RESEARCH BULLETIN 1007 DECEMBER, 1974

UNIVERSITY OF MISSOURI-COLUMBIA
COLLEGE OF AGRICULTURE
AGRICULTURAL EXPERIMENT STATION
ELMER R. KIEHL, DIRECTOR

Some Considerations for Interpretation of Soil Tests for Phosphorus and Potassium

(Publication authorized December, 1974)

COLUMBIA, MISSOURI

ACKNOWLEDGEMENTS

Among those persons responsible for establishing and maintaining the field experiments and for providing much of the field experimental information referred to in this discussion are Mr. Robert Light, formerly of the Department of Agronomy, University of Missouri-Columbia, and Dr. E. M. Kroth, Department of Agronomy, University of Missouri-Columbia. The field experiments were carried out under the former Missouri AES Project 7033-3610, Relationships Between Soil Test Values, Crop Yields, and Fertilizer Treatments.

CONTENTS

Some Considerations for Interpretation of Soil Tests for Phosphorus and Potassium

T. R. Fisher*

The primary function of chemical soil test measurements should be to determine the adequacy of the fertility of the soil for growing plants. Although an assessment of possible limitations on crop performance imposed by the fertility factors is the initial evaluation sought, soil test measurements are most often expressed to the user only in the amounts of nutrients present in the soil. Information he needs to evaluate the existing level of crop performance in relation to what may be possible if fertility limitations are removed is most often lacking. Sufficiency categories such as high, medium, and low have been devised in various forms in an attempt to relate soil test measurements to crop yields; however, more specific measurements would be desirable, especially if a mathematical treatment were sought.

The following discussion will consider a procedure for expressing and reporting soil test measurements of phosphorus and potassium in terms of expected crop yields. This is not a new concept. Bray used a percentage yield concept in Illinois in the 1940s (1). More recently Hatfield in North Carolina (3) has described the use of a "nutrient index" which relates soil test measurements to crop performance. Cope, in Alabama (2) has reported on a similar procedure utilizing a "fertility index" which expresses the percentage sufficiency of a nutrient based upon that amount necessary to provide 100 percent relative yield.

It is believed that considerable progress has been achieved by these investigators toward a more lucid expression of soil test measurements. One purpose of this discussion is to further extend these concepts and to describe a theoretical development of procedures which would provide farmers with understandable choices relative to soil fertility levels.

Additionally, this discussion will consider a procedure for computing the amounts of phosphorus and potassium fertilizer required to change soil test

---

*Formerly Associate Professor, Department of Agronomy, University of Missouri-Columbia, Columbia, Missouri, 65201.

values from some measured value to other values necessary for higher levels of yield. These quantities of phosphorus or potassium would be considered as *corrective treatment*. Once the new soil test value or relative yield level possibility is established as a result of the corrective treatment, *maintenance* quantities would be recommended independently of soil test measurements and be based upon the amounts the kind of crop uses and the yield levels expected or attained.

Finally, some considerations will be presented which indicate that full *corrective treatments* of phosphorus and potassium can be established profitably by corrective treatments.

SOIL TEST VALUE—CROP YIELD RELATIONSHIPS**

The relationship between soil test measurements or values and crop yields will be developed employing the basic premise that a soil test value exists at which maximum yield occurs. Additionally, at values less than this soil test value, the magnitude of the change in yield produced by a given change in soil test value is a function of the difference between that soil test value which provides maximum yield and an observed soil test value.

These relationships may be expressed as follows:

$$(1) \quad dy/dx = C(X - x)$$

y = an observed yield

x = an observed soil test value at which an observed yield occurs

X = the soil test value at which maximum yield occurs

C = a proportionality constant

Equation 1 may be integrated to provide the general equation of a curve describing the relationship between yield and soil test value.

$$(2) \quad y = -\frac{C}{2}(X - x)^2 + C_1$$

When an observed soil test value, x, equals that which provides maximum yield, the integration constant, C_1, may be evaluated as being equal to the maximum yield, Y. This value for C_1 may be substituted into equation 2.

$$(3) \quad y = Y - \frac{C}{2}(X - x)^2$$

**Appreciation is expressed to Dr. C. M. Woodruff for major contributions in developing these relationships. Additionally, appreciation is expressed to Dr. J. R. Brown for calling to the author's attention some of the early relevant research.

If the value for x is x_0 when the yield is zero, the following value for C is obtained.

$$C = \frac{2Y}{(X - x_o)^2}$$

This value for C may be substituted into equation 3.

$$(4) \quad y = Y - Y\left(\frac{X - x}{X --x_o}\right)^2$$

Equation 5 may be derived by rearranging equation 4.

$$(5) \quad y = \frac{Yx_o^2 - 2YXx_o}{(X - x_o)^2} + \frac{2YX}{(X - x_o)^2} x - \frac{Y}{(X - x_o)^2} x^2$$

This is an equation of a second degree polynomial of the type $y = a + bx - cx^2$. Since it has a theoretical basis described by the differential equation 1, the coefficients of the equation have physiological meaning.

If it is assumed that the yield is zero when the soil test value is zero, equation 5 becomes:

$$(6) \quad y = \frac{2Y}{X} x - \frac{Y}{X^2} x^2$$

Although soil test values of zero for phosphorus or potassium are rare and experimental observations on natural soils are essentially non-existant, crops in which the entire plant is harvested (forages) would be expected, intuitively, to have zero yield at zero soil test levels. It is conceivable that some grain crops might have zero yield at soil test levels above zero. Such a situation might be one in which sufficient nutrient is available to provide for some degree of vegetative growth but not for grain production. Equation 4 or 5 would be appropriate in this instance in which x_0 would have some positive value. It is suspected, however, that soil test levels for either phosphorus or potassium would closely approach zero in such instances.

Since equation 6 describes a parabola opening downward with the point $dy/dx = 0$ occuring at $Y = 100$ and $x = X$, soil test values greater than that providing maximum yield would cause decreasing values of yield. Experimental observations strongly suggest that yields do not decline at this point but remain at a maximum over a rather wide range of increasing soil test values. For this reason the use of the equation will be restricted to soil test values equal to or less than that providing maximum yield.

A decided advantage of the second degree polynomial equation for describing the soil test value-crop yield relationship is that it identifies a maximum yield rather than the maximum yield being approached asymptotically as with the logarithmic Mitscherlich type equation. Bray had no alternative but to identify 95, 98, or 99 percent yield levels, and never 100 percent levels. The implication that a readily identifiable maximum yield does not exist is hardly acceptable.

As early as 1912 and later through the 1920s, several papers appeared in the German literature which supported this viewpoint and, indeed, presented some evidence that the polynomial type equation fitted Mitscherlich's own data as well as the logarithmic type which he proposed. A summary and bibliography of this early work was published by R. Stewart in 1932 (4).

A fundamental difference between the development of equations 5 and 6 and the one proposed by Mitscherlich is the initial premise which provides the differential equation. The one proposed by Mitscherlich and employed by many investigators following him is based upon the concept that as the nutrient supply increases, the yield increases in proportion to the amount by which the observed yield is less than the maximum yield. This relationship may be expressed as follows:

$$dy/dx = C\ (Y - y)$$

y = observed yield

Y = maximum yield

x = amount of nutrient in the soil

C = a proportionality constant

Upon integration the familiar Mitscherlich equation may be derived.

The relationship discussed here could be described analogously; as the soil test value increases, the yield increases in proportion to the reduction of the difference between the observed test value and the soil test value which provides the maximum yield. This relationship is expressed by the differential equation 1.

Precise field experimental information which identifies the relationships between crop yields and soil test measurements is not abundant. Figures 1, 2, and 3 illustrate observations from field experiments on a Baxter silt loam soil near Purdy in Southwest Missouri with an alfalfa-orchard grass forage mixture from 1966 through 1970. The experiments consisted of two separate studies, one concerned with phosphorus, the other with potassium. In each instance, plots were established with initial plow-down treatments of 0, 100, 200, and 400 pounds per acre of either P_2O_5 or K_2O, establishing a range of soil test values from an untreated plot to one receiving the highest treatment. Superimposed on each of these treatments were annual top-dressed treatments of 0, 25, 50, and 100 pounds per acre of P_2O_5 or K_2O. Lime and nutrient elements other than phosphorus in the one instance and potassium in the other, were maintained at a nearly adequate level. Each year the forage yield was measured as a hay harvest. Soil samples were collected to a depth of six inches from each plot.

