Conjugate points along lightlike geodesics of Lorentzian manifolds

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Lecce, June 12th, 2013
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Outline

Section 1
Riemann and Lorentz manifolds

Let M be a differentiable n-manifold and g a nondegenerate symmetric $(0,2)$-tensor of constant index ν on M. (semi-Riemannian manifold)

- If $\nu = 0$, then (M, g) is called a Riemann manifold.

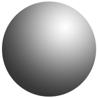
  At every point $p \in M$, T_pM is endowed with an inner product as the Euclidean n-dimensional space $(\mathbb{E}^n)$ has.

- If $\nu = 1$, then (M, g) is called a Lorentzian manifold.

  ... T_pM is endowed with a scalar product as the Lorentz-Minkowski n-dimensional space $(\mathbb{L}^n)$ has.

$$\mathbb{L}^n = (\mathbb{R}^n, \langle \ , \ \rangle), \quad \langle x, y \rangle = -x_1 y_1 + x_2 y_2 + \ldots + x_n y_n.$$

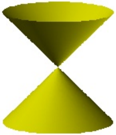
Spheres in T_pM (Riemann)

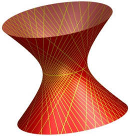


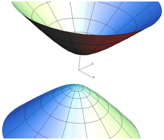
$$g(v,v) = r^2 > 0$$

"Spheres" in T_pM (Lorentz)

lightlike... spacelike... and timelike tangent vectors







$g(v,v) = 0,\ v \neq 0$ $\quad$ $g(v,v) = r^2 > 0$ or $v = 0$ $\quad$ $g(v,v) = -r^2 < 0$

The "miracle" of semi-Riemannian Geometry

...there is a unique affine connection ∇ with no torsion and compatible with the metric tensor g. ∇ is called the Levi-Civita connection.

Tullio Levi-Civita (1873-1941)

A curve γ is said to be a geodesic whenever $\nabla_{\gamma'}\gamma' = 0$.
Riemann and Ricci curvature tensors

- $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$
- $\mathrm{Ric}(v,w) = \mathrm{trace}\{R(\ ,v)w\}$

Scalar curvature

- $S(p) = \sum_{i=1}^{n} \epsilon_i \mathrm{Ric}(e_i, e_i), \quad p \in M$

Sectional curvature

If $\Pi = \mathrm{Span}\{x, y\} \subset T_pM$ is a two dimensional linear space such that $g\mid_\Pi$ is nondegenerate,

$$K(\Pi) = \frac{g(R(x, y)y, x)}{g(x, x)g(y, y) - g(x, y)^2}$$

Lightlike sectional curvature (only for Lorentzian manifolds)

Fix a timelike vector field $\mathcal{T}$, that is $g(\mathcal{T}, \mathcal{T}) < 0$. Let us consider $\Pi \subset T_pM$ a two dimensional linear space such that $g\mid_\Pi$ is degenerate[1] ,

$$\mathcal{K}_{\mathcal{T}}(\Pi) = \frac{g(R(x, v)v, x)}{g(x, x)},$$

where $\Pi = \mathrm{Span}\{v, x\}$ with $g(v, v) = 0$ and $g(v, \mathcal{T}_p) = 1$.

[1] S. G. Harris, A triangle comparison theorem for Lorentz manifolds, *Indiana Univ. Math. J.*, **31**(1982), 289–308.

Riemannian manifolds

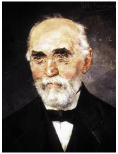
1. Every M admits a Riemannian metric.
2. For M connected, geodesically complete $\Leftrightarrow$ complete as metric space. (Hopf-Rinow)
   $d(p,q) = \mathrm{Inf}\Big\{ L(\alpha) = \int_a^d g(\alpha', \alpha')^{1/2} dt : \ \alpha \in \Omega(p,q) \Big\}.$
   M compact $\Rightarrow$ complete and $\mathrm{Iso}(M)$ compact.
3. M connected and geodesically complete $\Rightarrow$ geodesically connected.

Bernhard Riemann (1826-1866)

There is no Hopf-Rinow type theorem in Lorentzian geometry!!

