(\Uparrow, \Downarrow)$ is a Galois connection, $g_1 \in \{g_2\}^{\Uparrow\Downarrow}$ (i.e. $\{g_1\} \subseteq \{g_2\}^{\Uparrow\Downarrow}$) implies $\{g_2\}^{\Uparrow} \subseteq \{g_1\}^{\Uparrow}$. Moreover, by item 1, both $\{g_1\}^{\Uparrow}$ and $\{g_2\}^{\Uparrow}$ are full consistent and, therefore, $\{g_1\}^{\Uparrow} = \{g_2\}^{\Uparrow}$.
3. In the same way that occurs in the classical framework, since $(\Uparrow, \Downarrow)$ is a Galois connection between $(2^G, \subseteq)$ and $(2^{M \cup \overline{M}}, \subseteq)$, for any $X \subseteq G$, we have that $X^{\Uparrow} = \left(\bigcup_{g \in X} \{g\}\right)^{\Uparrow} = \bigcap_{g \in X} \{g\}^{\Uparrow}$. □

The above elementary lemmas lead to the following theorem emphasizing a significant difference with respect to the classical construction and it focuses on how the inclusion of new objects influences the structure of mixed concept lattice.

Theorem 2. *Let $\mathbb{K} = \langle G, M, I\rangle$ be a formal context, g_0 be a new object, i.e. $g_0 \notin G$, and $Y \subseteq M$ be the set of attributes that g_0 satisfies. Then, there exists $g \in G$ such that $\{g\}^{\Uparrow} = \{g_0\}^{\Uparrow}$ if and only if there exists an isomorphism between $\mathcal{B}^\sharp(G, M, I)$ and $\mathcal{B}^\sharp(G \cup \{g_0\}, M, I \cup \{\langle g_0, m\rangle \mid m \in Y\})$.*

That is, if a new different object (an object that differs at least in one attribute from each object in the context) is added to the formal context then the mixed concept lattice changes.

Proof. Obviously, if there exists $g \in G$ such that $\{g\}^{\Uparrow} = \{g_0\}^{\Uparrow}$, from Lemma 2 g and g_0 have exactly the same attributes, and moreover the lattices $\mathcal{B}^\sharp(G, M, I)$ and $\mathcal{B}^\sharp(G \cup \{g_0\}, M, I \cup \{\langle g_0, m\rangle \mid m \in Y\})$ are isomorphic.

Conversely, if the mixed concept lattices are isomorphic, there exists $X \subseteq G$ such that the closed set $X^{\Uparrow}$ in $\mathcal{B}^\sharp(G, M, I)$ coincides with $\{g_0\}^{\Uparrow}$. Thus, in the mixed concept lattice $\mathcal{B}^\sharp(G \cup \{g_0\}, M, I \cup \{\langle g_0, m\rangle \mid m \in X\})$, by Lemma 2, we have that $\{g_0\}^{\Uparrow} = X^{\Uparrow} = \cap_{g \in X} \{g\}^{\Uparrow}$. Moreover, since $\{g_0\}^{\Uparrow}$ is a full consistent set, $X \neq \varnothing$ because of, by Lemma 1, $\varnothing^{\Uparrow} = M \cup \overline{M}$. Therefore, for all $g \in X$ (there exists at least one $g \in X$), $g_0 \in \{g\}^{\Uparrow}$ and, by Lemma 2, $\{g\}^{\Uparrow} = \{g_0\}^{\Uparrow}$. □

Example 1. Let $\mathbb{K}_1 = (\{g1, g2\}, \{a, b, c\}, I_1)$ and $\mathbb{K}_2 = (\{g1, g2, g3\}, \{a, b, c\}, I_2)$ be formal contexts where I_1 and I_2 are the binary relations depicted in Table 4. Note that $\mathbb{K}_2$ is built from $\mathbb{K}_1$ by adding the new object $g3$. In the classical framework, the concept lattices $\mathcal{B}(\{g1, g2\}, \{a, b, c\}, I_1)$ and $\mathcal{B}(\{g1, g2, g3\}, \{a, b, c\}, I_2)$ are isomorphic. See Figure 1.