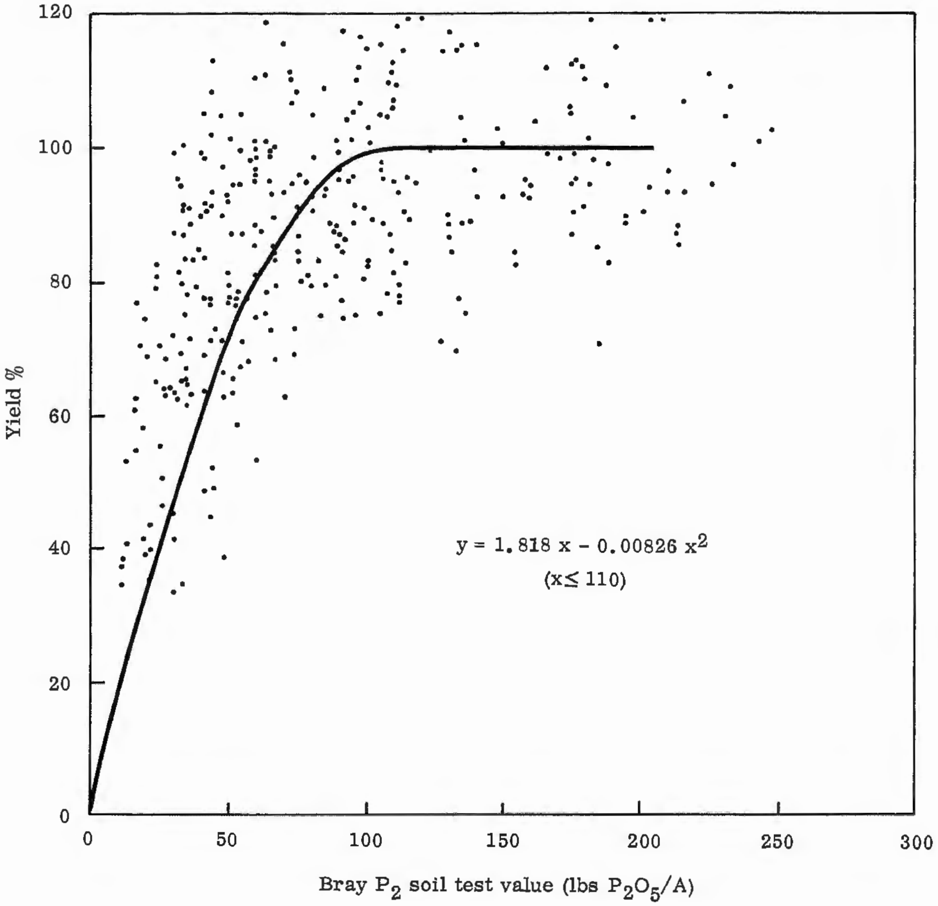

Figure 1. The relationship between the percentage yield of alfalfa-orchard grass hay and the phosphorus soil test value as measured by the Bray P_2 extracting reagent. (Purdy, Mo., 1966-1970)

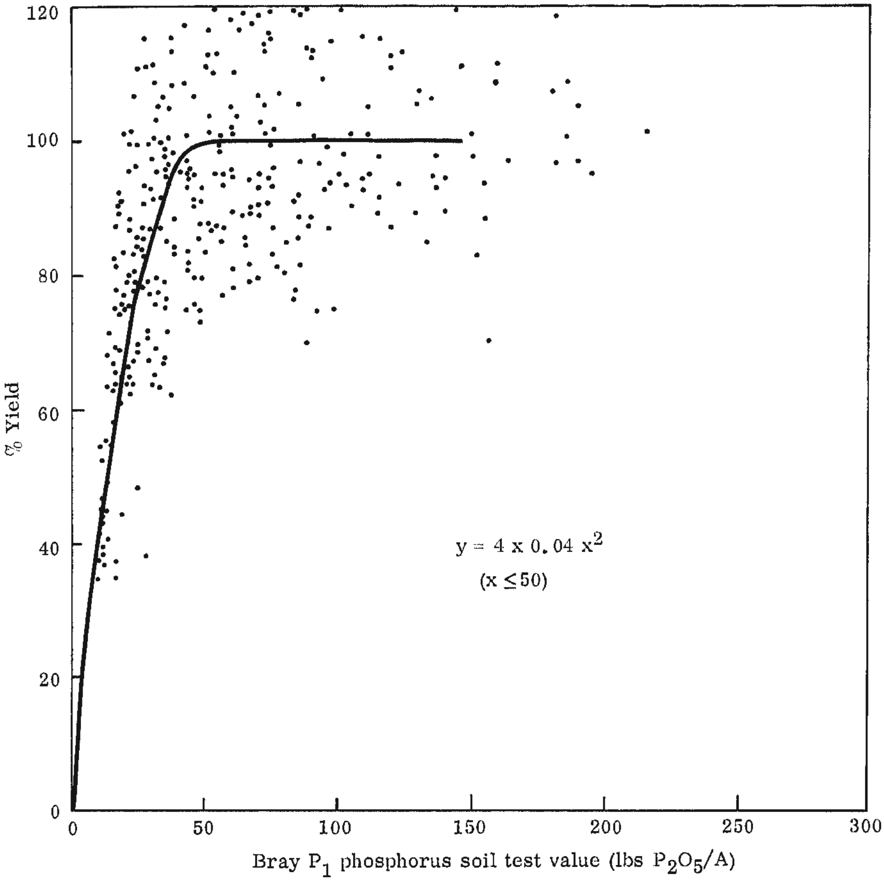

Figure 2. The relationship between the percentage yield of alfalfa-orchard grass hay and the phosphorus soil test value as measured by the Bray P1 extracting reagent. (Purdy, Mo., 1966-1970)

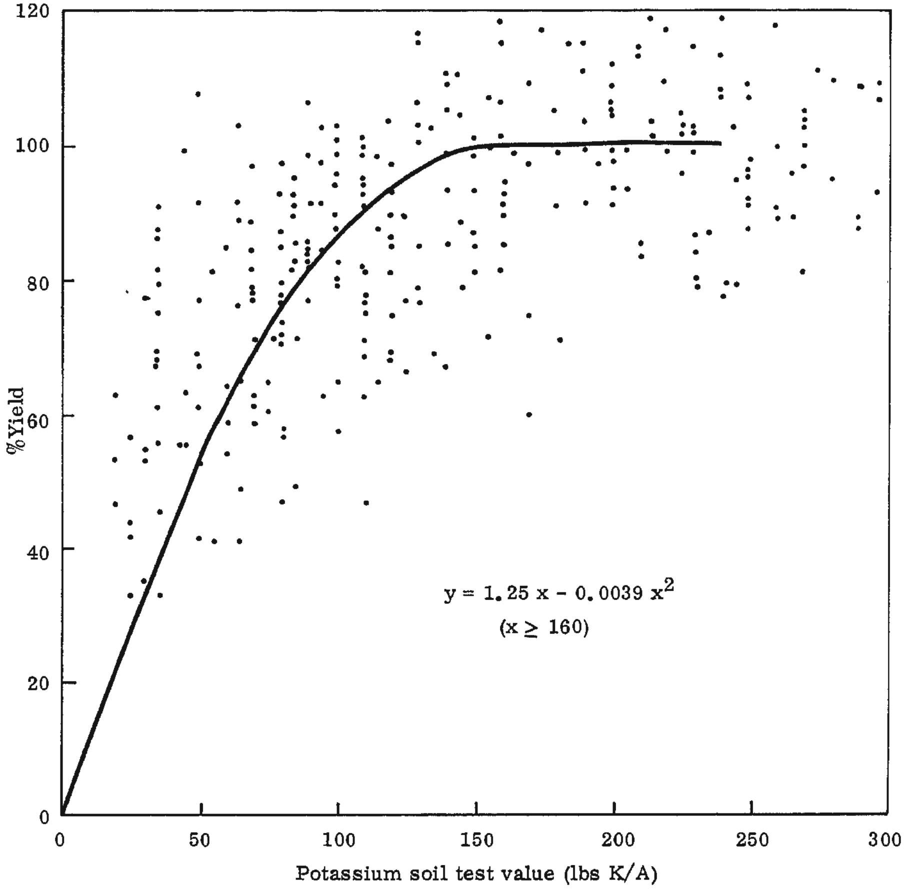

Figure 3. The relationship between the percentage yield of alfalfa-orchard grass hay and the exchangeable potassium soil test value. (Purdy, Mo., 1966-1970)

The soil samples were tested by the routine procedures of the Missouri soil testing program. The yield observations for each year were converted to percentage yield values by first establishing a 100 percent value. The 100 percent was established by computing the mean of all observations which occurred at the highest soil test values and appeared to occur beyond the $dy/dx = 0$ point on a curve defined by a second degree polynomial such as equation 6.

Each point on the graphs illustrated in Figures 1, 2, and 3 represents an annual plot observation of percentage yield and the corresponding soil test value. The curves which are fitted to these observations are not necessarily those of the best statistical fit since in each instance they are constrained to pass through the point $x = 0$ and $y = 0$. A casual inspection, however, strongly suggests (1) that the second degree polynomial equation can define the crop yield-soil test value relationship, (2) that crop yields attain a maximum value which extends over a rather wide range of soil test values for phosphorus and potassium, and (3) as soil test values for phosphorus and potassium approach zero, yields also approach zero.

FERTILITY INDEX—BRAY P_2 PHOSPHORUS SOIL TEST

To express soil test measurements in terms more closely related to expected crop yields, the term *fertility index* will be employed in this discussion. The fertility index will be that percentage crop yield defined by equation 6 that would correspond to a given soil test value when the maximum yield, Y, is 100 percent.

Application of the fertility index to the Bray P_2 phosphorus soil test employed in the Missouri soil testing program requires that a soil test value be established at which maximum yield occurs. No corrective phosphorus treatment is recommended for any crop at soil test values above 150 pounds P_2O_5 per acre under current Missouri fertilizer guidelines. This soil test value is considered to be one above which no increase in yield is observed.

The Purdy experiments indicated that maximum yields of alfalfa-orchard grass forage harvested for hay occurred near a soil test value of 110 pounds P_2O_5/A. This is somewhat less than that normally considered to be required. It may be the result of the annual top-dressing of phosphate fertilizer having caused an accumulation of phosphate in the surface two or three inches of the soil. This might result in a higher proportion of the applied material contributing to yield rather than satisfying fixation requirements of the soil. In any event, the top-dressing procedure is a common method of fertilizer application to forages and should be a part of soil test calibration studies.

Observations in Illinois seem to indicate that soil test values for phosphorus should be somewhat higher to attain maximum yields of wheat than for most other crops, although these observations are based upon the Bray P_1 test.

For purposes of illustration here it will be assumed that a Bray P_2 soil test value of 150 pounds P_2O_5 per acre will permit maximum yields of all crops on all soils of Missouri. It is recognized that this value may differ among some crops and soil regions within the state which future investigations may define more precisely.