Lorentzian manifolds

1. M admits a Lorentzian metric if and only if M is not compact or $\chi(M) = 0$.
2. There are compact Lorentzian manifolds which are not (geodesically) complete and $\mathrm{Iso}(M)$ may be non compact.
3. ...and complete does not imply geodesically connected.
4. If M is homogeneous and compact, then M is complete.



Hendrik Antoon Lorentz (1853-1928)

Two remarkable results in Lorentzian geometry...

$\mathcal{T} \in \mathfrak{X}(M)$ is said to be conformal when $\mathcal{L}_{\mathcal{T}} g = 2\sigma\, g$ and Killing if $\sigma = 0$.

- Every compact Lorentzian manifold (M, g) which admits a timelike conformal vector field $\mathcal{T}$ is geodesically complete.[2]
- Every compact Lorentzian manifold (M, g) with constant sectional curvature $K = c$ is geodesically complete. [3]

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[2] A. Romero and M. Sánchez, Completeness of compact Lorentz manifolds admiting a timelike conformal-Killing vector field, *Proc. Amer. Math. Soc.*, **123** (1995), 2831–2833.

[3] B. Klinger, Completude des varietes lorentziennes á courbure constante, *Math. Ann.*, **306**(1996), 353–370.

...and two amazing results.

- There is no compact Lorentzian manifold (M, g) with constant sectional curvature $K = c > 0$.
  - For $n = 2$ is a direct consequence of the Lorentzian "Gauss-Bonnet formula".
  - For $n \geq 3$ we have $\pi_1(M) = \Gamma$ is finite[4] $\Rightarrow M \approx \mathbb{S}^n_1/\Gamma$.
- Let (M, g) an $n(\geq 3)$-dimensional Lorentz manifold. Assume the sectional curvature K is bounded from below or from above. Then K is a constant.[5]

[4] E. Calabi and L. Markus, Relativistic space forms, *Ann. of Math.*, **75**(1962), 63–76.
[5] R. Kulkarni, The values of sectional curvature in indefinite metrics, *Comment. Math. Helv.*, **54**(1979), 173–176.

Lorentzian Geometry is the mathematical theory of General Relativity.

" A gravitational field may be effectively modelled by some Lorentzian metric g defined on a suitable Lorentzian manifold "

$$\boxed{\mathrm{Ric} - \frac{1}{2} S\, g + \Lambda\, g = 8\pi\, T}$$

The viewpoint of Global Differential Geometry began around 1970.

- **Singularity Theory**.
- **Causality Theory**.

Nowadays, the study of geometrical problems arisen in Lorentzian Geometry have become a proper branch of Differential Geometry.

Section 2
Conjugate points along lightlike geodesics

Let (M, g) be a semi-Riemannian manifold with Levi-Civita connection ∇ and curvature tensor R. Fix γ a geodesic $(\nabla_{\gamma'}\gamma' = 0)$.

- $J \in \mathfrak{X}(\gamma)$ is said to be a Jacobi vector field when

$$\frac{\nabla^2 J}{dt^2} + R(J, \gamma')\gamma' = 0.$$

- $\gamma(a)$ and $\gamma(b)$, $(a \neq b)$, are conjugate points along γ if there is a Jacobi vector field $J \neq 0$ such that

$$J(a) = 0, \quad J(b) = 0.$$

- When $\gamma(a)$ and $\gamma(b)$ are conjugate points, there is a variation $x : [a, b] \times (-\delta, \delta) \to M$ of γ such that every longitudinal curve is a geodesic and the transversal curves $x_a(t) = x(a, t)$ and $x_b(t) = x(b, t)$ satisfy

$$x_a'(0) = x_b'(0) = 0.$$

Conjugate points in Riemannian geometry

Let (M, g) be a connected Riemannian manifold.

1. If $\gamma(0) = p$ and $\gamma(a)$ are conjugate points and γ is arc length parametrized, then

$$d(p, \gamma(a+\epsilon)) < a + \epsilon = L(\gamma\mid_{[0,a+\epsilon]}).$$

2. $A = \{s > 0 : d(p, \gamma(s)) = s\} \subset \mathbb{R} \Rightarrow A = (0, r]$ or $A = (0, +\infty)$.

   $\gamma(r)$ is called a cut point of p along γ.