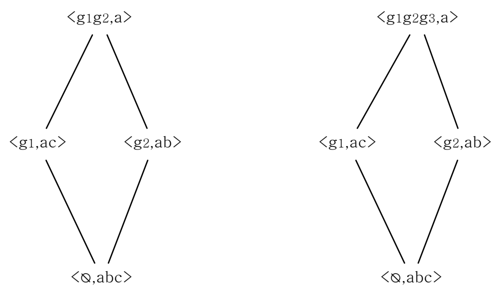
I_1	a	b	c
$g1$	×		×
$g2$	×	×	

I_2	a	b	c
$g1$	×		×
$g2$	×	×	
$g3$	×		

Table 4. The formal contexts $\mathbb{K}_1$ and $\mathbb{K}_2$

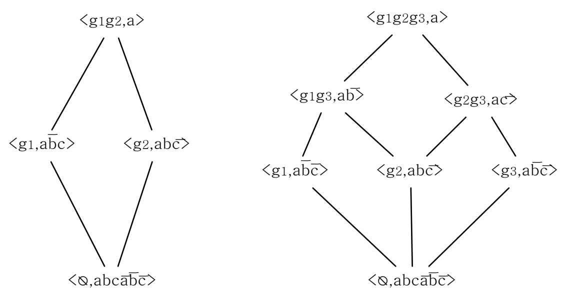
However, the lattices of mixed concepts cannot be isomorphic because the new object $g3$ is not a repetition of one existing object. See Figure 2.

The following theorem characterizes the atoms of the new concept lattice $\mathcal{B}^\sharp$.



$\mathcal{B}(\{g1, g2\}, \{a, b, c\}, I_1)$ $\mathcal{B}(\{g1, g2, g3\}, \{a, b, c\}, I_2)$

Fig. 1. Lattices obtained in the classical framework



$\mathcal{B}^\sharp(\{g1, g2\}, \{a, b, c\}, I_1)$ $\mathcal{B}^\sharp(\{g1, g2, g3\}, \{a, b, c\}, I_2)$

Fig. 2. Lattices obtained in the extended framework

Theorem 3. *Let $\mathbb{K} = \langle G, M, I\rangle$ be a formal context. The set of atoms in the lattice $\mathcal{B}^\sharp(G, M, I)$ is $\{\langle \{g\}^{\Uparrow\Downarrow}, \{g\}^{\Uparrow}\rangle \mid g \in G\}$.*

Proof. First, fixed $g_0 \in G$, we are going to prove that the mixed concept $\langle \{g_0\}^{\Uparrow\Downarrow}, \{g_0\}^{\Uparrow}\rangle$ is an atom in $\mathcal{B}^\sharp(G, M, I)$. If $\langle X, Y\rangle$ is a mixed concept such that $\langle \varnothing, M \cup \overline{M}\rangle < \langle X, Y\rangle \leq \langle \{g_0\}^{\Uparrow\Downarrow}, \{g_0\}^{\Uparrow}\rangle$, then $\{g_0\}^{\Uparrow} \subseteq Y = X^{\Uparrow} \subsetneq M \cup \overline{M}$. By Lemma 2, $\{g_0\}^{\Uparrow} \subseteq X^{\Uparrow} = \bigcap_{g\in X}\{g\}^{\Uparrow}$. Moreover, for all $g \in X \neq \varnothing$, by Lemma 2, both $\{g_0\}^{\Uparrow}$ and $\{g\}^{\Uparrow}$ are full consistent sets and, since $\{g_0\}^{\Uparrow} \subseteq \{g\}^{\Uparrow}$, we have $\{g_0\}^{\Uparrow} = \{g\}^{\Uparrow}$. Therefore, $\{g_0\}^{\Uparrow} = X^{\Uparrow} = Y$ and $\langle X, Y\rangle = \langle \{g_0\}^{\Uparrow\Downarrow}, \{g_0\}^{\Uparrow}\rangle$.

Conversely, if $\langle X, Y\rangle$ is an atom in $\mathcal{B}^\sharp(G, M, I)$, then $X \neq \varnothing$ and there exists $g_0 \in X$. Since $(\Uparrow, \Downarrow)$ is a Galois connection, $\{g_0\}^{\Uparrow} \supseteq X^{\Uparrow} = Y$ and, therefore, $\langle \{g_0\}^{\Uparrow\Downarrow}, \{g_0\}^{\Uparrow}\rangle \leq \langle X, Y\rangle$. Finally, since $\langle X, Y\rangle$ is an atom, we have that $\langle X, Y\rangle = \langle \{g_0\}^{\Uparrow\Downarrow}, \{g_0\}^{\Uparrow}\rangle$. □

The following theorem establishes the characterization of the mixed concept lattice, proving that atoms and join irreducible elements are the same notions.

Theorem 4. *Let $\mathbb{K} = \langle G, M, I\rangle$ be a formal context. Any element in $\mathcal{B}^\sharp(G, M, I)$ is $\vee$-irreducible if and only if it is an atom.*

Proof. Obviously, any atom is $\vee$-irreducible. We are going to prove that any $\vee$-irreducible element belongs to $\{\langle \{g\}^{\Uparrow\Downarrow}, \{g\}^{\Uparrow}\rangle \mid g \in G\}$. Let $\langle X, Y\rangle$ be a $\vee$-irreducible element. Then, by Lemma 2, $Y = X^{\Uparrow} = \bigcap_{g\in X}\{g\}^{\Uparrow}$. Let X' be the smaller set such that $X' \subseteq X$ and $Y = \bigcap_{g\in X'}\{g\}^{\Uparrow}$. If X' is a singleton, then $\langle X, Y\rangle \in \{\langle \{g\}^{\Uparrow\Downarrow}, \{g\}^{\Uparrow}\rangle \mid g \in G\}$.