When the values of 150 pounds P_2O_5 per acre and 100 percent yield are substituted into equation 6 for X and Y, respectively, it becomes:

$$y = 1.333x - 0.00444\ x^2 \qquad (7)$$

The curve which equation 7 describes is illustrated in Figure 4. The following selected relationships between fertility index (% yield) and soil test value would exist.

Fertility index (y)	lb. P_2O_5/A (x)
100	150
95	116
90	102
75	75
50	44

Since equation 6 is actually the equation of a parabola opening downward as pointed out earlier, soil test values greater than that providing a fertility index of 100 would cause decreasing values of the fertility index. In order to provide a fertility index for soil test values greater than X, the soil test value could be expressed as a percent of X, being always greater than 100. For example, when the soil test value is greater than 150 in this instance, the following selected values of the fertility index would apply.

Soil test value lb. P_2O_5/A	Fertility index
150	100
175	117
200	133
225	150
275	184
300	200

The fertility index for any value of x greater than X could be computed from the following equation.

$$\text{Fertility index} = 100\ x / X \qquad (x > X)$$

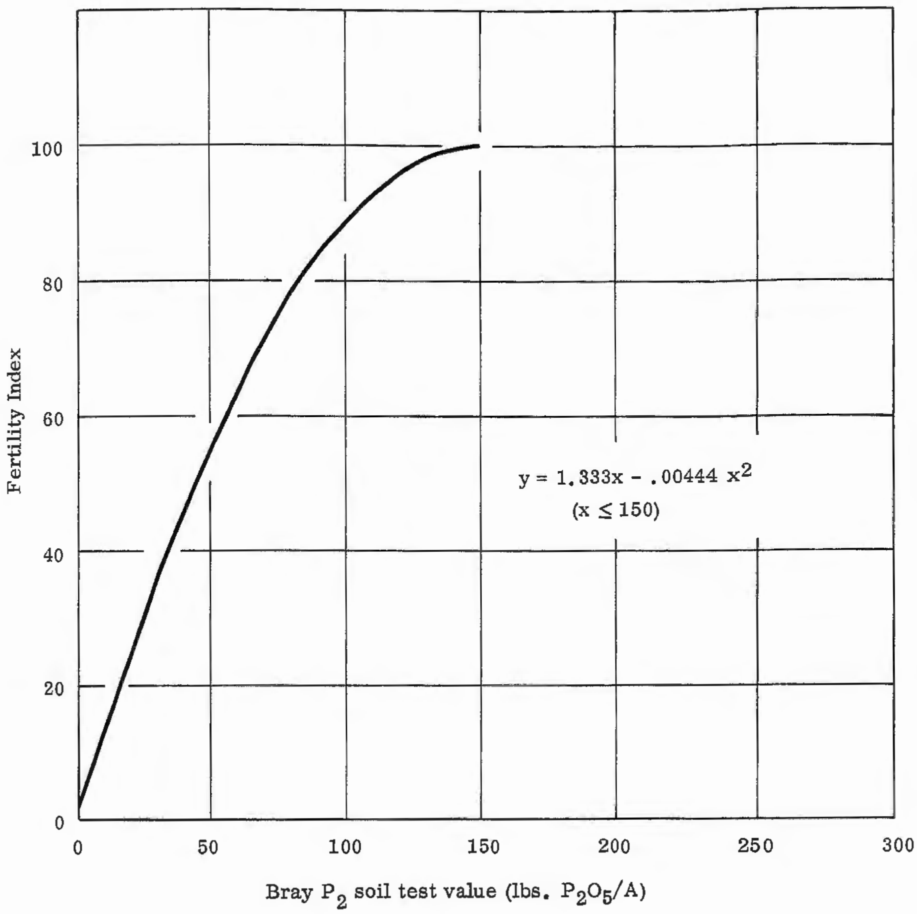

Figure 4. The relationship between the fertility index and the Bray P_2 soil test value when 100% yield occurs at a test value of 150 lbs P_2O_5/A.

The reporting and use of the fertility index could operate in the following manner. If a farmer's soil sample tested 60 pounds P_2O_5 per acre, the fertility index would be computed from equation 7 as follows:

$y = 1.333\ x - 0.00444x^2$

$y = 1.333\ (60) - 0.00444\ (60^2)$

$y = 64$

This fertility index of 64 would be reported to the farmer. He may decide that he desires to operate at a fertility index of 90. By interchanging the variables in equation 7 so as to avoid the awkward solution of x in a quadratic equation, the soil test value required to operate at the 90 percent yield level could be computed as follows:

(8) $x = X - 0.1\ X\ (Y - y)^{\frac{1}{2}}$

x = soil test value at a desired fertility index or yield level

X = soil test value at maximum yield

y = desired fertility index or yield level

Y = 100% yield

$x = 150 - 15(100 - 90)^{\frac{1}{2}}$

$x = 102.6$ lb. P_2O_5/A

The farmer could be informed that by establishing a soil test value of 102 pounds P_2O_5/A he could operate at the 90% yield level with respect to phosphate.

Other appropriate values of X could be utilized including those which would be applicable to the Bray P_1 test.

FERTILITY INDEX—EXCHANGEABLE POTASSIUM

A fertility index for exchangeable potassium may be developed in a manner similar to that for the Bray P_2 test for phosphorus. An added component, however, relates to the increasing soil test value providing maximum yield that is associated with increasing cation exchange capacity.

This consideration is currently in use in Missouri and has been for a number of years. A less detailed adjustment for variations in cation exchange capacity is currently in use in Illinois. It provides for a recommendation that soil test values be built up to 260 pounds K per acre in regions with low cation exchange capacity soils ($<$ 12 me/100 gm) and 300 pounds K per acre in regions with high cation exchange capacity soils ($>$ 12 me/100 gm).

Other states neighboring Missouri provide no adjustment for variations in cation exchange capacity. The soil test recommendations in use in Kansas consider a test value of 320 pounds K per acre to be adequate while those in both Nebraska and Iowa consider 300 pounds K per acre to be adequate and to require only maintenance quantities of potassium fertilizer.

As with the phosphorus, the soil test value for exchangeable potassium which appeared to provide maximum yield in the field studies was less than that normally considered to be required. Figure 3 indicates a soil test value near 160 pounds K per acre was adequate, which approximates a percentage potassium saturation of 2.3 percent for the 8 to 10 me/100 gm cation exchange capacity soil located there. The positional effect of the top-dressed potassium fertilizer in the soil layers may have influenced yield response in a manner similar to that suggested with regard to the phosphate. Future investigations may define these relationships more precisely.

For purposes of consideration here, the relationship between the soil test value for exchangeable potassium which provides maximum yield and the cation exchange capacity (CEC) will be that expressed by equation 9.

$$X = 220 + 5\,\mathrm{CEC} \tag{9}$$

X = lbs K/A

Some selected values of exchangeable potassium which would provide maximum yield as computed from equation 9 and the corresponding percentage potassium saturation values follow.

CEC	lbs K/A	% sat.
4	240	7.70
8	260	4.16
12	280	3.00
16	300	2.40
20	320	2.05

These values are similar to those currently in use in Missouri and at intermediate cation exchange capacity values (12 - 16 me/100 gms) are similar to those employed in neighboring states.

Since equation 9 identifies soil test values of exchangeable potassium which produce maximum yields, this value can replace X in equation 6 for computing the fertility index corresponding to given soil test values of exchangeable potassium.

$$y = \frac{2Y}{220 + 5\,\mathrm{CEC}}\,x - \frac{Y}{(220 + 5\,\mathrm{CEC})^2}\,x^2 \tag{10}$$

Several selected values follow for illustrative purposes.

Fertility index	Cation exchange capacity (me/100 gms) 4	10	16	22
	---------------- lbs K/A			----------------
100	240	270	300	330
95	186	210	233	256
90	164	185	205	226
75	120	135	150	165
50	70	79	88	97

Figure 5 further illustrates these relationships.

As with the fertility index for phosphorus, soil test values for potassium greater than that value providing a fertility index of 100 will be encountered. For soil test values greater than those provided by equation 9, the following equation would provide fertility index values always greater than 100.

$$\text{fertility index} = \frac{100\ x}{220 + 5\ \text{CEC}}$$

Several selected values are illustrated as follows.

Fertility index	Cation exchange capacity (me/100 gms) 4	10	16	22
	-------- lbs K/A test value --------			
100	240	270	300	330
125	300	338	375	413
150	360	406	450	495

For the purpose of computing soil test values which would exist at chosen fertility index values, the following modification of equation 8 may be used.