   - "The first cut point arrives before than the first conjugate point"
   - (Klingenberg, 1959) Assume q is a cut point of p and $d(q, p) = d(q, C(p))$. If q is not conjugate along a minimizing geodesic connecting p to q, then q is the midpoint of a geodesic loop, starting and ending at p.

Conjugate points in Lorentzian geometry...

Let (M, g) be a connected Lorentzian manifold.
Conjugate points are classified into spacelike, timelike and lightlike.

A causality Theorem

Let γ be a lightilke geodesic starting at $\gamma(0) = p$. Assume there is a conjugate point along γ strictly before to $\gamma(b) = q$. Then there is a timelike curve from p to q.

1. If $\gamma(0)$ and $\gamma(a)$ are conjugate points along a lightlike geodesic γ, then there is variation x of γ with longitudinal curves lightlike geodesics too.
2. Every Lorentz surface has no conjugate points on its lightlike geodesics.
3. A Lorentz manifold of constant sectional curvature has no conjugate point on lightlike geodesics. The converse is not true.

...and conformal changes of the metric.

For a Lorentzian metric g consider $g^f = e^{2f} g$...

$$\nabla^f_X Y = \nabla_X Y + Xf\, Y + Yf\, X - g(X,Y)\nabla f, \quad X, Y \in \mathfrak{X}(M).$$

...and let γ be a lightlike geodesic...

$$\nabla^f_{\gamma'}\gamma' = 2\gamma'(f)\gamma' \Rightarrow \gamma \text{ is a } g^f - \text{pregeodesic.}$$

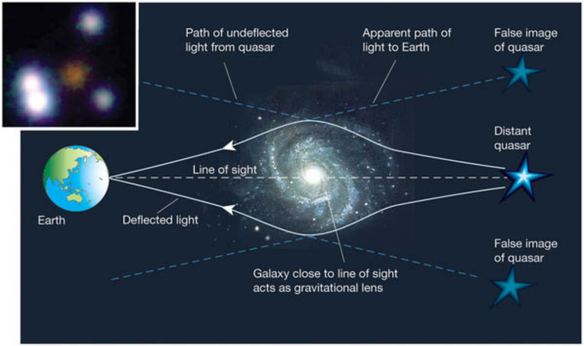
Assume $\gamma \circ \tau$ is a g^f-geodesic.

$$p = \gamma(0) = \gamma \circ \tau(s_0), \quad q = \gamma(a) = \gamma \circ \tau(s_1)$$

p and q are conjugate along γ if and only if are conjugate along $\gamma \circ \tau$.

The lightlike conjugate locus is a conformal invariant

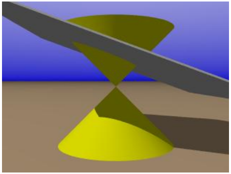
Physical interpretation... gravitational lensing



Section 3
Integral Inequality

Lightlike congruence associated to a timelike vector field $\mathcal{T}$

$$C_{\mathcal{T}}M = \{v \in TM : g(v,v) = 0 \text{ and } g(v,\mathcal{T}) = 1\}$$



$$(C_{\mathcal{T}}M)_p = T_pM \cap C_{\mathcal{T}}M$$

- $C_{\mathcal{T}}M$ can be seen as the bundle of lightlike directions of M.

- $\pi : C_{\mathcal{T}}M \to M$ is a fiber bundle with fiber $(C_{\mathcal{T}}M)_p \sim \mathbb{S}^{n-2}$.

A key result...

$C_{\mathcal{T}}M$ can be endowed with a Lorentzian metric $\widehat{g}$ in a such way that $\pi : C_{\mathcal{T}}M \to M$ is a Lorentzian submersion with spacelike fibers.

1. $(C_{\mathcal{T}}M)_p$ inherits a Riemannian metric and
2. $\pi_* : \left[(C_{\mathcal{T}}M)_p\right]^{\perp} \to T_pM$ is an isometry for every $p \in M$.

$$\boxed{\mathcal{T} \text{ conformal}, \quad \mathfrak{L}_{\mathcal{T}}g = 2\sigma\, g}$$

$$\Downarrow$$

- $C_{\mathcal{T}}M$ is invariant by the geodesic flow $\mathcal{Z}_g(v) = \frac{d\gamma_v'}{dt}\,|_0$, $v \in C_{\mathcal{T}}M$.
- $\mathrm{div}_{\widehat{g}}\mathcal{Z}_g = 0$.