Finally, we prove that X' is necessarily a singleton. In other case, a bipartition of X' in two disjoint sets Z_1 and Z_2 can be made satisfying $Z_1 \cup Z_2 = X'$, $Z_1 \neq \varnothing$, $Z_2 \neq \varnothing$ and $Z_1 \cap Z_2 \neq \varnothing$. Then, $Y = \bigcap_{g\in Z_1}\{g\}^{\Uparrow} \cap \bigcap_{g\in Z_1}\{g\}^{\Uparrow} = Z_1^{\Uparrow} \cap Z_2^{\Uparrow}$ and so $\langle X, Y\rangle = \langle Z_1^{\Uparrow\Downarrow}, Z_1^{\Uparrow}\rangle \vee \langle Z_2^{\Uparrow\Downarrow}, Z_2^{\Uparrow}\rangle$ and $Z_1^{\Uparrow} \neq Y \neq Z_2^{\Uparrow}$. However, it is not posible because $\langle X, Y\rangle$ is $\vee$-irreducible. □

As a final end point of this study, we may conclude that unlike in the classical framework, not every concept lattice may be linked with a formal context. Thus, lattices number 3 and 5 from Table 3 cannot be associated with a mixed formal concept. Both of them have one element which is not a atom but, at the same time, it is a join irreducible element in the lattice. More specifically, there does not exists a mixed concept lattice with three elements.

4 Conclusions

In this work we have presented an algebraic study of a general framework to deal with negative and positive information. After considering new forming-concept operators we prove that they constitutes a Galois connection. The main results of the work are devoted to establish the new relation among mixed concept lattices and mixed formal concepts. Thus, the most outstanding conclusions are that:

- the inclusion of a new (and different) object in a formal concept has a direct effect in the structure of the lattice, producing a different lattice.
- no any kind of lattice may be associated with a mixed formal context, which induces a restriction in the structure that mixed concept lattice may have.

References

1. R. Agrawal and R. Srikant. Fast Algorithms for Mining Association Rules in Large Databases. In *Proceedings of the 20th International Conference on Very Large Data Bases (VLDB)*, pages 487–499, Santiago de Chile, Chile, 1994. Morgan Kaufmann Publishers Inc.
2. Jean-François Boulicaut, Artur Bykowski, and Baptiste Jeudy. Towards the tractable discovery of association rules with negations. In *FQAS*, pages 425–434, 2000.
3. B. Ganter. Two basic algorithms in concept analysis. *Technische Hochschule, Darmstadt*, 1984.
4. Ghada Gasmi, Sadok Ben Yahia, Engelbert Mephu Nguifo, and Slim Bouker. Extraction of association rules based on literalsets. In *DaWaK*, pages 293–302, 2007.
5. J.L. Guigues and V. Duquenne. Familles minimales d implications informatives resultant d un tableau de donnees binaires. *Mathematiques et Sciences Sociales*, 95:5–18, 1986.
6. Heikki Mannila, Hannu Toivonen, and A. Inkeri Verkamo. Efficient algorithms for discovering association rules. In *KDD Workshop*, pages 181–192, 1994.
7. Rokia Missaoui, Lhouari Nourine, and Yoan Renaud. Generating positive and negative exact rules using formal concept analysis: Problems and solutions. In *ICFCA*, pages 169–181, 2008.
8. Rokia Missaoui, Lhouari Nourine, and Yoan Renaud. An inference system for exhaustive generation of mixed and purely negative implications from purely positive ones. In *CLA*, pages 271–282, 2010.
9. Rokia Missaoui, Lhouari Nourine, and Yoan Renaud. Computing implications with negation from a formal context. *Fundam. Inform.*, 115(4):357–375, 2012.
10. A. Revenko and S. Kuznetzov. Finding errors in new object intents. In *CLA*, pages 151–162, 2012.
11. J.M. Rodriguez-Jimenez, P. Cordero, M. Enciso, and A. Mora. Negative attributes and implications in formal concept analysis. *Procedia Computer Science*, 31(0):758 – 765, 2014. 2nd International Conference on Information Technology and Quantitative Management, {ITQM} 2014.
12. R. Wille. Restructuring lattice theory: an approach based on hierarchies of concepts. In *Rival, I. (ed.): Ordered Sets*, pages 445–470. Boston, 1982.