$$(11) \quad x = (220 + 5\ \text{CEC}) - (22 + 0.5\ \text{CEC})\ (100 - y)^{\frac{1}{2}}$$

x = soil test value at a given CEC and fertility index or percentage yield level

y = desired fertility index or percentage yield level

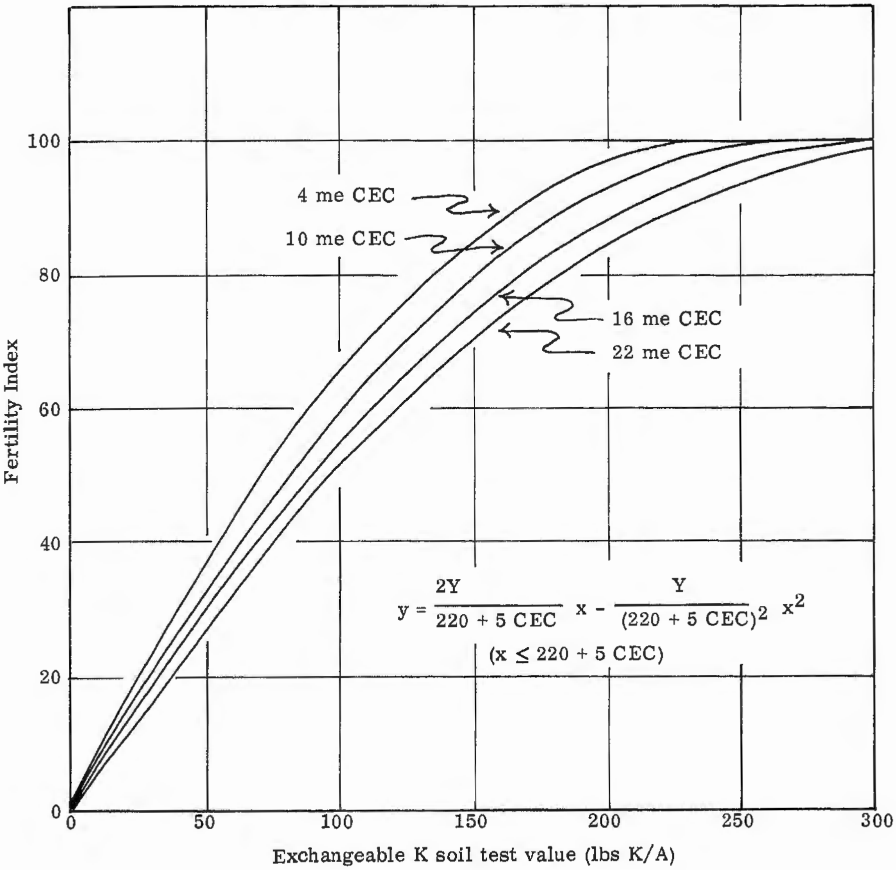

Figure 5. The relationship between the fertility index and exchangeable potassium soil test value as described by equation 10 at several cation exchange capacity values.

CORRECTIVE TREATMENTS—PHOSPHORUS

A corrective treatment will be considered to be that quantity of phosphorus or potassium fertilizer which is required to change an observed soil test value to some higher soil test value that corresponds to a selected higher percentage yield or fertility index. To determine corrective treatments, the relationship between changes in soil test values and amounts of fertilizer applied must be known.

Figure 6 illustrates the relationship observed between the net amounts of P_2O_5 applied as fertilizer or removed by cropping and Bray P_2 soil test values from the field studies near Purdy, Mo., described earlier. As one would

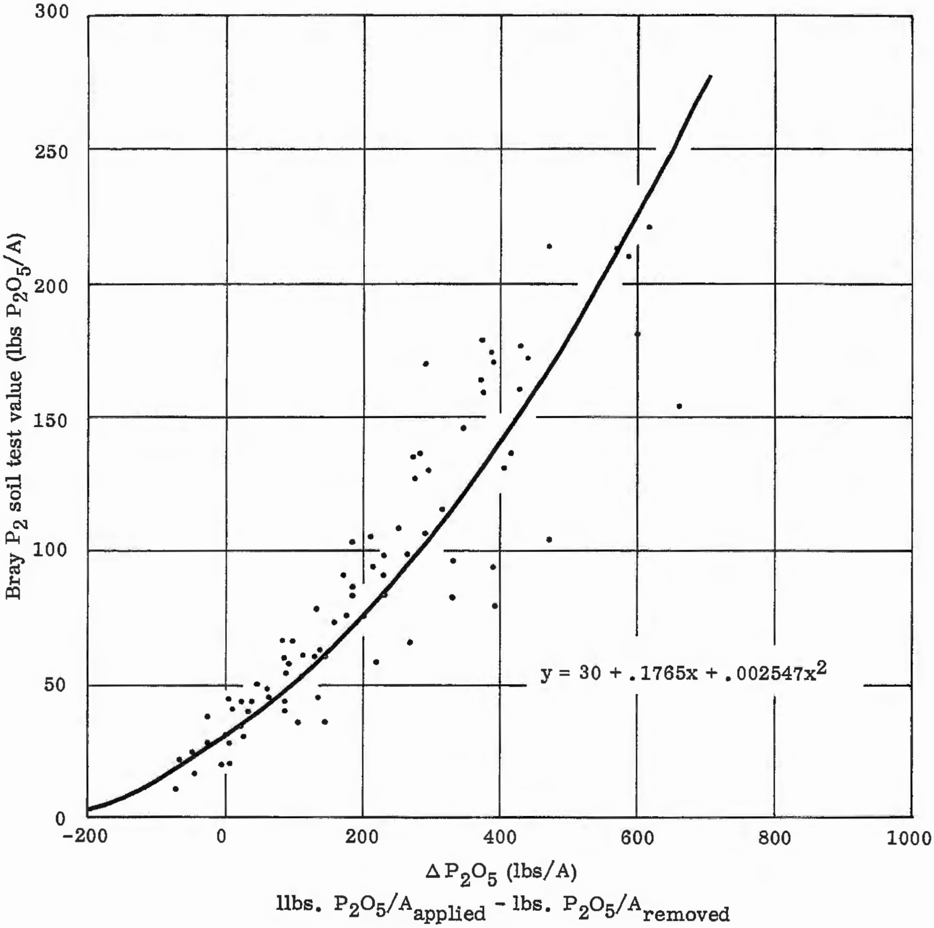

Figure 6. The relationship between the Bray P_2 soil test value and the net amounts of P_2O_5 applied as fertilizer or removed by cropping. (Purdy, Mo., 1966-1970)

expect, at low soil test values the changes in test values resulting from given additions or removals of phosphate were less than those at higher test values. A second degree polynomial was chosen to represent this relationship.

Figure 7 illustrates a relationship similar to that in Figure 6; however, the axes have been interchanged and the amounts of P_2O_5 added or removed have been shifted so that zero addition or removal coincides with zero soil test value. The slope of the curve at given soil test values is comparable to that in Figure 6. The equation of the curve in Figure 7 is as follows:

(12) $x_1 = 64\, x^{\frac{1}{2}}$

$x_1 = P_2O_5$ fertilizer equivalent of x

$x =$ soil test value, lbs P_2O_5/A

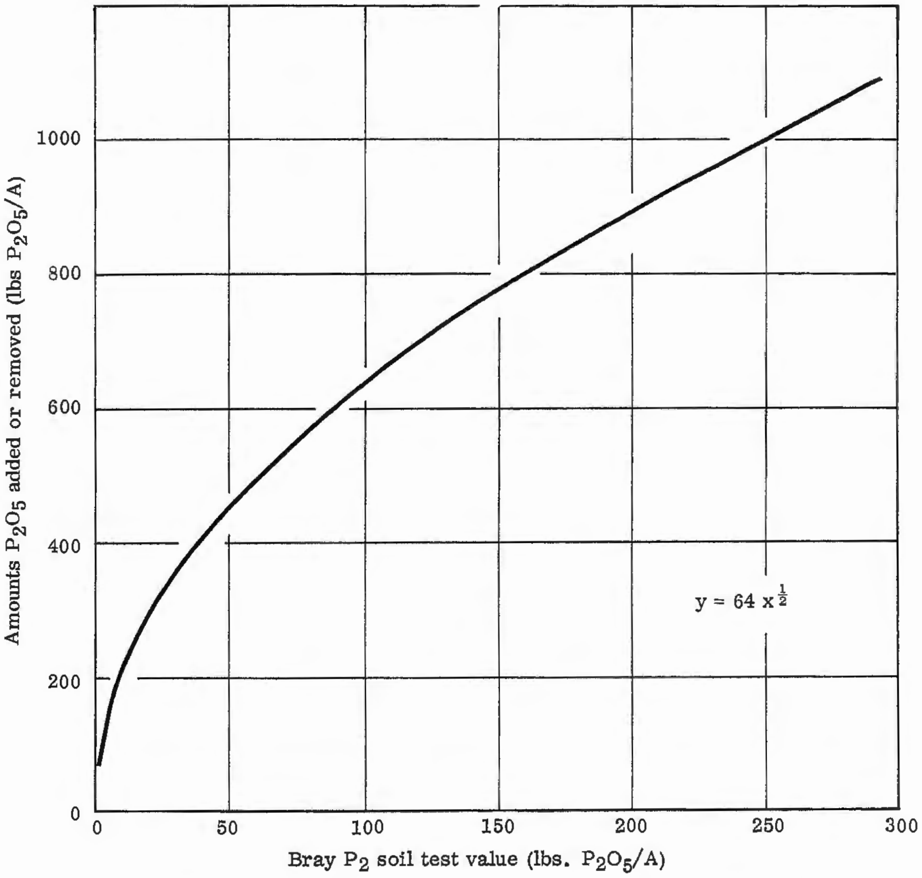

Figure 7. The relationship between the amounts of P_2O_5 applied as fertilizer or removed by cropping and the Bray P_2 soil test value. (zero addition or removal adjusted to correspond to zero soil test value.)

The derivative of equation 12 will provide the slope of the curve at any selected value of x.

$$dx_1/dx = 32/x^{\frac{1}{2}}$$

Following is a listing of the number of increments of P_2O_5 additions or removals required to change the soil test value one increment at selected increasing soil test levels.

Soil test value lbs P_2O_5/A	dx_1/dx
25	6.4
50	4.5
75	3.7
100	3.2
150	2.6
200	2.3
250	2.0
300	1.8

Equation 13 can be derived from Equation 12 and used to compute the quantity of P_2O_5 that would be required to change the soil test value from some observed value, x, to some desired value, x_d.

$$(13) \quad x_1 = 64\ (x_d^{\frac{1}{2}} - x^{\frac{1}{2}})$$

For example, if an observed soil test value should be 40 pounds P_2O_5 per acre and the user chose to operate at a percentage yield level or fertility index of 95, he would need to attain a soil test level of 116 pounds P_2O_5 per acre (equation 8). This value may be inserted into equation 13 for the value x_d.

$$x_1 = 64\ (116^{\frac{1}{2}} - 40^{\frac{1}{2}})$$

$$x_1 = 284 \text{ lbs } P_2O_5/A$$

The value 284 pounds P_2O_5 per acre would be reported as a corrective treatment. If a gradual build-up procedure were desired, one-fourth of this amount (71 pounds P_2O_5 per acre) could be applied in each of four years or an equivalent amount over some other period of time.