Fiber bundle of two dimensional degenerate linear tangent spaces

$$\mathcal{D}^+(M) = \Big\{ \Pi : \Pi \text{ is an oriented two dimensional degenerate linear space in } T_pM, \ p \in M \Big\}.$$

We have two natural fiber bundes,

$$\mathfrak{p} : \mathcal{D}^+(M) \to C_{\mathcal{T}}(M), \quad \mathfrak{p}(\Pi) = \Pi \cap C_{\mathcal{T}}M \quad (\text{fiber} \quad \mathbb{S}^{n-3}).$$

$$\pi \circ \mathfrak{p} : \mathcal{D}^+(M) \to M \quad (\text{fiber} \quad U\mathbb{S}^{n-2}).$$

Assume $n \geq 4$,

$$\begin{array}{ccc} \mathcal{D}^+(M) & \xrightarrow{\mathcal{K}_{\mathcal{T}}} & \mathbb{R} \\ \mathfrak{p}\downarrow & w \nearrow & \uparrow f \\ C_K M & \xrightarrow{\pi} & M \end{array}$$

$\mathcal{K}_{\mathcal{T}} = 0$ if and only if (M, g) has constant sectional curvature.[6]

There exists w if and only if the Weyl tensor W vanishes.[7]

There exists f if and only if $W = 0$ and $\mathcal{K}^{\perp}$ is integrable, its integral submanifolds are totally umbilical and with constant sectional curvature.[8]

[6] S. G. Harris, A triangle comparison theorem for Lorentz manifolds, *Indiana Univ. Math. J.*, **31**(1982), 289–308.

[7] E. García-Río, D. Kupeli, Null and infinitesimal isotropy in semi-Riemannian geometry, *J. Geom. Phys.*, **13**(1994), 207–222. .

[8] H. Karcher, Infinitesimale Charakterisierung von Friedmann-Universen, *Arch. Math.*, **38**(1982), 58–64.

How can we define the length of a lightlike geodesic?

...a moment in Riemann geometry...

For a Riemannian manifold (M, g) a geodesic γ is arc length parametrized when

$$g(\gamma', \gamma') = 1 \Leftrightarrow \gamma'(0) \in UM. \quad \left(L(\gamma_{|[a,b]}) = b - a\right)$$

...come back to Lorentzian geometry...

$\mathcal{T}$ is timelike and conformal $\Leftrightarrow g(\nabla_X \mathcal{T}, Y) + g(X, \nabla_Y \mathcal{T}) = 2\sigma\, g(X, Y)$

$$\Downarrow$$

$$\gamma'(0) \in C_{\mathcal{T}} M \Leftrightarrow \gamma'(t) \in C_{\mathcal{T}} M \ \text{ for all } \ t \ !!$$

Fix a timelike conformal vector field $\mathcal{T}$. A lightlike geodesic γ is said to be $\mathcal{T}$-parametrized whenever $\gamma'(t) \in C_{\mathcal{T}} M$

Theorem. Let (M, g) be an $(n \geq 3)$-dimensional compact Lorentzian manifold and $\mathcal{T}$ a timelike conformal vector field.

- Assume there is $a \in (0, +\infty)$ such that $\gamma : [0, a] \to M$, with $\gamma'(0) \in C_K M$, has no conjugate point to $\gamma(0)$ in $[0, a)$.

Then,

$$\mathrm{Vol}(C_{\mathcal{T}} M) \geq \frac{a^2}{\pi^2(n-2)} \int_{C_{\mathcal{T}} M} \widehat{\mathrm{Ric}}\, d\mu_{\widehat{g}}.$$

The equality holds if and only if $(\mathcal{U} = h\mathcal{T},\ h = (-g(\mathcal{T}, \mathcal{T}))^{1/2})$.

$$\mathcal{K}_{\mathcal{U}} = \frac{-\pi^2}{a^2 g(\mathcal{T}, \mathcal{T})} \quad \Rightarrow \text{ there exists } f \text{ in the above diagram !!.}$$

$$\boxed{\int_M h^{n-2}\, d\mu_g \geq \frac{a^2}{\pi^2(n-1)(n-2)} \int_M \Big[n\,\mathrm{Ric}(\mathcal{U}, \mathcal{U}) + S \Big] h^n\, d\mu_g.}$$

Under the stronger assumption $\mathcal{T}$ is Killing...

