Advanced techniques to compute improper integrals using a CAS

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Contents

Motivation of the problem

Let us consider the following types of improper integrals:

$$\int_0^\infty f(t)\,\mathrm{d}t \qquad ; \qquad \int_{-\infty}^0 f(t)\,\mathrm{d}t \qquad \text{and} \qquad \int_{-\infty}^\infty f(t)\,\mathrm{d}t$$

Let F be an antiderivative of f. The basic approach to compute such integrals involves the following computations:

Motivation of the problem

$$\int_0^\infty f(t)\,\mathrm{d}t = \lim_{m\to\infty}\int_0^m f(t)\,\mathrm{d}t = \lim_{m\to\infty}\Big(F(m)-F(0)\Big)$$

$$\int_{-\infty}^0 f(t)\,\mathrm{d}t = \lim_{m\to-\infty}\int_m^0 f(t)\,\mathrm{d}t = \lim_{m\to-\infty}\Big(F(0)-F(m)\Big)$$

$$\int_{-\infty}^\infty f(t)\,\mathrm{d}t = \int_{-\infty}^0 f(t)\,\mathrm{d}t + \int_0^\infty f(t)\,\mathrm{d}t \qquad \text{or, if convergence,}$$

$$\int_{-\infty}^\infty f(t)\,\mathrm{d}t = \lim_{m\to\infty}\int_{-m}^m f(t)\,\mathrm{d}t = \lim_{m\to\infty}\Big(F(m)-F(-m)\Big)$$

(Cauchy principal value)

Motivation of the problem

But, what happens if an antiderivative F for f or the above limits do not exist?

For example, for

$$\int_0^{\infty} \frac{\sin(at)}{t} \mathrm{d}t \quad ; \quad \int_0^{\infty} \frac{\cos(at) - \cos(bt)}{t} \mathrm{d}t \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{\cos(bt)}{t^2 + a^2} \mathrm{d}t$$

the antiderivatives can not be computed. Hence, the above procedures cannot be used for these examples.

Our approach

In this work we will deal with advance techniques to compute this kind of improper integrals using a CAS.

Laplace and Fourier transforms or the Residue Theorem in Complex Analysis are some advance techniques which can be used for this matter.

We will introduce the file `ImproperIntegrals.mth`, developed in DERIVE 6, which deals with such computations.

Rule-Based Integrators

- Some CAS use different rules for computing integrations.
- For example RUBI system, a **ru**le-**b**ased **i**ntegrator developed by Albert Rich (see `http://www.apmaths.uwo.ca/~arich/`), is a very powerful system for computing integrals using rules.
- We will be able to develop new rules schemes for some improper integrals using `ImproperIntegrals.mth`.
- These new rules can extend the types of improper integrals that a CAS can compute.

Theoretical frames

To achieve this goal, we first establish the theoretical frames. Specifically, we will use:

- Laplace Transforms
- Fourier Transforms
- The Residue Theorem

Laplace Transform $\mathcal{L}$

We define the **Laplace Transform** of a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ as

$$\mathcal{L}\Big[f(t)\Big] = F_{\mathcal{L}}(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t$$

if such integral exists.

Improper integrals computation using Laplace Transform

According with the previous definition, evaluating in $s = 0$:

$$F_{\mathcal{L}}(0) = \int_0^\infty f(t)\,\mathrm{d}t$$

But normally, Laplace Transform exists for $s > 0$. Therefore, in order to compute an improper integral we can use the following result:

$$\int_0^\infty f(t)\,\mathrm{d}t = \lim_{s\to 0^+} \mathcal{L}\big[f(t)\big]$$

if both, the Laplace Transform and the limit exist.

Fourier Transform $\mathcal{F}$

We define the **Fourier Transform** of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ as

$$\mathcal{F}\Big[f(t)\Big] = F_{\mathcal{F}}(s) = \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-i\,st}\,\mathrm{d}t \qquad\qquad s \in \mathbb{R}$$

if such integral exists.

Improper integrals computation using Fourier Transform

According with the previous definition, evaluating in $s = 0$:

$$\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t = \mathcal{F}\big[f(t)\big]\Big|_{s=0}$$

if the Fourier Transform exists.

The Residue Theorem

Let $\mathcal{C}$ be a closed piecewise smooth positive oriented curve.

Let f be an analytic function with a finite number of isolated singularities z_1, z_2, ..., z_n inside $\mathcal{C}$.

Let $\operatorname*{Res}_{z=z_k} f(z)$ be the residue of f in z_k.