Equation 13 provides for the increasing efficiency of applied P_2O_5 in changing the phosphorus soil test value as it increases, at least until a 1:1 ratio is attained at slightly above a test value of 1,000 pounds P_2O_5 per acre.

The value of the constant in equation 13 was derived empirically from the field studies described earlier. This value may be different for other soils; however, it seems to fit previous observations and perhaps would not vary greatly among Missouri soils having an appreciable cation exchange capacity. Additional confirmation of this would seem warranted.

Equation 13 has an added convenience in that it enables computation of the amount of P_2O_5 that can be removed by cropping if the soil test value should be excessively high. For example, if the soil test value should be 200 pounds P_2O_5/A and a value of 150 lbs P_2O_5/A produces maximum yield, the equation provides a negative value.

$$x_1 = 64\ (150^{\frac{1}{2}} - 200^{\frac{1}{2}}) = -121 \text{ lbs } P_2O_5/A$$

Over a four year period maintenance amounts of P_2O_5 could be reduced by 30 pounds P_2O_5 per acre per year at which time the soil test value would have declined to around 150 P_2O_5 per acre.

For further illustration, the amounts of P_2O_5 required to change soil test values from several observed values to values corresponding to several desired yield levels will be considered. Equation 8 predicts the soil test value that must be attained to provide for a chosen yield level or fertility index. The right hand portion of equation 8 may replace the value x_d in equation 13. When a soil test value of 150 pounds P_2O_5 per acre provides for maximum yield, equation 14 may de derived.

$$(14) \quad x_1 = 64\{[150 - 15\ (100 - y)^{\frac{1}{2}}]^{\frac{1}{2}} - x^{\frac{1}{2}}\}$$

x_1 = lbs P_2O_5/A to be added or removed

y = percentage yield or fertility index desired ($\leq$100)

x = observed soil test value

Following is a list of quantities of P_2O_5 to be added as fertilizer or removed by cropping in order to attain soil test values corresponding to 90, 95, and 100 percent yield levels, as predicted by equation 14 at several observed soil test values.

Soil test value lbs P_2O_5/A	90% yield level	95% yield level	100% yield level
		---------- lbs P_2O_5/A ----------	
10	444	487	582
25	326	369	464
50	194	237	332
102	0	43	138
116	- 43	0	95
150	-138	- 95	0
200	-258	-216	-121
300	-462	-419	-324

The negative numbers indicate the quantity of P_2O_5/A that can be removed by cropping to allow the soil test value to decline to that corresponding to the selected percentage yield level.

Equation 14 may be combined with an expression providing maintenance quantities of P_2O_5 based upon expected or attained yield levels. For example, alfalfa and alfalfa-grass hay or silage contain approximately 14 pounds P_2O_5 per ton. If a four-year build-up and maintenance program is desired, equation 14 can be modified to provide values as follows:

$$(15) \quad x_1 = 64/4 \left\{ \left[150 - 15 (100 - y)^{\frac{1}{2}} \right]^{\frac{1}{2}} - x^{\frac{1}{2}} \right\} + 14 \text{ (yield T/A)}$$

Equation 15 would vary for other crops in the final term in which corresponding yield units and pounds of P_2O_5 per unit would need to be inserted.

CORRECTIVE TREATMENTS—POTASSIUM

The procedure for computing corrective treatments for potassium may be developed in a manner similar to that for phosphorus. Figure 8 illustrates the relationship observed from the field studies described earlier that existed between the net amounts of potassium applied as fertilizer or removed by cropping and soil test values for exchangeable potassium.

Figure 9 illustrates the same relationship as that in Figure 8; however, the axes have been interchanged and the amounts of potassium added or removed have been converted to K_2O and shifted so that zero addition or removal coincides with zero soil test value.

The equation of the curve in Figure 9 is as follows.

$$(16) \quad x_1 = 75.5\, x^{\frac{1}{2}}$$

x_1 = lbs K_2O/A added or removed

x = lbs K/A soil test value

Following is a listing of increments of K_2O additions or removals required to change the exchangeable K soil test value one increment at selected soil test levels as computed from the derivative of equation 16.

Soil test value lbs K/A	dx_1/dx
25	7.55
50	5.34
100	3.78
150	3.08
200	2.67
300	2.18

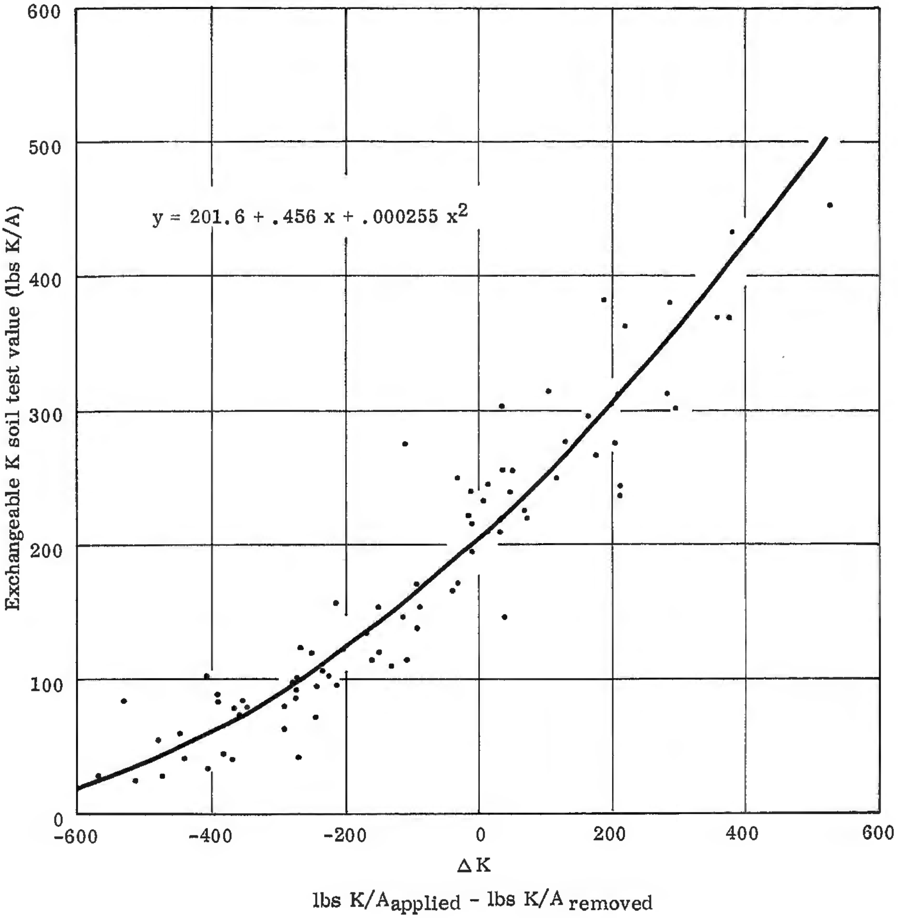

Figure 8. The relationship between the exchangeable K soil test value and the net amount of K applied as fertilizer or removed by cropping. (Purdy, Mo., 1966-1970)

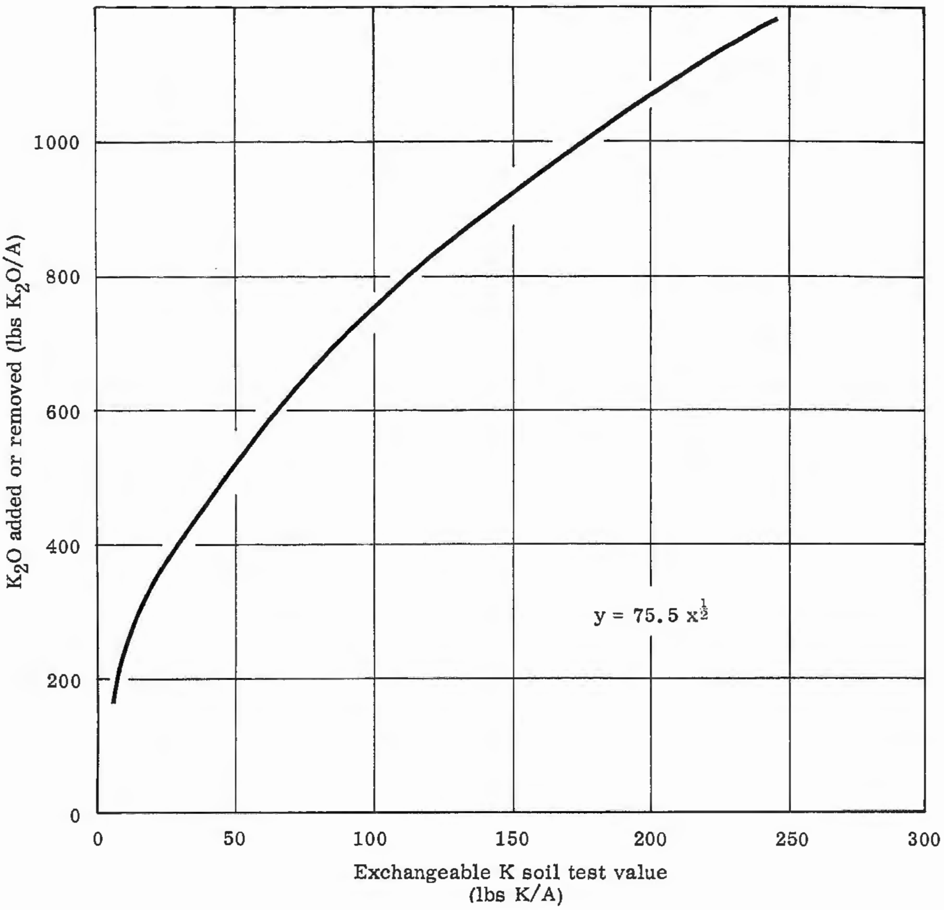

Figure 9. The relationship between the amounts of K_2O applied as fertilizer or removed by cropping and the exchangeable K soil test value. (zero addition or removal adjusted to correspond to zero soil test value.)