Theorem.

$$\int_M h^{n-2}\, d\mu_g \geq \frac{a^2}{\pi^2(n-1)(n-2)} \int_M Sh^n\, d\mu_g.$$

The equality holds if and only if $g(\mathcal{T},\mathcal{T})$ is constant and the universal cover of (M,g) is (globally) isometric to

$$\left(\mathbb{R}\times\mathbb{S}^{n-1}\left(\frac{ah}{\pi}\right), -g_0+g_R\right).$$

Section 4
Lorentzian odd dimensional spheres

Even dimensional spheres has no Lorentz metric!! ($\chi(\mathbb{S}^{2n}) = 2$!!)

How can we endowed an odd dimensional sphere $\mathbb{S}^{2n+1}$ with a Lorentz metric?

First method...

$\mathbb{R}^{2n+2} \approx \mathbb{C}^{n+1} \Leftrightarrow (x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}) \approx (z_1 = x_1 + \mathbf{i}y_1, \ldots, x_{n+1} + \mathbf{i}y_{n+1})$

$$\mathbb{S}^{2n+1} = \left\{ (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : \sum_{j=1}^{n+1} z_j \cdot \overline{z_j} = 1 \right\}$$

Consider the vector field $\xi \in \mathfrak{X}(\mathbb{S}^{2n+1})$ given by $\xi(p) = \mathbf{i}\, p$.

- $g_R(\xi, \xi) = 1$.
- ξ is Killing for g_R $\boxed{\Phi_t(p) = e^{\mathbf{i}t}\, p}$

$$g_\xi(X, Y) = g_R(X, Y) - 2g_R(X, \xi)g_R(Y, \xi), \quad X, Y \in \mathfrak{X}(\mathbb{S}^{2n+1}).$$

...second method

Let us consider the Hopf bundle

$$\tau : (\mathbb{S}^{2n+1}, g_R) \to (\mathbb{C}P^n, g_{FS}), \;\; \tau(z_1, ..., z_{n+1}) = [z_1, ..., z_{n+1}].$$

The vertical distribution is given by

$$\mathcal{V}(p) = \{v \in T_p\mathbb{S}^{2n+1} : \tau_*(v) = 0\} = \mathrm{Span}(\xi(p)).$$

$\mathcal{V} = \xi^\perp$ defines a connection on the principal $\mathbb{S}^1$-bundle $\tau : \mathbb{S}^{2n+1} \to \mathbb{C}P^n$. The corresponding 1-form satisfies

$$\omega : T\mathbb{S}^{2n+1} \longrightarrow \mathfrak{s}^1 = \mathbf{i}\mathbb{R}, \quad X \in T_p\mathbb{S}^{2n+1} \longmapsto \omega(X) = \mathbf{i}g_R(X, \xi_p).$$

$$g_\omega(X, Y) = g_{FS}(\tau_*(X), \tau_*(Y)) + \omega(X) \cdot \omega(Y).$$

Two ways provide the same Lorentzian metric on $\mathbb{S}^{2n+1}$

Properties of $g = g_\xi = g_\omega$

1. $\tau : (\mathbb{S}^{2n+1}, g) \to (\mathbb{C}P^n, g_{FS})$ is a semi-Riemannian submersion with timelike fibers.
2. ξ is Killing and timelike with $g(\xi, \xi) = -1 \;\Rightarrow\; \widetilde{\nabla}_\xi \xi = 0$.
3. The Levi-Civita connection of g is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y - 2g_R(X,\xi)\nabla_Y \xi - 2g_R(Y,\xi)\nabla_X \xi,$$

   where ∇ is the Levi-Civita connection of g_R.
4. For every $n \geq m$ the natural inclusion $\mathbb{S}^{2n+1} \to \mathbb{S}^{2m+1}$ is totally geodesic.