Then:

$$\begin{aligned}\oint_{\mathcal{C}} f(z)\,\mathrm{d}z &= 2\pi\, i \left(\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z)\right) \\ &= 2\pi\, i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)\end{aligned}$$

Improper integrals computation using the Residue Theorem

Let f be an analytic function with a finite set of isolated singularities $S_{\mathcal{P}}$ inside $\mathcal{P} \equiv \mathcal{I}\mathrm{m}(z) \geq 0$ none of them being on the real axis.

Let $\mathrm{CPV}(I)$ the Cauchy Principal Value of integral I

Then:

1. If $\lim_{z\to\infty} z\,f(z) = 0 \implies$

$$\mathrm{CPV}\left(\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x\right) = \oint_{\mathcal{P}} f(z)\,\mathrm{d}z = 2\pi\, i \sum_{z_k\in S_{\mathcal{P}}} \operatorname*{Res}_{z=z_k} f(z).$$

Improper integrals computation using the Residue Theorem

2. If $\lim_{z\to\infty} f(z) = 0 \implies$

$$\mathrm{CPV}\left(\int_{-\infty}^{\infty} f(x)\cos(ax)\,\mathrm{d}x\right) = \mathcal{R}\mathrm{e}\left(\oint_{\mathcal{P}} f(z)\,\mathrm{e}^{i\,az}\,\mathrm{d}z\right) = \mathcal{R}\mathrm{e}\left(2\pi\, i \sum_{z_k\in S_{\mathcal{P}}} \operatorname*{Res}_{z=z_k} f(z)\,\mathrm{e}^{i\,az}\right)$$

$$\mathrm{CPV}\left(\int_{-\infty}^{\infty} f(x)\sin(ax)\,\mathrm{d}x\right) = \mathcal{I}\mathrm{m}\left(\oint_{\mathcal{P}} f(z)\,\mathrm{e}^{i\,az}\,\mathrm{d}z\right) = \mathcal{I}\mathrm{m}\left(2\pi\, i \sum_{z_k\in S_{\mathcal{P}}} \operatorname*{Res}_{z=z_k} f(z)\,\mathrm{e}^{i\,az}\right)$$

Improper integrals computation using Laplace Transform

- $$\mathcal{L}\left[\frac{\sin(at)}{t}\right] = \frac{\pi}{2} - \operatorname{atan}\left(\frac{s}{a}\right) \qquad \Longrightarrow$$
$$\int_0^\infty \frac{\sin(at)}{t}\,\mathrm{d}t = \lim_{s\to 0^+}\left(\frac{\pi}{2} - \operatorname{atan}\left(\frac{s}{a}\right)\right) = \frac{\pi}{2}$$

- $$\mathcal{L}\left[\frac{\cos(at)-\cos(bt)}{t}\right] = \ln\left(\sqrt{\frac{s^2+b^2}{s^2+a^2}}\right) \qquad \Longrightarrow$$
$$\int_0^\infty \frac{\cos(at)-\cos(bt)}{t}\mathrm{d}t = \lim_{s\to 0^+} \ln\left(\sqrt{\frac{s^2+b^2}{s^2+a^2}}\right) = \ln\left(\left|\frac{b}{a}\right|\right)$$

Improper integrals computation using Fourier Transform

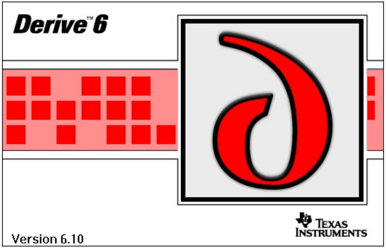
- $\mathcal{F}\left[\mathrm{e}^{-|a|t^2}\right] = \sqrt{\frac{\pi}{|a|}}\,\mathrm{e}^{-\frac{s^2}{4|a|}} \qquad \Longrightarrow$

$$\int_{-\infty}^{\infty} \mathrm{e}^{-|a|t^2}\,\mathrm{d}t = \left.\sqrt{\frac{\pi}{|a|}}\,\mathrm{e}^{-\frac{s^2}{4|a|}}\right|_{s=0} = \sqrt{\frac{\pi}{|a|}}$$

- $\mathcal{F}\left[\frac{\cos(bt)}{a^2+t^2}\right] = \frac{\pi}{2|a|}\left(\mathrm{e}^{-|a|\cdot|s-b|} + \mathrm{e}^{-|a|\cdot|s+b|}\right) \qquad \Longrightarrow$

$$\int_{-\infty}^{\infty} \frac{\cos(bt)}{a^2+t^2}\mathrm{d}t = \left.\frac{\pi}{2|a|}\left(\mathrm{e}^{-|a|\cdot|s-b|} + \mathrm{e}^{-|a|\cdot|s+b|}\right)\right|_{s=0} = \frac{\pi\cdot\mathrm{e}^{-|ab|}}{|a|}$$

Improper integrals computation using the Residue Theorem



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