The quantity of K_2O required to change the potassium soil test value from some observed value, x, to some desired value, x_d, can be computed with a modification of equation 13 as follows.

$$(17) \quad x_1 = 75.5\,(x_d^{\frac{1}{2}} - x^{\frac{1}{2}})$$

The value of the constant in equation 17 was determined empirically and may vary among kinds of soils; however, for purposes here it is considered to be adequately representative for all Missouri soils.

As in the discussion of phosphorus, an expression can be derived which would predict the quantity of K_2O required to be added to or removed from the soil to change the soil test value from that corresponding to an observed percentage yield or fertility index to another value chosen on the basis of a desired yield level.

Equation 11 predicts the soil test value at a given cation exchange capacity and selected percentage yield. The value of this equation may be substituted into equation 17 for the quantity x_d as follows.

$$(18) \quad x_1 = 75.5\,\{[(220 + 5\,CEC) - (22 + 0.5\,CEC)(100 - y)^{\frac{1}{2}}]^{\frac{1}{2}} - x^{\frac{1}{2}}\}$$

Equation 18 predicts the quantity of K_2O required to change an observed potassium soil test value, x, to another value corresponding to a selected percentage yield, y. When used for computing the quantity of K_2O required to change the soil test value from some low value to another higher selected value, x_1 may be considered as a corrective treatment. Some selected values follow for illustrative purposes.

Soil test value lbs K/A	90% yield level 10 me/100gm CEC	 16 me/100gm CEC	100% yield level 10 me/100gm CEC	 16 me/100gm CEC
	lbs K_2O/A			
25	649	704	863	930
50	493	547	709	774
100	272	326	486	552
150	102	156	316	383
200	- 41	14	173	242
300	-281	-226	- 67	0

The negative values indicate the quantity of K_2O which can be removed by cropping before the soil test value drops below that associated with the corresponding percentage yield level.

If the values predicted by equation 18 are to be considered in a four-year build-up and maintenance program, the equation may be modified by inserting a ¼ term and adding a term which incorporates the yield and the

K_2O content per yield unit of the harvested crop. For example, if alfalfa or alfalfa-grass mixtures harvested for hay or silage contain 60 pounds K_2O per ton, the equation is:

$$x_1 = \tfrac{1}{4}\ (75.5)\ \left\{\left[\ (220 + 5\ \mathrm{CEC}) - (22 + 0.5\ \mathrm{CEC})\ (100 - y)^{\frac{1}{2}}\right]^{\frac{1}{2}} - x^{\frac{1}{2}}\right\} + 60\ (\text{yield T/A})$$

A similar relationship for other crops may be developed. The final recommendation to the user would consist of two parts, that quantity which may be considered as a corrective treatment and that estimated as needed to sustain the level established by the corrective treatment.

ADDITIONAL CONSIDERATIONS RELATING TO THE ECONOMICS OF FERTILIZER USE

As discussed earlier equation 6 relates soil test values to percentage yield.

$$(6) \qquad y = \frac{2Y}{X}\,x - \frac{Y}{X^2}\,x^2$$

Also, equations 12 and 16 relate the amounts of P_2O_5 and K_2O fertilizer, respectively, required to establish given soil test values. When considering P_2O_5 fertilizer, equation 12 is as follows.

$$(12) \qquad x_1 = 64\ x^{\frac{1}{2}}$$

x_1 = lbs/A P_2O_5 fertilizer

x = lbs/A P_2O_5 soil test value

A relationship between percentage yield and the amount of P_2O_5 fertilizer required to establish given soil test values can be derived by solving equation 12 for x and substituting into equation 6 the equivalent values of P_2O_5 fertilizer corresponding to x.

$$x^{\frac{1}{2}} = \frac{x_1}{64}$$

$$x = \frac{x_1^{\,2}}{64^2}$$

$$x^2 = \frac{x_1^{\,4}}{64^4}$$

The above values for x and x^2 can be substituted into equation 6.

$$(19)\quad y = \frac{2Y}{64^2X} x_1^2 - \frac{Y}{64^4X^2} x_1^4$$

When Y = 100 and X = 150 equation 19 becomes:

$$y = 3.26 \times 10^{-4} x_1^2 - 2.65 \times 10^{-10} x_1^4$$

This relationship is illustrated in Figure 10. At low soil test values and yield levels, increments of applied P_2O_5 fertilizer contribute little to yield, a large proportion being fixed in the soil. As additional increments are added an increasing proportion contributes to yield increases until an almost linear relationship exists. Finally, as maximum yield levels are approached additional fertilizer increments produce declining yield increments.

When yield and soil test value are expressed as their monetary equivalents in equation 6, its derivative provides a particular usefulness. At any given monetary equivalent of the soil test value which is equal to or less than that providing maximum yield, the incremental change in monetary value of the crop yield produced by an incremental change in the soil test value monetary equivalent can be computed.

The percentage yield predicted by equation 6 can be converted into a monetary value by establishing a maximum yield and converting yield from percent into yield units. Multiply this value by the market price per yield unit. The monetary equivalent of yield, y_2, at a given soil test value, x, would be predicted by the following modification of equation 6.

$$(20)\quad y_2 = 0.01\, a\, Y_1 \left(\frac{2Y}{X} x - \frac{Y}{X^2} x^2\right)$$

y_2 = yield equivalent in dollars

Y_1 = maximum yield in yield units

a = market price per unit of yield

The soil test value can be converted into a monetary equivalent by utilizing equation 12 or 16 to compute the quantity of P_2O_5 or K_2O fertilizer required to establish a given soil test value and then multiplying this value by the market cost of the fertilizer. This modification of equation 12 follows when P_2O_5 fertilizer is being considered.

$$(12)\quad x_1 = 64\, x^{\frac{1}{2}}$$

x_1 = lbs/A P_2O_5 fertilizer

x = lbs/A P_2O_5 soil test value

$$(21)\quad x_2 = 64\, b\, x^{\frac{1}{2}}$$

x_2 = monetary equivalent of P_2O_5 fertilizer

b = market cost/lb of P_2O_5 fertilizer

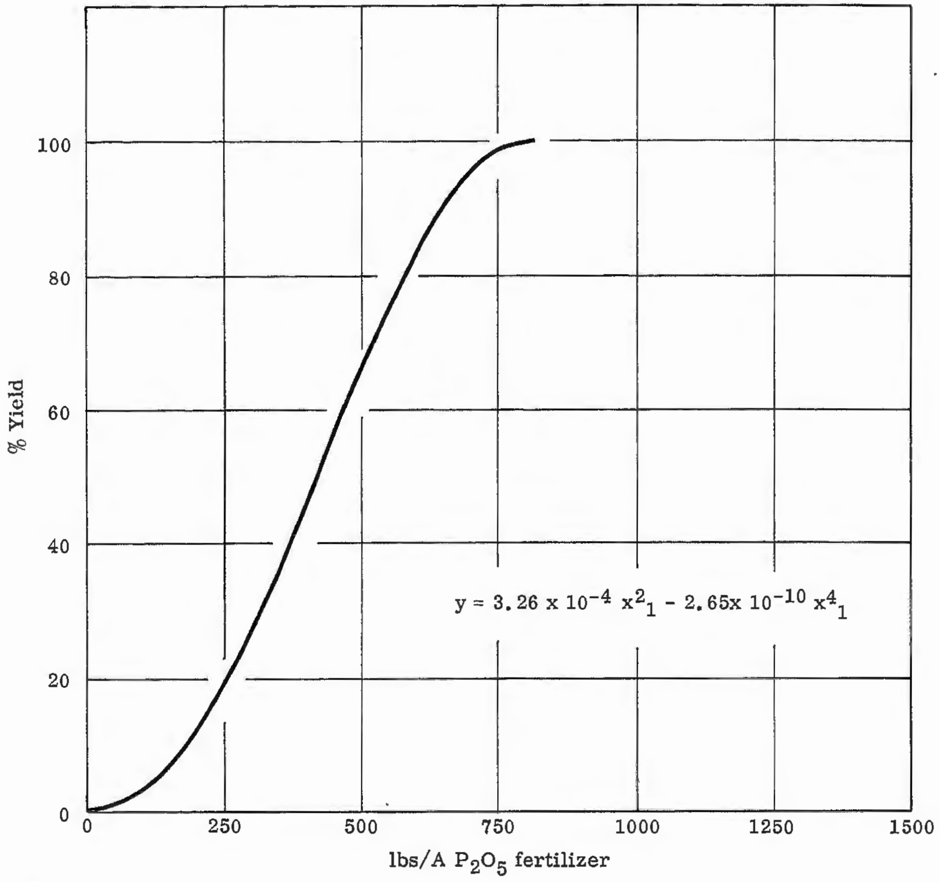

Figure 10. The relationship between percentage yield and the amount of P_2O_5 fertilizer required to establish corresponding soil test values as described by equation 19.

The relationships between yield returns and fertilizer costs can be determined by combining equations 20 and 21.