....by a kind permission: algebraical properties...

Consider the special unitary group,

$$SU(n+1) = \left\{ A \in \mathcal{M}_{n+1}(\mathbb{C}) : A\overline{A}^T = I, \ \det(A) = 1 \right\}$$

$SU(n+1) \times \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}$ acts transitively by isometries of g_R and ξ is invariant $(A(\xi(p)) = \xi(A(p)))$

For g we have...

1. $SU(n+1) \times \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}$ acts transitively by isometries of the Lorentzian metric g
2. The isotropy group at $(0, ..., 0, 1) \in \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ is

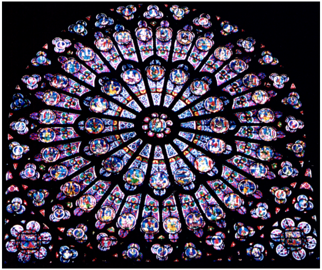
$$SU(n) = \left\{ A \in SU(n+1) : A(e_{n+1}) = e_{n+1} \right\}$$

3. $\mathbb{S}^{2n+1} = SU(n+1)/SU(n)$ as a Lorentzian manifold!!!
4. ... what is the isometry group of $(\mathbb{S}^{2n+1}, g)$?

...more properties of homogeneity...

- For every $p \in \mathbb{S}^{2n+1}$ and $u, v \in (C_\xi \mathbb{S}^{2n+1})_p$, there exists $A \in SU(n+1)$ such that
  1. $A(p) = p$.
  2. $A(\xi_p) = \xi_p$.
  3. $A(u) = v$.

$\mathbb{S}^{2n+1}$ is said to be spatially isotropic with respect to ξ.



Notre Dame (Paris)

Riemann versus Lorentz odd dimensional spheres

Every $(2n+1)$-dimensional sphere $\mathbb{S}^{2n+1}$ is a Riemannian symmetric space endowed with g_R. The global symmetry at every point $p \in \mathbb{S}^{2n+1}$ is given by

$$s_p(x) = -x + 2g_R(p, x)p$$

s_p is not an isometry for g!!

Still a bit more

- … Lorentzian odd dimensional spheres are not symmetric spaces.

Lightlike geodesics of $\mathbb{S}^{2n+1}$

The homogeneity properties reduces our computations to a single point.

Fix $p_0 = (1, 0, \ldots, 0) \in \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$,

$$(C_\xi \mathbb{S}^{2n+1})_{p_0} = \Big\{ v = (-\mathbf{i}, z_2, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : \sum_{j=2}^{n+1} z_j \overline{z_j} = 1 \Big\}.$$

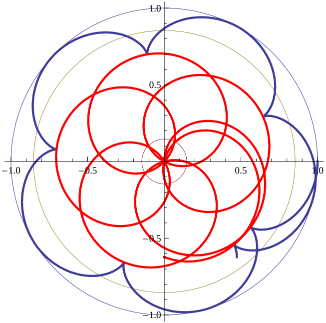
Let us write $\gamma_v = (\Theta_1^v, \ldots, \Theta_{n+1}^v)$ with $\Theta_k^v : \mathbb{R} \to \mathbb{C}$, $(1 \leq k \leq n+1)$. We get,

$$\Theta_1^v(t) = \frac{2-\sqrt{2}}{4} e^{(-2-\sqrt{2})\mathbf{i}t} + \frac{2+\sqrt{2}}{4} e^{(-2+\sqrt{2})\mathbf{i}t},$$

$$\Theta_k^v(t) = \frac{\sqrt{2}\mathbf{i}z_j}{4} \Big[e^{(-2-\sqrt{2})\mathbf{i}t} - e^{(-2+\sqrt{2})\mathbf{i}t} \Big] \quad (2 \leq k \leq n+1).$$

$$\Theta_1^v(t) = \frac{2-\sqrt{2}}{4} e^{(-2-\sqrt{2})\mathbf{i}t} + \frac{2+\sqrt{2}}{4} e^{(-2+\sqrt{2})\mathbf{i}t},$$

$$\Theta_k^v(t) = \frac{\sqrt{2}\mathbf{i}z_j}{4}\left[e^{(-2-\sqrt{2})\mathbf{i}t} - e^{(-2+\sqrt{2})\mathbf{i}t}\right] \quad (2 \leq k \leq n+1).$$