Equation 21 can be solved for x and x^2 as follows:

$$x^{\frac{1}{2}} = \frac{x_2}{64\ b}$$

$$x = \frac{{x_2}^2}{64^2b^2}$$

$$x^2 = \frac{{x_2}^4}{64^4b^4}$$

When the values above for x and x^2 are substituted into equation 20, it becomes:

$$(22) \quad y_2 = 0.01\ a\ Y_1\ (\frac{2Y}{64^2b^2X}\ {x_2}^2 - \frac{Y}{64^4b^4X^2}\ {x_2}^4)$$

Figure 11 illustrates the relationships between the cost of P_2O_5 fertilizer and the return from corn yield represented by equation 22 when the following parameters are considered.

$Y = 100\%$

$Y_1 = 150$ bu/A

$X = 150$ lbs P_2O_5/A

$a = \$1.40$, $\$2.10$, and $\$2.80$/bu

$b = \$0.20$/lb P_2O_5

The derivative of equation 22 offers a means of computing the slope of the curve at any value of x_2.

$$(23) \quad dy_2/dx_2 = 0.01\ a\ Y_1\ (\frac{4Y}{64^2b^2X}\ x_2 - \frac{4Y}{64^4b^4X^2}\ {x_2}^3)$$

Equation 23 becomes:

$$(24) \quad dy_2/dx_2 = a\ (2.44 \times 10^{-2}\ x_2 - 1 \times 10^{-6}\ {x_2}^3)$$

When

a = price/bu corn

Y_1 = maximum yield = 150 bu/A

$Y = 100\%$

$b = \$0.20$/lb P_2O_5 fertilizer

$X = 150$ lbs P_2O_5/A

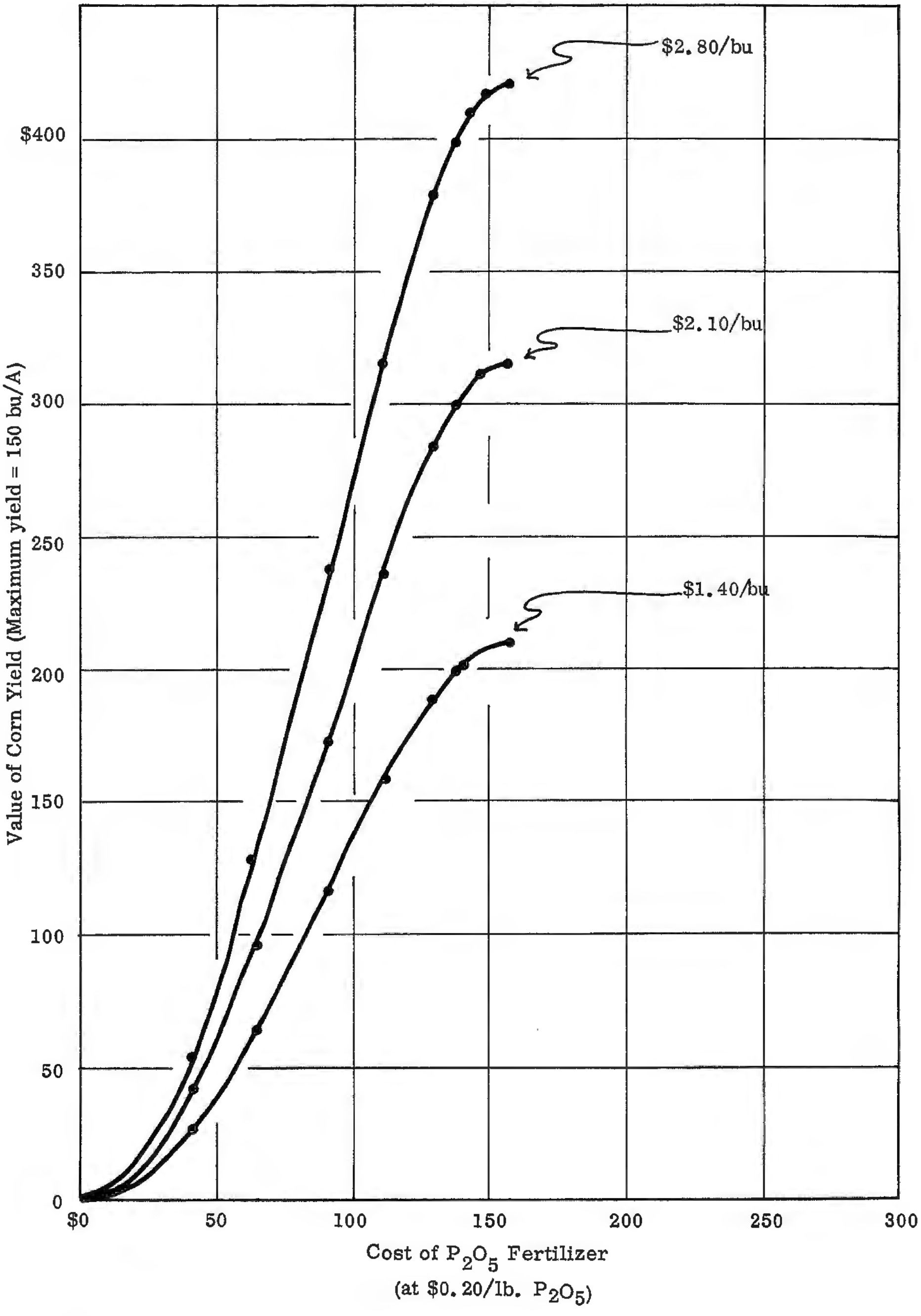

Figure 11. The relationship between the monetary value of corn yield at three price levels and the cost of P_2O_5 fertilizer as described by equation 21.

Following are values of dy_2/dx_2 as computed by equation 24 for selected values of market price of corn, a, and P_2O_5 fertilizer costs, x_2, and P_2O_5 soil test values, x. The percentage yield levels corresponding to x and x_2 are also listed for additional perspective.

x	x_2	y	dy_2/dx_2 (a=\$1.40)	dy_2/dx_2 (a=\$2.10)	dy_2/dx_2 (a=\$2.80)
10	\$ 40.47	12.9%	1.29	1.93	2.58
25	64.00	30.5	1.82	2.74	3.65
50	90.51	54.2	2.06	3.10	4.13
75	110.85	75.0	1.90	2.85	3.79
102.6	129.60	90.0	1.41	2.11	2.81
116	137.86	95.0	1.07	1.61	2.15
119	139.62	96.1	0.96	1.45	1.93
130	145.93	98.0	0.67	1.00	1.29
135	148.71	99.0	0.52	0.77	1.03
140	151.45	99.6	0.35	0.53	0.70
150	156.76	100.0	0.00	0.00	0.00

These values for dy_2/dx_2 may be most usefully interpreted as being the ratios of the return in yield per unit of cost for P_2O_5 at a particular soil test value, x, or fertilizer cost, x_2, or percentage yield level, y. With reference to Figure 11, all three examples illustrated indicate a return greater than cost until rather high values of soil test level and percentage yield are reached, with the exception of very low values. Considering only the upper portions of the curves, at the point at which dy_2/dx_2 equals one, the return equals the cost. Beyond this point the cost of the fertilizer exceeds the return. For the examples illustrated, the point at which dy_2/dx_2 equals one is at a fertilizer cost of about \$139.62, a soil test value of 119 pounds P_2O_5 per acre, and a 96 percent yield level when corn is worth \$1.40 per bushel; \$145.93, 130 pounds per acre and 98 percent yield when corn is worth \$2.10 per bushel, and \$148.71, 135 pounds per acre, and 99 percent yield when corn is worth \$2.80 per bushel.

Equation 23 can be further interpreted and utilized in the following manner. When dy_2/dx_2 equals one, the value of x_2 is that fertilizer cost at which an incremental return equals an incremental cost at any selected values of Y_1, x, a, and b. The utility of equation 23, however, would seem to be greater if this point could be identified in units of soil test value rather than as fertilizer cost.

Again consider equation 23.

$$(23)\quad dy_2/dx_2 = 0.01\ a\ Y_1 \left(\frac{4Y}{64^2 b^2 X} x_2 - \frac{4Y}{64^4 b^4 X^2} x_2{}^3\right)$$

Let $dy_2/dx_2 = 1$

Also recalling equation 21

$$(21)\quad x_2 = 64\ b\ x^{\frac{1}{2}}$$

$$x_2{}^3 = 64^3 b^3 x^{3/2}$$

The above values for x_2 and $x_2{}^3$ can be substituted into equation 23, and after simplification gives the following.

$$dy_2/dx_2 = 1 = \frac{aY_1}{b}\left(\frac{0.04\ Y}{64X}\ x^{\frac{1}{2}} - \frac{0.04\ Y}{64X^2}x^{3/2}\right)$$

Upon further simplification and rearrangement when Y = 100 percent and X = 150 pounds P_2O_5 per acre the following results.

$$(25)\quad a/b = \frac{1}{Y_1\ (4.167 \times 10^{-4}\ x^{\frac{1}{2}} - 2.778 \times 10^{-6}\ x^{3/2})}$$

When soil test values, x, are substituted into equation 25 and a value for maximum yield, Y_1, we can compute the ratio of market price per unit to cost per pound of fertilizer at which an incremental yield return equals an incremental fertilizer cost.

It would be more helpful, however, to be able to easily identify a given soil test value at which, at a given ratio of market price per yield unit and cost per pound of fertilizer, and at a given maximum yield possibility, an incremental yield return would equal an incremental fertilizer cost.

A graphic solution is convenient which avoids the awkward solution of x in equation 25 at given values of a/b. When a series of convenient values for x are substituted into equation 25, and for selected values of Y_1, the corresponding values of a/b can be computed and plotted versus the soil test value as illustrated in Figure 12.

Figure 12 may be useful in the following way. If the market price of corn is \$2.60 per bushel and the cost of P_2O_5 fertilizer is \$0.20 per pound, the ratio of these values would be 2.60 ÷ 0.20 or 13. If the maximum yield possibility for a given soil should be estimated to be 100 bushels per acre, the soil test value at which an incremental return equals the incremental cost would occur at two points: one at approximately 3 pounds per acre and another at approximately 125 pounds per acre. The latter is designated as point A on Figure 12. The soil test value of 125 pounds P_2O_5 per acre would provide a 97.4 percent yield.

If the ratio a/b should be greater, for example, \$3.00 ÷ \$0.15 = 20, the soil test value at which an incremental return equals an incremental cost would be near 135 pounds P_2O_5 per acre or near 99 percent yield level. Similar relationships for maximum yield possibilities greater than 100 bushels per acre are illustrated by other curves on the graph.

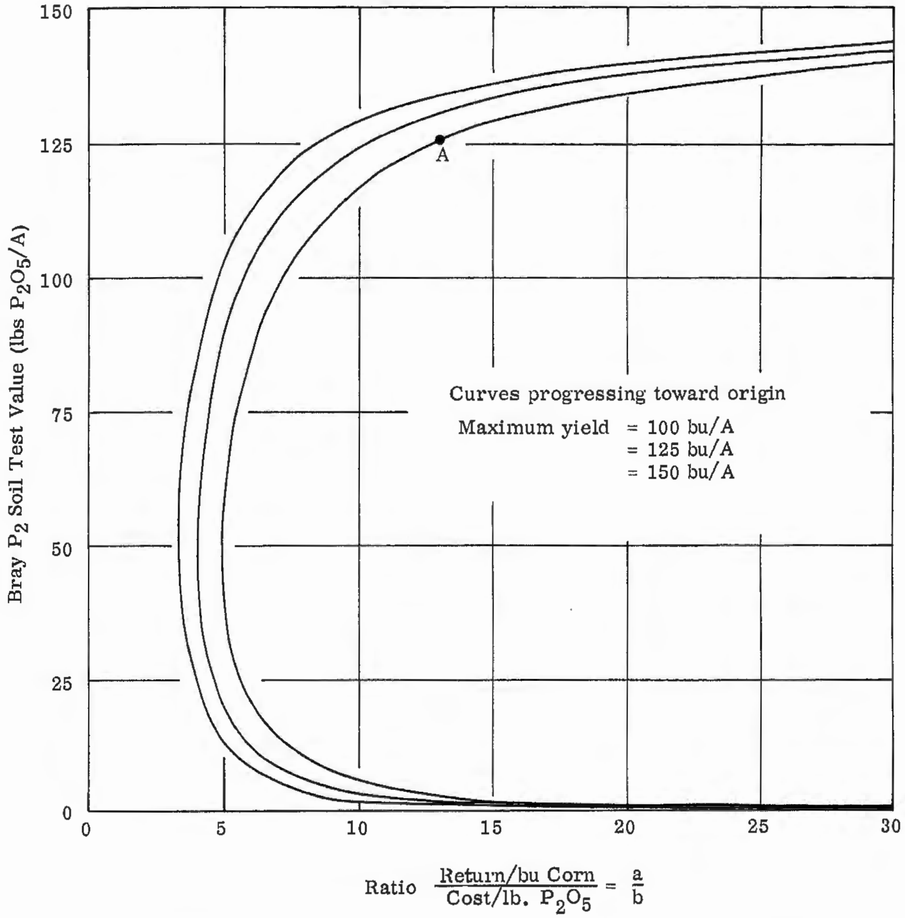

Figure 12. Relationships between Bray P_2 soil test values and ratios of the selling price of corn and the purchase cost of P_2O_5 fertilizer at which an incremental yield return equals the incremental fertilizer cost for producing it. Several maximum yield possibilities are considered.

It becomes apparent that any point on Figure 12 corresponding to values of a/b and soil test level which fall within the concavity outlined by any of the curves, is a point at which incremental returns are greater than the incremental costs necessary to produce the returns. Conversely, any point falling on the convex side of any of the curves is one at which an incremental return is less than the corresponding incremental cost.

For example, at a maximum yield possibility of 100 bushels per acre, and an a/b ratio of less than 5, there is no soil test value that can be established that would allow an incremental return equal to or greater than the incremental cost. Such a ratio would be one less than that provided by values of $1.00 per bushel for corn and $0.20 per pound for fertilizer. This can also be illustrated by Figure 11. The situation just described would be represented by a curve which at no point on it would the slope be equal to or greater than one.

It should be pointed out, however, that these considerations are being applied to a yield possibility of 100 bushels per acre the first year only, following a corrective treatment. In the example just cited, another interpretation would be that the cost of the corrective treatment could not be recovered the first year.

A situation could exist, however, in which it were desired to apply a corrective treatment of P_2O_5 fertilizer to that point at which an incremental return of 10 percent of the cost of the fertilizer increment were acceptable. This would represent a reasonable return on many kinds of investments. In this instance the a/b term in equation 25 would become a/0.1 b. In the example cited above in which corn was valued at $1.00 per bushel and P_2O_5 $0.20 per pound, P_2O_5 fertilizer could be applied until a soil test value of approximately 144 pounds P_2O_5 per acre was attained before the incremental return would be less than 10 percent of the incremental fertilizer cost. This would be a yield level above 99 percent.

An additional perspective emerges when one again considers that the relationships described above involve the return from yield during the first year only following corrective fertilizer treatments. If maintenance fertilizer treatments can be considered operational expense like pesticides, seed, and fuel, then once the corrective treatment is applied, yield levels are sustained during following years by maintaining soil test levels.

In effect then, Y_1, the maximum yield term in equation 25, can be doubled when two years are considered, or tripled when three years are considered, etc. Figure 13 illustrates these additional considerations when two or more years are involved by the inclusion of curves which could represent the accumulation of maximum yield values over some number of years.

From Figure 13 one must conclude that if farming is to continue for some period of several years, it would be profitable to operate at or very near that soil test value which provides maximum yield. A condition that would need to exist would be that at some point on a curve such as illustrated

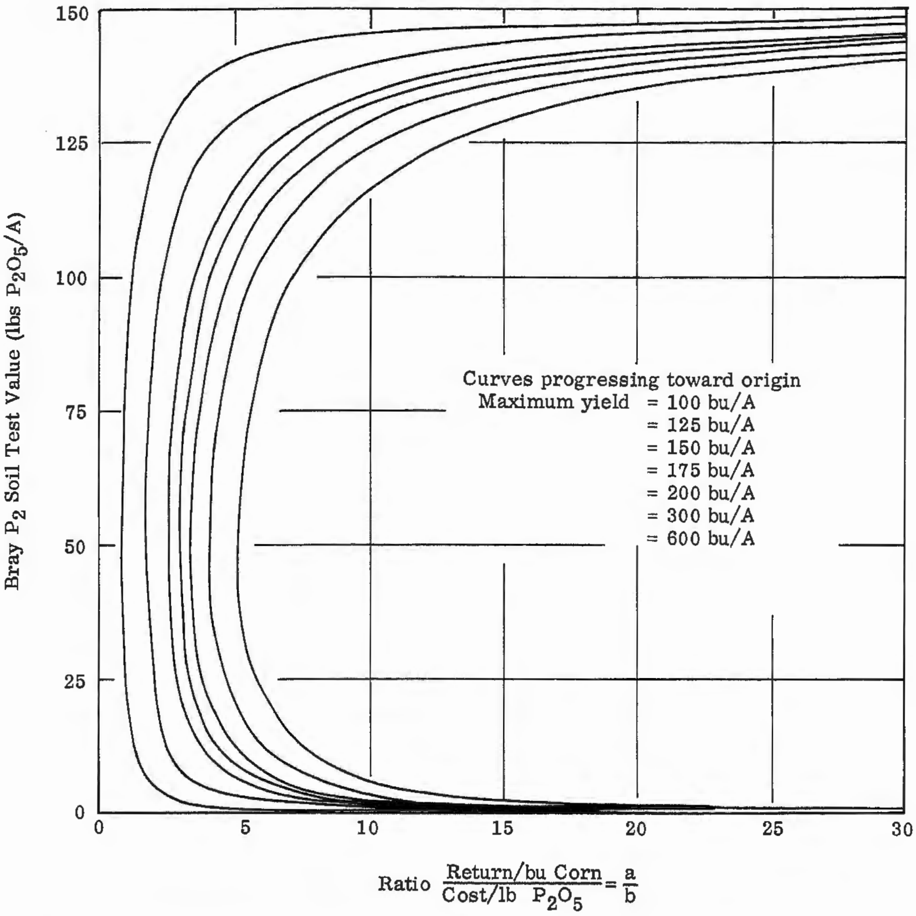

Figure 13. Relationships between Bray P$_2$ soil test values and ratios of the selling price of corn and the purchase cost of P$_2$O$_5$ fertilizer at which an incremental yield return equals the incremental fertilizer cost for producing it. Several maximum yield possibilities are considered.

in Figure 11, the slope must be greater than one. In this discussion no consideration of costs associated with seeding, harvesting, transporting, storage, or interest on money that would be incurred with increased yields was attempted. The effect of such costs would be to reduce the value of the crop yield or increase the cost of the fertilizer, either of which could be incorporated into the relationships portrayed in Figure 11.

LITERATURE CITED

1. Bray, Roger H., 1944. Soil Plant Relations: I. The quantitative relation of exchangeable potassium to crop yields and to crop response to potash additions. Soil Science 58:305-324.
2. Cope, J. T., 1973. Use of a fertility index in soil test interpretation. Communications in Soil Science and Plant Analysis 4(2), 137-146.
3. Hatfield, A. L., 1972. Soil Test Reporting: A nutrient index system. Communications in Soil Science and Plant Analysis 3(5), 425-436.
4. Stewart, R., 1932. The Mitscherlich, Wiessmann, and Neubauer methods of determining the nutrient content of soils, Imperial Bureau of Soil Science, Technical Communication No. 25, London.