Two properties of lightlike geodesics

1. Every γ_v is injective!!
   Assume $v \in (C_\xi \mathbb{S}^{2n+1})_{p_0}$ and $t \neq 0$ with $\gamma_v(t) = p_0$.

$$\Theta_1^v(t) = 1 \Leftrightarrow t = 0.$$

2. Let us consider $u, v \in (C_\xi \mathbb{S}^{2n+1})_{p_0}$ with $u \neq v$.

$$u = (-\mathbf{i}, w_2, \ldots, w_{n+1}), \;\; v = (-\mathbf{i}, z_2, \ldots, z_{n+1}) \in (C_\xi \mathbb{S}^{2n+1})_{p_0},$$

with $w_j \neq z_j$.

$$\gamma_u(t) = \gamma_v(t) \Leftrightarrow e^{(-2-\sqrt{2})\mathbf{i}t} = e^{(-2+\sqrt{2})\mathbf{i}t} \Leftrightarrow t = \frac{m\,\pi}{\sqrt{2}}, \;\; m \in \mathbb{Z}.$$

All the lightlike geodesics starting at p_0 will meet at $t = \pi/\sqrt{2}$.

A Morse-Schönberg type result for lightlike sectional curvature

Let $\gamma : [0, a] \to M$ be a lightlike geodesic such that $\mathcal{K}_{\mathcal{T}}(\Pi) \leq \delta$, for all $\Pi \in \mathcal{D}^{+}(M)$ with $\gamma'(t) \in \Pi$.

- Assume $\gamma(0)$ and $\gamma(a)$ are conjugate points along γ.

Then,

$$a \geq \frac{\pi}{\sqrt{\delta}}.$$

Theorem. Let γ_v be a lightlike geodesic of $\mathbb{S}^{2n+1}$ with $v \in (C_\xi \mathbb{S}^{2n+1})_p$.

1. The first conjugate point to $\gamma_v(0) = p$ is $\gamma_v(\frac{\pi}{2\sqrt{2}})$.
2. The lightlike conjugate locus of every point $p \in \mathbb{S}^{2n+1}$ is a topological $(2n-1)$-dimensional sphere.

Sketch of the proof.

We have,

$$\mathrm{Ric}(\xi,\xi) = 2n \text{ and } S = 2n(2n+3)$$

$$\Downarrow \big(\text{Integral inequality and homogeneity properties}\big)$$

- There exists $a \in (0, +\infty)$ such that every lightlike geodesic γ_v reaches its first conjugate point at $\gamma_v(a)$ and

$$a^2 \leq \frac{(2n-1)\pi^2}{4(n+1)}.$$

- Let us consider $\Pi \in \mathcal{D}^+(\mathbb{S}^{2n+1})$ and $v \in \Pi \bigcap (C_\xi \mathbb{S}^{2n+1})_p$.
  If $\Pi = \mathrm{Span}\{v, x\}$ and $v = -\xi_p + y$ then

  $$\mathcal{K}_\xi(\Pi) = 2\mathcal{K}_{FS}(\tau_*(x), \tau_*(y)) \Rightarrow 2 \leq \mathcal{K}_v(\Pi) \leq 8.$$

  Therefore, from the Morse-Schönberg type result,

  $$\frac{\pi^2}{8} \leq a^2 \leq \frac{(2n-1)\pi^2}{4(n+1)}.$$

- For $\mathbb{S}^3$, we have $a = \frac{\pi}{2\sqrt{2}}$.
- Taking into account that $\mathbb{S}^3$ is totally geodesic in all $\mathbb{S}^{2n+1}$, we get the first affirmation.
- Finally, take $u, v \in (C_\xi \mathbb{S}^{2n+1})_p$ with $u \neq v$.

  $$\gamma_u(t) = \gamma_v(t) \Leftrightarrow t = \frac{k\pi}{\sqrt{2}}, \; k \in \mathbb{Z}$$

  The second part is obtained from $a < \frac{\pi}{\sqrt{2}}$.

Thank you very much for your kind attention